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A survey on tube surfaces in Galilean 3-space

3-Boyutlu Galilean uzayında tüp yüzeyler üzerine bir araştırma

Yazar(lar) (Author(s)): *Fatma ALMAZ[1], Mihriban ALYAMAÇ KÜLAHCI[2]*

ORCID[1]: 0000-0002-1060-7813

ORCID[2]: 0000-0002-8621-5779

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A Survey on Tube Surfaces in Galilean 3-Space

Highlights

- *The Clairaut's theorem can be expressed for geodesic movement on tube surface defined in a coordinate system adapted to one parameter group of symmetries*
- *The specific energy and the angular momentum can be given on tubular surfaces in Galilean 3-space*
- *The conditions of being geodesic on the tubular surface can be given with the help of Clairaut's theorem*

Graphical Abstract

In this paper, the tube surfaces generated by the curve defined in Galilean 3-space are examined and some certain results of describing the geodesics on the surfaces are also given. Furthermore, the conditions of being geodesic on the tubular surface are obtained with the help of Clairaut's theorem, which allows us to constitute the specific energy. The physical meaning of the specific energy and the angular momentum is of course related with the physical meaning itself. Our results show that the specific energy and the angular momentum obtained on tubular surfaces can be expressed using arbitrary geodesic curve in Galilean space. In addition, some characterizations are given for these surfaces, with the obtained mean and Gaussian curvatures.

Aim

We consider the tube surfaces in Galilean 3-space to express specific kinetic energy and angular momentum on surfaces.

Design & Methodology

We indicate physical concepts on tube surface. Considering differential geometry formulas, we express them in Galilean 3-space.

Originality

All findings in the paper are original.

Findings

We define the tube surfaces using the arbitrary curve in Galilean 3-space. We calculate the specific kinetic energy and angular momentum on tube surface. Also, we give the geodesic equations on this surface.

Conclusion

The tubular surface and some certain results of describing the geodesics given on the surfaces are examined. Furthermore, we have explored the conditions of being geodesic, in which the curve can be chosen to be the curve defined in G_3, which allows us to constitute the specific energy, our results show that the specific energy and the angular momentum obtained on tubular surfaces can be expressed using arbitrary geodesic curve in Galilean space.

Declaration of Ethical Standards

The author of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

A Survey on Tube Surfaces in Galilean 3-Space

Araştırma Makalesi / Research Article

Fatma ALMAZ[1*], Mihriban ALYAMAÇ KÜLAHCI[2]

[1,2] Firat University, Faculty of Sciences, Department of Mathematics 23119 Elazığ, Turkey

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ABSTRACT

In this study, the tube surfaces generated by the curve defined in Galilean 3-space are examined and some certain results of describing the geodesics on the surfaces are also given. Furthermore, the conditions of being geodesic on the tubular surface are obtained with the help of Clairaut's theorem, which allows us to constitute the specific energy. The physical meaning of the specific energy and the angular momentum is of course related with the physical meaning itself. Our results show that the specific energy and the angular momentum obtained on tubular surfaces can be expressed using arbitrary geodesic curve in Galilean space. In addition, some characterizations are given for these surfaces, with the obtained mean and Gaussian curvatures.

Keywords: Galilean space, tube surface, geodesic curve, specific kinetic energy, specific angular momentum.

3-Boyutlu Galilean Uzayında Tüp Yüzeyler Üzerine Bir Araştırma

ÖZ

Bu çalışmada, Galilean 3-uzayında tanımlanan eğri tarafından üretilen tüp yüzeyleri incelenmiş ve yüzeyler üzerindeki jeodeziklerin açıklanmasının bazı sonuçları da verilmiştir. Ayrıca, tüp yüzeyde jeodezik olma koşulları, spesifik enerjiyi oluşturmak için Clairaut's teoremi yardımıyla elde edildi. Spesifik enerjinin ve açısal momentumun fiziksel anlamı elbette ki fiziksel anlamın kendisiyle ilişkilidir. Sonuçlarımız tüp yüzeylerde elde edilen spesifik enerjinin ve açısal momentumun Galilean uzayında keyfi jeodezik eğri kullanılarak ifade edilebildiğini göstermektedir. Ayrıca, elde edilen ortalama ve Gauss eğrilikleri elde edilerek, bu yüzeyler için bazı karakterizasyonlar verildi.

Anahtar Kelimeler: Galilean uzay, tüp yüzeyi, jeodezik eğri, spesifik kinetik enerji, spesifik açısal momentum.

1. INTRODUCTION

In recently, many researchers have begun to examine the curves and surfaces in Galilean space and afterwards pseudo-Galilean space. Geodesics have mostly studied in Riemannian geometry, metric geometry and general relativity by a lot of mathematicians. More surely, a curve on a surface is called to be geodesic if its geodesic curvature is zero. The geodesic equations are given by constant of motion due to energy, many approaches that reflect important use of energy idea are introduced in many books according to concerned topics. However, it seems attractive to use the relativistic energy in defining the central force problem. Furthermore, the equation of motion including the energy and angular momentum are a natural topic using by many applications.

In [3], the differential features of tubular surfaces were given by the author. In [5], the definition of parallel surface was given in Galilean space, the first and the second fundamental forms of parallel surfaces and connection between the curvatures of the parallel surfaces in Galilean space was also determined by the authors. In [6], the Darboux vectors of ruled surfaces were investigated and relationships between Darboux and Frenet vectors of each type of ruled surfaces were obtained by C. Ekici and M. Dede in pseudo-Galilean space. In [7], the problem of constructing a family of surfaces was analyzed from a given spacelike (or timelike) geodesic curve by the authors taking the Frenet frame of the curve in Minkowski space and they expressed the family of surfaces as a linear combination of the components of this frame and the necessary and sufficient conditions were also given. In [9], the twisted surfaces according to the supporting plane and type of rotations in pseudo-Galilean were investigated. In [10], the rotation surfaces in 4-dimensional pseudo-Euclidean spaces were studied by the authors. Also, the description of rotational surfaces in 4-dimensional (4D) Galilean space was expressed by the authors using a curve and matrices in G_4, [1]. In [11], the weighted mean and weighted Gaussian curvatures of surfaces of revolution in Galilean 3-space with density were expressed by the authors. In [15], the characterizations of helix for a curve with respect to the Frenet frame were obtained by authors in G_3. In [16], the authors investigated some curves in plane and in Galilean plane G_2. Furthermore, they defined the slant helix and gave the some characterization of slant helices in Galilean space G_3. In [18], the author studied surfaces of revolution in G_3 and characterized surfaces of revolution in G_1^3 as to the position vector field and Gauss map. Furthermore, some studies and

*Sorumlu Yazar (Corresponding Author)
e-posta : fb_fat_almaz@hotmail.com

results about surfaces in G_3 were given by the authors in [3,8,13,23]. In [24], the author established Frenet-Serret frame of a curve in the G_4 and he obtained the mentioned curve's Frenet-Serret equations. Also, he proved that tangent vector of a curve in G_4 satisfing a vector differential equation of fourth order.

The mass m of the particle whose motion traces out a geodesic path is unconnected in this problem, these physical features as energy and momentum that they include the mass as well proportioned factor will instead by changed by the "specific" features supplied by dividing out the mass. Therefore, since the kinetic energy is $E = mW^2/2$, the specific kinetic energy $\left(E = W^2/2\right)$ is divided by the mass m. Hence, both the specific energy and speed are constant for an affine parametrization of the geodesic. In [21,22], the system of two second order geodesic equations it is expressed that a standard physics technique of partially integrating them can be used and so reducing them to two first order equations by taking two constants of the movement that it expose from the equations of movement and in these references some conclusions in a constant energy (and in a constant angular momentum and rotational symmetry) are given according to time translation. Therefore, we can say that the specific energy of the particle is constant because of the point of view of its motion in space as the physical approach according to references [21,22], it is only accelerated perpendicular to the surface. If a force is accountable for this acceleration, that is to say the normal force which supplies the particle on the surface, since it is perpendicular to the velocity of the particle. Hence, we can say that its energy and specific energy E must be constant. Resembling the speed must be constant along a geodesic according to this cause, the existence of this constant is a result of the one parameter rotational group of symmetries of the surface, as a constant of the movement introduces a new thing since the surface is invariant under any 1-parameter group of symmetries. Mathematically, this is a constant obtained by Clairaut for geodesic movement on surface defined in a coordinate system adapted to this 1-parameter group of symmetries, [17].

In this study, we try to express specific energy and specific angular momentum on tube surfaces in Galilean space and that the speed is constant along a geodesic is shown according to Clairaut's theorem. Furthermore, using some parameters, the geodesic formulaes are given.

2. PRELIMINARIES

The scalar product of the vectors $U = (u_1, u_2, u_3)$, $V = (v_1, v_2, v_3)$ in G_3 is expressed as

$$\langle U, V\rangle_{G_3} = \begin{cases} u_1 v_1, & \text{if } u_1 \neq 0 \vee v_1 \neq 0 \\ u_2 v_2 + u_3 v_3, & \text{if } u_1 = 0 \wedge v_1 = 0 \end{cases}. \quad (1)$$

The cross product of Galilean space is given as

$$U \times V = \begin{cases} \begin{pmatrix} 0, \\ v_1 u_3 - v_3 u_1, \\ v_2 u_1 - v_1 u_2 \end{pmatrix}, \text{if } u_1 \neq 0 \vee v_1 \neq 0 \\ \begin{pmatrix} v_3 u_2 - v_2 u_3, \\ 0, \\ 0 \end{pmatrix}, \text{if } u_1 = 0 \wedge v_1 = 0 \end{cases}, \quad (2)$$

[12].

Let $\delta : I \subset \mathbb{R} \to G_3$ be an unit speed curve given by $\delta(x) = (x, y(x), z(x))$, where x is a Galilean invariant parameter. The vectors of the Frenet-Serret frame are defined as

$$t(x) = \delta'(x) = (1, y'(x), z'(x)),$$
$$n(x) = \frac{\delta''(x)}{\|\delta''(x)\|} = \frac{1}{\kappa(x)}(0, y''(x), z''(x)),$$
$$b(x) = t(x) \times n(x) = \frac{1}{\kappa(x)}(0, -z''(x), y''(x)),$$

where the real valued functions $\kappa(x) = \|t'(x)\|$ and $\tau(x) = \|n'(x)\|$ are curvatures of the curve δ. The curvature and torsion of the curve δ are defined by

$$\kappa(x) = \|\delta''(x)\|;$$
$$\tau(x) = \frac{1}{\kappa^2(x)} \det\left(\delta'(x), \delta''(x), \delta'''(x)\right).$$

For the curve in G_3, Frenet-Serret equations are written as follows

$$t' = \kappa n, \; n' = \tau b, \; b' = -\tau n. \quad (3)$$

The equation of a surface $\Omega = \Omega(\xi, v)$ in G_3 is given by

$$\Omega(\xi, v) = (x(\xi, v), y(\xi, v), z(\xi, v)). \quad (4)$$

Then the unit isotropic normal vector field η on $\Omega(\xi, v)$ is given by

$$\eta = \frac{\Omega_\xi \times \Omega_v}{\left\|\Omega_\xi \times \Omega_v\right\|}, \quad (5)$$

where the partial differentiations with respect to ξ and v will be denoted as follows

$$\Omega_\xi = \frac{\partial \Omega(\xi, v)}{\partial \xi}, \; \Omega_v = \frac{\partial \Omega(\xi, v)}{\partial v}. \quad (6)$$

On the other hand, the isotropic unit vector ς on the tangent plane of the surface is defined as

$$\varsigma = \frac{x_v \Omega_\xi - x_\xi \Omega_v}{w}, \tag{7}$$

where $x_\xi = \frac{\partial x(\xi,v)}{\partial \xi}$, $x_v = \frac{\partial x(\xi,v)}{\partial v}$ and $w = \left\| \Omega_\xi \times \Omega_v \right\|$. Let us define

$$\begin{aligned} &g_1 = x_\xi,\ g_2 = x_v,\ g_{ij} = g_i g_j; \\ &g^1 = \frac{x_v}{w};\ g^2 = \frac{x_\xi}{w}; \\ &g^{ij} = g^i g^j;\ i, j = 1, 2 \end{aligned} \tag{8}$$

$$\begin{aligned} &h_{11} = \left\langle \Omega^*_\xi, \Omega^*_\xi \right\rangle,\ h_{12} = \left\langle \Omega^*_\xi, \Omega^*_v \right\rangle; \\ &h_{22} = \left\langle \Omega^*_v, \Omega^*_v \right\rangle, \end{aligned} \tag{9}$$

where Ω^*_ξ and Ω^*_v are the projections of the vectors Ω_ξ and Ω_v onto the yz -plane, respectively, and the first fundamental form ds^2 of the surface $\Omega(\xi, v)$ is given by

$$\begin{aligned} ds^2 &= ds_1^2 + ds_2^2 = \left(g_1 d\xi + g_2 dv\right)^2 \\ &+ \varepsilon\left(h_{11} d\xi^2 + 2h_{12} d\xi dv + h_{22} dv^2\right), \end{aligned} \tag{10}$$

[12]. In this case, the coefficients of ds^2 are denoted by g^*_{ij}. That is, the function can be represented in terms of g_i and h_{ij} as follows

$$w^2 = g_1^2 h_{22} - 2g_1 g_2 h_{12} + g_2^2 h_{11}. \tag{11}$$

The Gaussian curvature and the mean curvature of a surface are defined by means of the coefficients of the second fundamental form L_{ij}, which are the normal components of $\Omega_{ij}(i, j = 1, 2)$. So that,

$$\Omega_{ij} = \sum^2 \Gamma^k_{ij} \Omega_k + L_{ij}\eta, \tag{12}$$

where Γ^k_{ij} is the Christoffel symbols of the surface and L_{ij} are given by

$$\begin{aligned} L_{ij} &= \frac{1}{g_1}\left\langle g_1 \Omega^*_{ij} - g_{ij}\Omega^*_1, \eta \right\rangle \\ &= \frac{1}{g_2}\left\langle g_2 \Omega^*_{ij} - g_{ij}\Omega^*_2, \eta \right\rangle, \end{aligned} \tag{13}$$

from this, the Gaussian curvature K and the mean curvature H of the surface are given as

$$\begin{aligned} K &= \frac{L_{11}L_{22} - L_{12}^2}{w^2}, \\ H &= \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}^2}{w^2}, \end{aligned} \tag{14}$$

[19].

Definition 1. Let $\gamma(s) = (\rho(s), 0, h(s))$ be a regular parametrized plane curve with $x = \rho(s) > 0,\ z = h(s)$. Then, the surface of revolution is created by rotating the curve γ around the z axis yielding a surface parametrized by

$$\begin{aligned} &\sigma(u,v) = (\rho(u)\cos v, \rho(u)\sin v, h(u)); \\ &u \in I, 0 \le v \le 2\pi, \end{aligned}$$

[12, 17].

Definition 2. Let $\delta\ :\ I \subset \mathrm{R} \to M$ be a curve given by

$$\delta(s) = \left(x\left(\xi(s), v(s)\right), y\left(\xi(s), v(s)\right), z\left(\xi(s), v(s)\right)\right), \tag{15}$$

which is an arc length parametrized geodesic on a surface of revolution. We need the differential equations satisfied by $\left(\xi(s), v(s)\right)$. Denote the differentiation with respect to s by an overdot. From the Lagrangian:

$$L = \dot{\xi}^2 + \rho^2 \dot{v}^2, \tag{16}$$

we obtain the Euler-Lagrange equations

$$\begin{aligned} &\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial \xi}{\partial s}}\right) = \frac{\partial L}{\partial \xi};\ \frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v}, \\ &\ddot{\xi} = \rho\rho' \dot{v}^2,\ \frac{d}{ds}\left(\rho \dot{v}^2\right) = 0, \end{aligned} \tag{17}$$

so that is a constant of the motion, [12, 17].

Definition 3. A vector $v = (v_1, v_2, v_3)$ is said to be a non-isotropic if $v_1 \neq 0$. If $v_1 = 0$, the vector $v = (v_1, v_2, v_3)$ is said to be isotropic and all unit isotropic vectors are denoted as $v = (1, v_2, v_3)$, [12].

Theorem 1. (Clairaut's Theorem) Let δ be a geodesic on a surface of revolution S, let ρ be the distance function of a point of S from the axis of rotation, and let θ be the angle between δ and the meridians of S. The $\rho\sin\theta$ is constant along δ. Conversely, if $\rho\sin\theta$ is constant along some curve δ on the surface, and if no part of δ is part of some parallel of S, then δ is a geodesic, [17].

3. THE MATHEMATICAL APPROACH ON TUBE SURFACE IN G_3

In this section, we try to express the tube surfaces generated by the position vector $\rho(\xi)$ of an arbitrary curve according to mathematical approach in G_3.

Let us denote by the vector ρ connecting the point from the parametrized curve $\delta(\xi)$ from the surface. Also, the position vector of a point on the surface is given as $R = \delta(\xi) + \rho$, since ρ lies in the Euclidean normal plane of the curve $\delta(\xi)$, the points at a distance A from a point of $\delta(\xi)$ form a Euclidean circle in G_3. Thus, it is easy to write that

$$\rho = A(\cos v \overrightarrow{n} + \sin v \overrightarrow{b}), \tag{18}$$

where A is a constant radius of a Euclidean circle of the Galilean frenet frame, v is the Euclidean angle between the isotropic $\vec{n}$ and $\vec{\zeta}$, [4]. Combining R and (18), we can define a tubular surface with constant radius A in term of the Galilean Frenet frame as

$$\Omega(\xi, v) = \delta(\xi) + A(\cos v \overrightarrow{n} + \sin v \overrightarrow{b}), \tag{19}$$

where $\vec{n}$ is the unit isotropic normal vector of the surface along a curve $\delta(\xi)$.

3.1. Clairaut's Theorem on Tubular Surfaces in Galilean 3-Space

In this subsection, we use the position vector $\delta(\xi)$ of an arbitrary curve in G_3 (see [2]) and using the Clairaut's theorem in G_3, the tube surface generated by this curve are characterized.

Theorem 2. In [2], the position vector $\delta(\xi)$ of an arbitrary curve with curvature $\kappa(\xi)$ and torsion $\tau(\xi)$ in the Galilean space G_3 is computed from the natural representation form

$$\delta(\xi) = \begin{pmatrix} \xi, \\ \int\left(\int \kappa(\xi)\cos\left(\int \tau(\xi)d\xi\right)d\xi\right)d\xi, \\ \int\left(\int \kappa(\xi)\sin\left(\int \tau(\xi)d\xi\right)d\xi\right)d\xi \end{pmatrix}.$$

Theorem 3. Let $\delta : I \subset \mathrm{R} \to G_3$ be a regular isotropic curve with curvatures $\kappa(\xi) \neq 0$ in G_3 and let $\Omega(\xi, v)$ be the tubular surface generated by the position vector $\delta(\xi)$ of an arbitrary curve in G_3. Then, the following statements hold:

1) The tubular surface is given by

$$\Omega(\xi, v) = \xi\vec{t} + \left(\int f(\xi)d\xi + A\cos v\right)\vec{n} + \left(\int g(\xi)d\xi + A\sin v\right)\vec{b},$$

where $f(\xi)$ and $g(\xi)$ are the differential functions.

2) The Gaussian curvature K and the mean curvature H of the surface Ω are given by

$$K = \frac{\begin{array}{c} -\cos v\left(2\kappa + \xi\kappa' + f'(\xi) - 2\tau g(\xi)\right) \\ -\sin v\left(2\xi\kappa + g'(\xi) + 2\tau f(\xi)\right) \\ -\left(\tau'\sin v + \tau^2\cos v\right)\int f(\xi)d\xi \\ +\left(\tau'\cos v - \tau^2\sin v\right)\int g(\xi)d\xi \end{array}}{A} - 2\tau^2;$$

$$H = \frac{1}{2A}$$

and this family of the tube surface has constant mean curvature.

3) The first fundamental form for the surface Ω is given by

$$I = \begin{pmatrix} d\xi^2 & 0 \\ 0 & d\xi^2 + A^2dv^2 \end{pmatrix}.$$

4) The curve $\delta(\xi)$ is a geodesic on the surface $\Omega(\xi, v)$ if and only if the following parameters hold

$$v = \frac{c_2}{2A^2}s + d_2 \text{ or } v = \frac{1}{A}\int \sin\theta ds \text{ and}$$

$$\xi = \int \cos\theta ds \text{ or } \xi = \frac{c_1}{4}s + d_1,$$

where

$$f(\xi) = \int \kappa(\xi)\cos\left(\int \tau(\xi)d\xi\right)d\xi,$$

$$g(\xi) = \int \kappa(\xi)\sin\left(\int \tau(\xi)d\xi\right)d\xi, \; c_i, d_i \in \mathbb{R}_0.$$

Proof. The tube surface generated by the position vector $\delta(\xi)$ of an arbitrary curve in G_3 is parametrized by

$$\Omega(\xi, v) = \delta(\xi) + A(\cos v\vec{n} + \sin v\vec{b}), \tag{20}$$

where v is angle between the isotropic vectors $\vec{n}$ and $\vec{A}$ and using the following the curve

$$\delta(\xi) = \begin{pmatrix} \xi, \int\left(\int \kappa(\xi)\cos\left(\int \tau(\xi)d\xi\right)d\xi\right)d\xi, \\ \int\left(\int \kappa(\xi)\sin\left(\int \tau(\xi)d\xi\right)d\xi\right)d\xi \end{pmatrix},$$

we can write the tubular surface as

$$\begin{aligned}\Omega(\xi,v) &= \xi\vec{t} + \left(\int f(\xi)d\xi + A\cos v\right)\vec{n} \\ &+ \left(\int g(\xi)d\xi + A\sin v\right)\vec{b},\end{aligned} \tag{21}$$

where

$$\begin{aligned}f(\xi) &= \int \kappa(\xi)\cos\left(\int \tau(\xi)d\xi\right)d\xi; \\ g(\xi) &= \int \kappa(\xi)\sin\left(\int \tau(\xi)d\xi\right)d\xi.\end{aligned}$$

Then, we get partial derivatives of $\Omega(\xi,v)$ with respect to ξ and v as follows

$$\begin{aligned}\Omega_\xi &= \vec{t} + \begin{pmatrix}\xi\kappa + f(\xi) \\ -\tau\int g(\xi)d\xi \\ -\tau A\sin v\end{pmatrix}\vec{n} \\ &+ \begin{pmatrix}g(\xi) \\ +\tau\int f(\xi)d\xi \\ +\tau A\cos v\end{pmatrix}\vec{b} = N_\xi\end{aligned} \tag{22}$$

$$\Omega_v = -A\sin v\vec{n} + A\cos v\vec{b} = AN_v. \tag{23}$$

Also, for these vectors, the vector cross product is found as

$$\begin{aligned}&\Omega_\xi \times \Omega_v = -A\cos v\vec{n} - A\sin v\vec{b} \\ &\Rightarrow \left\|\Omega_\xi \times \Omega_v\right\| = A\end{aligned} \tag{24}$$

and from previous equations, by using (24), the unit isotropic normal vector η of $\Omega(\xi,v)$ is found as

$$\eta = -\cos v\vec{n} - \sin v\vec{b}, \tag{25}$$

furthermore, from (7), we can write that

$$N_v = \frac{\Omega_v}{A} = -A\sin v\vec{n} + A\cos v\vec{b},$$

since $\vec{n}$ and $\vec{b}$ are the isotropic vectors, we can find

$$\begin{aligned}&x(\xi,v) = \xi;\ x_\xi = 1 = g_1;\ x_v = 1 = g_2; \\ &g_{11} = 1,\ g_{12} = 0,\ g_{22} = 0; \\ &g^1 = 0,\ g^2 = \frac{-1}{A};\end{aligned} \tag{26}$$

$$h_{11} = 1,\ h_{12} = 0,\ h_{22} = A^2. \tag{27}$$

If we substitute (26) and (27) into (10), the coefficients of the first fundamental form of the special tube surface with the Galilean Frenet frame in G_3, and for isotropic vectors since $\varepsilon = 1$, are obtained as

$$\begin{aligned}&I = d\xi^2 + \varepsilon\left(d\xi^2 + A^2dv^2\right) \\ &\Rightarrow I = 2d\xi^2 + A^2dv^2.\end{aligned} \tag{28}$$

Moreover, to compute the second fundamental form of $\Omega(\xi,v)$, we have to calculate the following equations

$$\begin{aligned}\Omega_{\xi\xi} &= \begin{pmatrix}2\kappa + \xi\kappa' + f'(\xi) - 2\tau g(\xi) \\ +\tau^2\left(\int f(\xi)d\xi + A\cos v\right) \\ -\tau'\left(\int g(\xi)d\xi + A\sin v\right)\end{pmatrix}\vec{n} \\ &+ \begin{pmatrix}2\xi\kappa + g'(\xi) + 2\tau f(\xi) \\ +\tau^2\left(\int g(\xi)d\xi + A\sin v\right) \\ +\tau'\left(\int f(\xi)d\xi + A\cos v\right)\end{pmatrix}\vec{b} \\ \Omega_{vv} &= -A\sin v\vec{b} - A\cos v\vec{n}; \\ \Omega_{\xi v} &= -A\tau\sin v\vec{b} - A\tau\cos v\vec{n}.\end{aligned} \tag{29}$$

From (13) and (29), the coefficients of the second fundamental form are calculated as

$$\begin{aligned}L_{11} &= -\cos v\begin{pmatrix}2\kappa + \xi\kappa' + f'(\xi) - 2\tau g(\xi) \\ +\tau^2\int f(\xi)d\xi - \tau'\int g(\xi)d\xi\end{pmatrix} \\ &\quad -\sin v\begin{pmatrix}2\xi\kappa + g'(\xi) + 2\tau f(\xi) \\ +\tau^2\int g(\xi)d\xi + \tau'\int f(\xi)d\xi\end{pmatrix} - \tau^2 A \\ L_{11} &= -\cos v\left(2\kappa + \xi\kappa' + f'(\xi) - 2\tau g(\xi)\right) \\ &\quad -\sin v\left(2\xi\kappa + g'(\xi) + 2\tau f(\xi)\right) \\ &\quad -\left(\tau'\sin v + \tau^2\cos v\right)\int f(\xi)d\xi \\ &\quad +\left(\tau'\cos v - \tau^2\sin v\right)\int g(\xi)d\xi - \tau^2 A\end{aligned}$$

$$L_{22} = A;\ L_{12} = \tau A = 0. \tag{30}$$

Thus, the Gaussian curvature K and the mean curvature H are expressed as

$$K = \frac{\begin{aligned}&-\cos v\begin{pmatrix}2\kappa + \xi\kappa' \\ + f'(\xi) - 2\tau g(\xi)\end{pmatrix} \\ &-\sin v\left(2\xi\kappa + g'(\xi) + 2\tau f(\xi)\right) \\ &-\begin{pmatrix}\tau'\sin v \\ +\tau^2\cos v\end{pmatrix}\int f(\xi)d\xi \\ &+\begin{pmatrix}\tau'\cos v \\ -\tau^2\sin v\end{pmatrix}\int g(\xi)d\xi\end{aligned}}{A} - 2\tau^2 \tag{31}$$

$$H = \frac{1}{2A}. \tag{32}$$

Also, the first fundamental form has two variable parameters with $h_{12} = 0$. Moreover, it is important to note that, the coordinates of parametrization are orthogonal and since the first fundamental form is diagonal. Therefore, that means this surface has an orthonormal basis, thus is possible to generate Clairaut's theorem to it. So, for the isotropic vectors since $\varepsilon = 1$, we have the Lagrangian equation

$$2d^2\xi + A^2 d^2 v = L \text{ or } 2\dot{\xi}^2 + A^2 \dot{v}^2 = L \quad (33)$$

and a geodesic on the surface $\Omega(\xi, v)$ is given by the Euler-Lagrangian equations

$$\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial \xi}{\partial s}}\right) = \frac{\partial L}{\partial \xi}; \quad \frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v}. \quad (34)$$

1) For $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial \xi}{\partial s}}\right) = \frac{\partial L}{\partial \xi} = 0,$ we obtain $\frac{\partial L}{\frac{\partial \xi}{\partial s}} = 4\dot{\xi} =$ constant, which means $\xi = \frac{c_1}{4}s + d_1,$ where $d_1, c_1 \in \mathbb{R}_0$.

2) For $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v} = 0,$ we can find $\frac{\partial}{\partial s}\left(2A^2\dot{v}\right) = 0,$

which means that $2A^2\dot{v}$ is constant along the geodesic and we have

$$v = \frac{c_2}{2A^2}s + d_2. \quad (35)$$

Let $\delta(\xi)$ be a geodesic on the surface $\Omega(\xi, v)$, it is given by $(\xi(s), v(s))$ and also let θ be the angle between $\dot{\delta}$ and a meridian, where N_ξ is the vector pointing along meridians of Ω and N_v is the vector pointing along meridians of Ω. Hence, we can say that $\{N_\xi, N_v\}$ is a orthonormal basis and hence a unit vector $\dot{\delta}$ tangent to $\Omega(\xi, v)$ can be written as

$$\dot{\delta} = N_\xi \cos\theta + N_v \sin\theta = \dot{\xi}\Omega_\xi + \dot{v}\Omega_v$$

$$= \dot{\xi} N_\xi + \dot{v} A N_v.$$

We see that $A\dot{v} = \sin\theta$, hence we can write $2A^2\dot{v} = 2A\sin\theta$ being a constant along $\delta(\xi)$. On the contrary, $\delta(\xi)$ is a curve with $2A^2\dot{v} = 2A\sin\theta$ in G_3 which is a constant, the second Euler-Lagrange equation is satisfied, differentiating L and substituting this into the second equation yields the first Euler Lagrange equation. Hence, we obtain

$$v = \int \frac{\sin\theta}{A} ds. \quad (36)$$

Furthermore, for $\xi = \frac{c_1}{4}s + d_1$, we have $\dot{\xi} = \frac{c_1}{4}$ is constant along the geodesic and we see that $\dot{\xi} = \cos\theta$, hence we can write as $4\dot{\xi} = 4\cos\theta$ being a constant along $\delta(\xi)$. On the contrary, $\delta(\xi)$ is an curve with $4\cos\theta$ that is a constant, the first Euler Lagrange equation is supplied, differentiating L and substituting this into the second equation yields the second Euler Langrange equation. So, we get

$$\xi = \int \cos\theta ds \text{ or } \xi = \int \cos\theta ds + c_8, \quad (37)$$

where $c_i, d_i \in \mathbb{R}$.

Theorem 4. The general equations of geodesics on the tube surfaces generated by the isotropic position vector $\delta(\xi)$ of an arbitrary curves in G_3 are given by

1) For the parameter $v = \frac{c_2}{2A^2}s + d_2$ or $v = \frac{1}{A}\int \sin\theta ds$,

$$\frac{d\xi}{dv} = \frac{A\sqrt{4A^2L - c_3}}{c_2\sqrt{2}} \text{ or } \frac{d\xi}{dv} = \frac{A\sqrt{L - \sin^2\theta}}{\sqrt{2}\sin\theta} \quad (38a)$$

2) For the parameter $\xi = \frac{c_1}{4}s + d_1$ or $\xi = \int \cos\theta ds$,

$$\frac{dv}{d\xi} = \frac{\sqrt{2}}{Ac_1}\sqrt{8L - c_4} \text{ or } \frac{dv}{d\xi} = \frac{\sqrt{L - 2\cos^2\theta}}{A\cos\theta} \quad (38b)$$

where $c_i, d_i \in \mathbb{R}_0$.

Proof. In order to obtain the general equation of geodesics, we should consider the Euler Lagrange equations in (17).

1) For $v = \frac{c_2}{2A^2}s + d_2$ or $v = \frac{1}{A}\int \sin\theta ds$, we explain the equation of geodesic, from the solving of the differential equations in $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v}$, we obtain $\dot{v} = \frac{c_2}{2A^2\varepsilon}$ or $\dot{v} = \frac{\sin\theta}{A}$. If we put the value of $\dot{v}$ at $2\dot{\xi}^2 + A^2\dot{v}^2 = L$, we can write

$$2\left(\frac{d\xi}{dv}\frac{dv}{ds}\right)^2 + A^2\left(\frac{dv}{ds}\right)^2 = L, \quad (39)$$

we can obtain the general equation of geodesics on $\Omega(\xi, v)$ as $\frac{d\xi}{dv} = \frac{A\sqrt{4A^2L - c_3}}{c_2\sqrt{2}}$.

2) For $\xi = \int \cos\theta ds$ or $\xi = \frac{c_1}{4}s + d_1$, from the solving of the differential equations in $\frac{\partial}{\partial s}\left(\frac{\partial L}{\partial \frac{\partial \xi}{\partial s}}\right) = \frac{\partial L}{\partial \xi}$, we obtain

$\dot{\xi} = \frac{c_1}{4}$; $\dot{\xi} = \cos\theta.$ If we put the value of $\dot{\xi}$ at the lagrangian equation, we get

$$2\left(\frac{d\xi}{ds}\right)^2 + A^2\left(\frac{dv}{d\xi}\frac{d\xi}{ds}\right)^2 = L. \tag{40}$$

Hence, we obtain the general equation of geodesics on $\Omega^2(\xi, v)$ as $\frac{dv}{d\xi} = \frac{\sqrt{2}\sqrt{8L - c_4}}{Ac_1}$. Also, for the parameters $\dot{v} = \frac{\sin\theta}{A}$, $\dot{\xi} = \cos\theta$ and the equations of geodesics are given by

$$\frac{d\xi}{dv} = \frac{A\sqrt{L - \sin^2\theta}}{\sqrt{2}\sin\theta}; \frac{dv}{d\xi} = \frac{\sqrt{L - 2\cos^2\theta}}{A\cos\theta},$$

Where $c_i, d_i \in \mathrm{R}$.

4. THE PHYSICAL APPROACH ON SPECIAL TUBE SURFACES GALILEAN 3-SPACE

In this section, we try to explain physically, thinking tracing out a geodesic by becoming clear the affine parameter $\xi(s)$ with the time, so that the figure is now of a point particle setting out on the surface, following a path called the trajectory of the particle. Suppose that $\Omega(\xi(s), v(s))$ is a parametrized curve on the surface as follows

$$\Omega(\xi(s), v(s)) = \xi(s)\vec{t} + \begin{pmatrix} \int f(\xi(s))d\xi \\ + A\cos v(s) \end{pmatrix}\vec{n} + \begin{pmatrix} \int g(\xi(s))d\xi \\ + A\sin v(s) \end{pmatrix}\vec{b}. \tag{41}$$

Also, the tangent vector of the geodesic curve is called as the velocity defined by

$$\vec{W} = \frac{d\Omega(\xi(s), v(s))}{ds} = W^{\xi}\Omega_{\xi} + W^{v}\Omega_{v}; \quad W = \left(g_{ij}\frac{dy^i}{ds}\frac{dy^j}{ds}\right)^{1/2}. \tag{42}$$

If we want to calculate the derivative of this tangent vector along the curve on the surface, we have to need the product and chain rules. Hence, using the chain rule, the tangent vector of the curve δ can be taken down as

$$\frac{d\Omega(\xi(s), v(s))}{ds} = \frac{d\xi(s)}{ds}\Omega_{\xi} + \frac{dv(s)}{ds}\Omega_{v} \tag{43}$$

$$\dot{\delta} = N_{\xi}\cos\theta + N_{v}\sin\theta = \dot{\xi}\Omega_{\xi} + \dot{v}\Omega_{v} = \dot{\xi}N_{\xi} + \dot{v}AN_{v}. \tag{44}$$

We think that $W^{\xi^*} = \sqrt{2}^{-1}W^{\xi} = W\cos\theta,$ which is the first axis, is the radial velocity while W^{v} is the horizontal angular velocity and $W^{v^*} = AW^{v} = W\sin\theta$, which is the second axis, is also the horizontal component of the velocity vector. The velocity can be symbolized in respect of polar coordinates in the tangent plane to do clear its norm and slope angle according to the radial direction on the surface. The role of the changeable is played by the speed in this velocity plane, while the angle θ express the side of the velocity according to the side Ω_{ξ^*} in this plane, the direction of the velocity with respect to the direction N_{ξ} is given by the angle θ. Also, we can say that the speed is constant along the geodesic for affinely parametrized geodesics and the mass m of the point particle that its movement follows a geodesic path is insufficient in this matter, physically these features which are defined as energy and momentum which necessitates the mass as a proportionality element will in place of the specific features found by partitioning out the mass. Thus, the specific kinetic energy is given as follows

$$E = \frac{\left(\sqrt{2E}\cos\theta\right)^2 + \left(\sqrt{2E}\sin\theta\right)^2}{2} = \frac{W^2}{2} = \frac{1}{2}\left(W^2\cos^2\theta + W^2\sin^2\theta\right) = \frac{1}{2}\left(2\left(\frac{d\xi}{ds}\right)^2 + A^2\left(\frac{dv}{ds}\right)^2\right)$$

using the right side of the previous equations we can say that both the specific energy and speed are constant along geodesic.

Theorem 5. Let $\Omega(\xi, v)$ be the tube surface generated by the isotropic position vector $\delta(\xi)$ of an arbitrary curves in G_3. Then, if $\delta(\xi)$ is a geodesic on the surface $\Omega(\xi, v)$, the following statements hold:

1) For the parameter $v = \frac{c_2}{2A^2}s + d_2$ (or $v = \frac{1}{A}\int \sin\theta ds$), the specific angular momentum ℓ is given by

$$\ell = AW\sin\theta$$

and the specific energy E is given by

$$E = \frac{1}{2}(2\left(\frac{d\xi}{ds}\right)^2 + \frac{\ell^2}{A^2}) = \frac{1}{2}(2\cos^2\theta + \frac{\ell^2}{A^2})$$
$$= \frac{\ell^2}{2A^2\sin^2\theta}.$$

2) For the parameter $\xi = \int \cos\theta ds$ (or $\xi = \frac{c_1}{4}s + d_1$), the specific angular momentum ℓ is given by

$$\ell = \frac{1}{\sqrt{2}} W\cos\theta$$

and the specific energy E is given by

$$E = \frac{1}{2}\left(8\ell^2 + A^2\left(\frac{dv}{ds}\right)^2\right) = \frac{1}{2}\left(8\ell^2 + A^2\sin^2\theta\right) \text{ or}$$
$$E = \left(\frac{\ell}{\cos\theta}\right)^2.$$

Proof. 1) For the equations $v = \frac{c_2}{2A^2}s + d_2$ or $v = \frac{1}{A}\int \sin\theta ds,$ we can define as in the case of circular movement around an axis with Radius $\left\|\vec{R}\right\| = A$ or $\vec{R} = A\vec{e}_1$, that is to say the velocity $V^{v^*} = A\frac{dv}{ds}$ in the angular side multiplied by the radius A of the circle. Physically, the specific angular momentum ℓ is given as following equation

$$\ell = \vec{e}_3.\left(\vec{R}\times_{G_3}\vec{W}\right) = AW\sin\theta, \qquad (45)$$

since $V^{v^*} = A\frac{dv}{ds} = W\sin\theta = \sqrt{2E}\sin\theta,$ we can write $A^2\frac{dv}{ds} = AW\sin\theta$, and we say that the specific angular momentum is constant along a geodesic. Therefore, we have

$$\ell = A^2\frac{dv}{ds} \Rightarrow \frac{dv}{ds} = \frac{\ell}{A^2} \text{ or } \ell = A\sqrt{2E}\sin\theta. \qquad (46)$$

This expression can be rewritten the changeable angular velocity dv/ds in the specific energy formula E, the constant specific energy that is given according to the radial motion and another constant of the motion is given by

$$E = \frac{1}{2}(2\left(\frac{d\xi}{ds}\right)^2 + \frac{\ell^2}{A^2}) = \frac{1}{2}(\frac{c_5}{2}\frac{c_5}{2} + \frac{\ell^2}{A^2})$$
$$= \frac{1}{2}(2\cos^2\theta + \frac{\ell^2}{A^2}) \qquad (47)$$

and from $\ell = A\sqrt{2E}\sin\theta,$ we find

$$\frac{\ell^2}{2A^2\sin^2\theta} = E.$$

2) For the equation $\xi = \int\cos\theta ds$ (or $\xi = \frac{c_1}{4}s + d_1$) we write $\frac{1}{\sqrt{2}}\dot{\xi} = \frac{1}{\sqrt{2}}\cos\theta$ being a constant along $\delta(\xi)$ and using this situation we explain in this physics language. Also, we can express as in the case of circular movement round an axis with radius $\left\|\vec{R}\right\| = \frac{1}{\sqrt{2}}$ or $\vec{R} = -\frac{1}{\sqrt{2}}\vec{e}_2,$ that is to say the velocity $W^{\xi^*} = \frac{1}{\sqrt{2}}W^{\xi}$ $= W\cos\theta = \frac{1}{\sqrt{2}}\frac{d\xi}{ds} = \sqrt{2E^2}\cos\theta$ in the angular direction multiplied by the radius $\frac{1}{\sqrt{2}}$ of the circle. The first geodesic equation is told that the specific angular momentum is constant along a geodesic as follows

$$\ell = \vec{e}_3(\vec{R}\times\vec{W}) = \frac{1}{\sqrt{2}}W\cos\theta \qquad (48)$$

since $\frac{1}{\sqrt{2}}\frac{d\xi}{ds} = W\cos\theta,$ we can write $\frac{1}{2}\frac{d\xi}{ds} = \frac{1}{\sqrt{2}}W\cos\theta$, and we say that the specific angular momentum ℓ is constant along a geodesic.

So, we can write

$$\ell = \frac{1}{2}\frac{d\xi}{ds} = \frac{1}{\sqrt{2}}W\cos\theta$$
$$\Rightarrow \frac{d\xi}{ds} = 2\ell \text{ or} \ell = \sqrt{E}\cos\theta \qquad (49)$$

and from $\ell = \sqrt{E}\cos\theta,$ we find

$$E = \left(\frac{\ell}{\cos\theta}\right)^2.$$

Hence, this statement can be rewritten the changeable angular velocity $d\xi/ds$ in the specific energy formula according to the constant angular momentum, the constant specific energy E that is given according to the radial motion and another constant of the motion is given by

$$E = \frac{1}{2}\left(8\ell^2 + A^2\left(\frac{dv}{ds}\right)^2\right)$$
$$= \frac{1}{2}\left(8\ell^2 + A^2\sin^2\theta\right). \qquad (50)$$

6. CONCLUSION

In putting forward consideration the mathematical problem of geodesics on a surface, there is an important advantage in conceptual comprehending that results from taking the point of view of a physicist by explaining parametrized geodesics as the paths traced out in time by the motion of a point on the surface, recognizing the parameter as the time, this combination of the constants of the motion is of course also constant along a geodesic. The existence of this constant is a conclusion of the one parameter rotational group of symmetries of the tubular surface, like this a constant of the movement introduces a new thing when the surface is invariant under any one parameter group of symmetries, which is seen in the variational approximate to the geodesic equations easily. Mathematically, this quantity is a constant obtained by Clairaut for geodesic movement on surface defined in a coordinate system adapted to this one parameter group of symmetries [17]. Thinking about energy levels in an impact potential for the decreased movement then supply to be an extremely useful tool in studying the treatment and features of the geodesics.

In this paper, the tubular surface and some certain results of describing the geodesics given on the surfaces are examined. Furthermore, we have explored the conditions of being geodesic, in which the curve can be chosen to be the curve defined in G_3, which allows us to constitute the specific energy, our results show that the specific energy and the angular momentum obtained on tubular surfaces can be expressed using arbitrary geodesic curve in Galilean space.

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DECLARATION OF ETHICAL STANDARDS

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

AUTHORS' CONTRIBUTIONS

Fatma ALMAZ: put forward the first idea on the stated title and wrote, analyzed and commented on data.

Mihriban ALYAMAÇ KÜLAHCI: analyzed the data and reinterpreted

CONFLICT OF INTEREST

There is no conflict of interest in this study.

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