

Research Article

Mathematics Teacher Candidates' Conceptual Knowledge of the Concept of Limit in Single-Variable Functions^{1, 2}

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Abstract

The aim of this study is to investigate teacher candidates' conceptual understanding of the concept of limit in single-variable functions. The study sample consisted of 30 students who were studying Primary School Mathematics Teaching at the Department of Mathematics and Science Education at a state university in Turkey and were enrolled in the Analysis I course in their second year. This study used a basic qualitative research design, and data were collected through open-ended questions and clinical interviews with focus students. The results revealed that the teacher candidates gave memorized answers to conceptual knowledge questions. The results showed that the teacher candidates' concept definitions were generally based on the right-left limit equation theorem and the dynamic form of the limit. However, the results of the clinical interviews indicated that teacher candidates avoided giving the formal definition of a limit.

Keywords: *Limit, single-variable functions, conceptual knowledge.*

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Matematik Öğretmen Adaylarının Tek Değişkenli Fonksiyonların Limit Kavramına Yönelik Kavramsal Bilgileri

Öz

Bu araştırmanın amacı öğretmen adaylarının tek değişkenli fonksiyonların limit kavramına yönelik kavramsal anlamalarının incelenmesi üzerinedir. Araştırma, Türkiye’ de bir devlet üniversitesi Matematik ve Fen Bilimleri Eğitimi Bölümü İlköğretim Matematik Öğretmenliği Programı ikinci sınıf Analiz 1 dersini alan otuz öğretmen adayının katılımı ile gerçekleştirilmiştir. Araştırmanın modeli temel nitel araştırma olup veri toplama araçları açık uçlu sorular ve odak öğrencilerle yapılan klinik görüşmeyi içermektedir. Araştırmadan elde edilen veriler incelendiğinde öğretmen adaylarının kavramsal bilgi içeren sorulara ezbere dayalı yanıtlar verdikleri görülmüştür. Sonuçlar, öğretmen adaylarının kavram tanımlarını genellikle sağ-sol limit eşitliği teoremine ve limitin dinamik formuna dayandırdığını gösterdi. Bununla birlikte, klinik görüşmelerin sonuçları, öğretmen adaylarının limitin formal tanımını vermekten kaçındıklarını göstermiştir.

Anahtar Sözcükler: *Limit, tek değişkenli fonksiyonlar, kavramsal bilgi.*

Introduction

When people learn new things, they construct this knowledge on their prior knowledge. Similarly, mathematical knowledge is built by adding to already existing knowledge. If new knowledge can be appropriately connected and merged with prior knowledge, then the understanding of that concept is constructed (Skemp, 1971). Learning takes place as soon as the connection occurs in mind. If the conceptual knowledge is like interconnected rings, each ring includes pieces of knowledge separately. As the number of interconnected rings increases, the set of rings is expanded, and the connected pieces of knowledge are enriched. Since each added ring will create more meaningful learning, the concept represented by the set of rings will create meaningfulness (Hiebert and Lefevre, 1986). The increase in conceptual knowledge occurs with the increase of the connections between pieces of knowledge.

Each symbol in the formal language of mathematics makes sense when aided by appropriate concepts (Schoenfeld, 1985). In the process of learning conceptual knowledge, the student is a problem solver who can use his or her creativity, intuition, and abilities efficiently in problem-solving and mathematical knowledge production. To this end, the conceptual learning dimension sees mathematics as a network of interconnected concepts, thoughts, and proposes that the student himself or herself constructs knowledge instead of memorizing and copying mathematical concepts and thoughts (Baki and Bell, 1997). In the context of Analysis course, Artigue (2000) states that the unifying role of the limit concept is more important than the role of solving problems, while Cornu (1991) defines the concept of limit as a concept that is located at the center of each subject of analysis.

Conceptual learning is a way to understand mathematics and science better, although it is not often seen in the existing system. We encounter two different solvers in the solution stage of mathematical problems: the master and the apprentice. While the master reaches the solution by applying conceptual knowledge in the solution of the problem, the apprentice investigates whether the problem is similar to the previously solved problems and tries to apply the solutions he or she can remember for the new problem. The conceptual learning view argues that mathematical knowledge can be transferred directly to the student by the teacher and that internalizing the correct mathematical knowledge can be ensured by the activities of the student

himself or herself (Baki, 1995; Cobb, 1986; Noss & Baki, 1996). Since the subjects in mathematics courses are not enriched with conceptual knowledge, rote-learning activities are used mostly instead of meaningful learning experiences. Many students are not aware of what concepts and mathematics at the heart of their operations actually mean. Learning mathematics does not mean memorizing ready knowledge and storing it into the mind. It means using this knowledge in problem-solving to reveal one's own thoughts. When the student starts to think mathematics together with the conceptual structure, his or her success also increases (Porter and Masingila, 2000). Conceptual knowledge involves mathematical concepts themselves and their mutual relationships. Since mathematical concepts are essentially relationships created in the human mind, a certain level of mental development must be achieved in order to acquire them.

Limit, which various concepts such as continuity, derivative, and integral are based on, is one of the most important and fundamental concepts of mathematics. The concept of limit is also critical because it is a mathematical concept where it is not possible for students to reach the result easily by using algebra and arithmetic methods (Cornu, 1991). Therefore, the concept of limit is considered as an indicator of the transition to advanced mathematical thinking (Tall, 1992).

Considerable research has been conducted on the concept of limit. While most studies focus on the difficulties experienced by students and the sources of these difficulties (Bezuidenhout, 2001; Cornu, 1991; Ferrini-Mundy & Graham, 1994; Monaghan, 1991; Sierpiska, 1987; Tall and Vinner, 1981), there are also studies about the process how the concept is learnt. (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Mamona-Downs, 2001; Przeniosla, 2004; Roh, 2007). Cornu (1991) and Tall and Vinner (1981) stated that students conceptualize the limit in two ways as informal (dynamic) and formal (static). The informal definition of the limit defined by Tall and Vinner (1981) as a dynamic form is based on " $x \rightarrow a \Rightarrow f(x) \rightarrow L$ " or, in other words, " x approaches a as $f(x)$ approaches L ". The formal definition of the limit is expressed as " $\lim_{x \rightarrow a} f(x) = L \Leftrightarrow$ for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\exists |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$ " (Kabael, Barak & Özdaş, 2015). We could suggest that students have difficulty in conceptualizing limits formally (Tall and Vinner, 1981), and they tend to interpret the formal definition of a limit as a formula (Przeniosla, 2004).

Some researchers state that students have difficulty in conceptualizing limits formally because of the quantifiers “for every” and “there exists” in the formal definition of limits (Cottrill et al., 1996; Tall and Vinner, 1981). Tall and Vinner (1981) argue that students cannot make sense of the quantifiers “for every” and “there exists” in the formal definition of limits, so they have difficulty in proving the existence of a limit. Williams (1991) states that students may tend to plot the function and substitute the point where the limit is investigated in the algebraic expression of the function to verify the limit of a function at a point. Cottrill et al. (1996) propose a cognitive model called genetic decomposition, which explains what it means to understand the definition of the concept of limit rather than describing the types of difficulties that students have. Szydlik, (2000), Tall and Vinner, (1981) and Williams (1991) state that students tend to conceptualize limits in the dynamic way. Williams (1991) and Szydlik (2000) suggest that students who define a limit in the dynamic way do this by investigating approximation of the images in the function of the points close to the point where the limit is investigated or by examining the approximations through the graph of the function.

Tall (1980) and Williams (1991) argue that the dynamic expression of a limit makes it difficult for students to conceptualize the formal definition. Tall (1980) also states that students understand the dynamic form of limits more easily and that students use the dynamic form of limits despite attempts to guide them through the formal conceptualization of limits during teaching. Przeniosla (2004), on the other hand, states that students give the relationships they establish for the concept of limit of a function as the definition of the limit of a function and that students cannot recognize the inconsistency between these definitions and the formal definition of a limit.

The difficulties that prevent students from conceptualizing limits have numerous sources. Sierpinska (1987) states that the difficulties experienced by students in the limiting process are closely related to students’ perception of the concept of infinite. In fact, infinity is a concept that students find it difficult to grasp (Cornu, 1991; Juter and Grevholm, 2006; Tall, 2001). Tall (1992) states that students can perceive the concept of infinity as a number.

Determining the conceptual knowledge of teacher candidates about the concept of limit of a single-variable function could shed light on the learning process about the concept of limit at

university level. Therefore, the aim of this study is to investigate the conceptual knowledge levels of teacher candidates about the concept of limit of a single-variable function.

Theoretical Framework

The term concept is defined as a set of words used to describe a certain concept (Tall and Vinner, 1981). The concept definition may be formal, as accepted by the mathematical authorities, or it may also be informal (dynamic), a subjective explanation of that concept by students (Kabael et al., 2015). Szydlik, (2000), Tall and Vinner, (1981) and Williams (1991) state that students tend to conceptualize limits in the dynamic way. Williams (1991) and Szydlik (2000) suggest that students who express a limit in the dynamic form do this by investigating approximation of the images in the function of the points close to the point in question or by examining the approximations through the graph of the function. Cottrill et al. (1996) and Tall and Vinner (1981) argue that the reason for students' difficulty in formally conceptualizing limits is the quantifiers "for every" and "there exists" in the formal definition of limits. Tall and Vinner (1981) further state that students cannot make sense of the quantifiers "for every" and "there exists" in the formal definition of limits, so they have difficulty in proving the existence of a limit.

Several factors cause the difficulties that prevent students' conceptual understanding of limits. Sierpiska (1987) states that the difficulties experienced by students in the limiting process are closely associated with students' perception of the concept of infinite. In fact, infinity is a concept that students find it difficult to comprehend (Cornu, 1991; Juter and Grevholm, 2006; Tall, 2001). Tall (1992) states that students might consider the concept of infinity as a number.

According to the results from a study on how 16/17-year-old learners' conceptualizations of real number, limit and infinity concepts progress over one year, learners' basic conceptualizations of infinity and limits barely change over a year (Monaghan, 1986). We could suggest that students have difficulty in formally conceptualizing limits (Tall and Vinner, 1981) and tend to interpret the formal definition of limits as a formula (Przeniosla, 2004). Some researchers state that students have difficulty in conceptualizing limits formally because of the quantifiers "for every" and "there exists" in the formal definition of limits (Cottrill et al., 1996; Tall and Vinner, 1981). Tall and Vinner (1981) argue that students cannot make sense of the

quantifiers “for every” and “there exists” in the formal definition of limits, so they have difficulty in proving the existence of a limit. One challenge is that students have difficulty in understanding the algebraic representation in the traditional $\epsilon - \delta$ definition of limits (Cornu, 1991; Cottrill et al., 1996; Eryvncck, 1981; Fernandez, 2004).

Fernandez (2004) states that students are confused about what are represented by ϵ and δ , the connections among variables in the definition, and the reason why $|x - c|$ has to be positive while $|f(x) - L|$ does not. In addition, evidence indicates that challenges students have in quantification also affect learners’ problems with the formal definition of limits (Cottrill et al., 1996; Dubinsky, Elterman, and Gong, 1988; Tall and Vinner, 1981). Research shows that students’ difficulties with limit definition are partly caused by the use of quantifiers (Cottrill et al., 1996; Dubinsky, Elterman, and Gong, 1988; Tall and Vinner, 1981). Various studies on the concept of limit investigate students’ misconceptions about limits. (Bezuidenhout, 2001, Davis and Vinner, 1986; Ferrini-Mundy and Graham, 1994; Monaghan, 1991; Tall and Vinner, 1981; Williams, 1991). However, only few of these studies address students’ understanding of the formal definition (Cornu, 1991; Cottrill et al., 1996; Eryvncck, 1981; Fernandez, 2004; Tall, 1992; Tall & Vinner, 1981; Vinner, 1991; Williams, 1991).

Investigating teacher candidates’ knowledge about the concept of limit could contribute to the efforts to determine the learning process regarding the concept of limit. Therefore, the purpose of this study is to investigate the conceptual understanding of teacher candidates about the definition of limit of a single-variable function. For this purpose, the study seeks answers to the following:

- Define the concept of limit of a single variable f function at a point.
- Show that $\lim_{x \rightarrow x_0} (f - g)(x) = L_1 - L_2$ if $\lim_{x \rightarrow x_0} f(x) = L_1$ and $\lim_{x \rightarrow x_0} g(x) = L_2$.

Method

The Research Model

The aim of this study was to investigate primary school mathematics teacher candidates’ conceptual knowledge of limit in single-variable functions. Data were collected, analyzed, and interpreted using a basic qualitative research approach, which is widely used in educational

research. In order to collect data, the teacher candidates were asked open-ended questions prepared by experts and reviewed by at least three experts.

Participants

The study sample consisted of 30 students who were studying Primary School Mathematics Teaching at the Department of Mathematics and Science Education at a state university in Turkey and were enrolled in the Analysis I course in their second year. The participants were selected according to their levels of achievement, as shown by their answers to the open-ended questions. A total of six students were selected for the study: two students were selected from each of the high, medium, and low achievement levels. The two students with a high level of achievement were coded as K1 and K2, respectively; the two students with a medium level of achievement were coded as K3 and K4, respectively; and the two students with a low level of achievement levels were coded as K5 and K6, respectively.

Data Collection Tools

As a part of the Analysis I course, the topic on limit in single-variable functions was discussed in detail. Following the relevant lesson, the teacher candidates were asked open-ended questions to see how they constructed the subject conceptually, and their answers to the open-ended questions were analyzed. The open-ended questions included conceptual knowledge questions about the concept of limit. The data needed to find answers to the sub-problems in the study were collected by clinical interviews with the focus students. All stages of this study were followed in accordance with research and publication ethics, and on 09.04.2018 ethical committee approval numbered 41452 was obtained from Anadolu University Ethics Committee.

The Research Process

As a part of Analysis I course, the topic on limit in single-variable functions was discussed in detail. Following the relevant lesson, the teacher candidates were asked open-ended questions to see how they constructed the subject conceptually, and their answers to the open-ended questions were analyzed. This study lasted 4 weeks and 24 hours in total. The open-ended

questions included conceptual knowledge questions about the concept of limit. The data needed to find answers to the sub-problems in the study were collected by clinical interviews with the focus students.

All stages of this study were followed in accordance with research and publication ethics, and on 09.04.2018 ethical committee approval numbered 41452 was obtained from Anadolu University Ethics Committee.

Analysis of Data

In this study, data were analyzed using the content analysis method. The main purpose of content analysis is to reach the concepts and relationships that can explain the collected data. The basic process in content analysis is to bring together similar data within the context of certain concepts and themes and present them to the reader in a clear and comprehensible way (Yıldırım and Şimşek, 2006). A content analysis generally involves four aspects: processing and coding qualitative research data obtained from documents, developing themes, categorizing codes and themes, and describing and interpreting the results. Therefore, the data obtained were classified into themes in terms of accuracy (i.e., accuracy of the participants' responses) and analyzed in the form of tables. The findings from the research are provided under the theme topics.

The components based on this classification are as follows:

Correct : An answer that includes all the components of the correct answer.

Partially Correct: An answer that does not include all the components of the correct answer.

Incorrect : An answer that includes irrelevant or inaccurate knowledge or that includes misconception or is irrational.

Those teacher candidates who were placed in the Formal Correct category responded to the question by giving the formal definition of the concept of limit: " $\lim_{x \rightarrow x_0} f(x) = L \Leftrightarrow \text{for } \forall \epsilon > 0, \text{ there exists } \exists \delta > 0 \text{ such that } |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$ ". On the other hand, the teacher candidates who were placed in the Informal Correct category responded to the question by giving the following definition: " $x \rightarrow x_0 \Rightarrow f(x) \rightarrow L$ " or, in other words, " $f(x)$ approaches L

as x approaches x_0 ". The teacher candidates who were placed in the Correct category for the concept of neighborhood responded to the question by giving the following expression "if (x) exists in ε neighborhood of point L when x exists in δ neighborhood point x_0 , the limit of function f at point x_0 is L ". The teacher candidates who were placed in Partially Correct category for the three categories did not give all the components of the answers mentioned above. Some of these teacher candidates expressed quantifiers or variables incorrectly. Those candidates who did not use the quantifiers \forall, \exists or who wrote random things instead of \forall were placed in the Formal Partially Correct category.

Findings

The conceptual knowledge questions, which were prepared in order to investigate the participants' conceptual understanding of limit of single-variable functions, were analyzed in separate categories. The first question asked to the teacher candidates is as follows:

"Define the limit concept of a single-variable real-valued function".

The aim of the first question asked to teacher candidates was to examine their answers for the definition of limit at a certain point of a function in single-variable functions in terms of conceptual knowledge. The participants' responses were categorized as correct, partially correct and incorrect responses. Table 1 shows the frequency and percentage distributions of the teacher candidates' responses.

Table 1
Frequency and percentage distributions of the teacher candidates' responses about the definition of the concept of limit

	Frequency	Percentage (%)
Correct	6	20
Partially correct	13	43
Incorrect	11	37
Total	30	%100

Among the teacher candidates, 20% gave correct answers to the question about the limit concept in single-variable functions. For further analysis, the teacher candidates' responses

were examined in three sub-categories: formal definitions, informal definitions, and definitions using neighborhood concept.

Table 2
Categorization of the answers to the concept of limit

	Correct		Partially correct		Incorrect		Total
	Frequency	Percentage	Frequency	Percentage	Frequency	Percentage	
		(%)		(%)		(%)	
Formal Definition	4	19	10	48	7	33	21
Informal Definition	2	17	3	25	7	58	12
Definition using neighborhood concept	2	18	2	18	7	64	11

Those teacher candidates who made a correct formal definition of the concept of limit used the sentence, “Let $y = f(x)$ function be defined in the neighborhood of point x_0 . L is the limit of f at the point x_0 if there exists the number $\exists \delta > 0$ so that $|f(x) - L| < \varepsilon$ when $|x - x_0| < \delta$ for the number $\forall \varepsilon > 0$.” For informal definitions, the teacher candidates made definitions by using their own sentences. Among the participants, 70% made formal definitions. In general, the participants preferred to make formal definitions, but only 19% of them were able to make correct formal definitions. It was also determined that they used quantifiers incompletely when making formal definitions.

Among the participants, 37% made a definition by referring to the concept of neighborhood. However, only 18% of the participants expressed the concept of the neighborhood correctly. Among all the participants, 7% of the teacher candidates preferred to use both formal and informal definitions.

In general, the participants were not able to clearly express the concepts and inter-conceptual relationships. This result is consistent with the findings of Baki’s 1998 study. The teacher candidates were divided into two groups when making a formal definition of limit in single-variable functions: those who used quantifiers and those who did not. Next, the responses of those candidates who used quantifiers were grouped into three sub-categories: correct use,

partially correct use, and incorrect use. Table 3 shows the frequency and percentage distributions of the teacher candidates' responses.

Table 3.
Use of quantifiers in limit definition

Use of quantifiers in limit definition		Frequency	Percentage (%)
Use of Quantifiers	Correct use	10	%53
	Partially correct use	4	%21
	Incorrect use	5	%26

Out of the participants who used quantifiers when making a definition of the concept of limit, 53% used them correctly. One of the most noticeable errors of the participants while making a formal definition of the concept of limit was their frequent use of the expression, “For at least one ε ”. Some of the teacher candidates incorrectly stated the place of quantifiers in their definitions. The participants' incorrect use of quantifiers suggests that they did not master the subject.

Table 4.
The participants who did not use quantifiers in limit definition

	Frequency	Percentage (%)
Those who did not use quantifiers	8	27

Out of the participants, 27% did not prefer to use quantifiers. Instead, they defined the concept of limit using informal definitions.

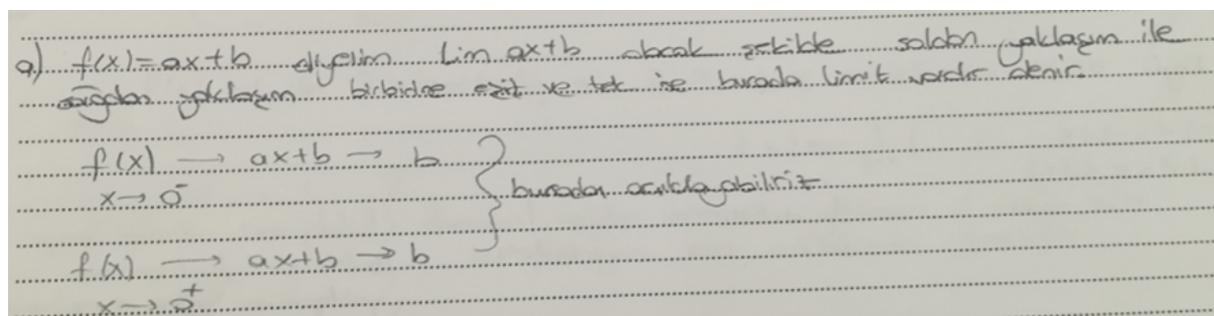


Figure 1. Example of a teacher candidate who made an informal definition

(English translation of Figure 1: Let $f(x) = ax + b$. $\lim ax + b$ approach from the left approach is equal to the right if there is a limit.)

As can be seen in the answer given in the Figure 1, this teacher candidate tried to make a definition with her own sentences and came up with an informal definition. However, the candidate's notations were incorrect and incomplete. Figures 1 show examples of teacher candidates placed in the Informal Partially Correct category.

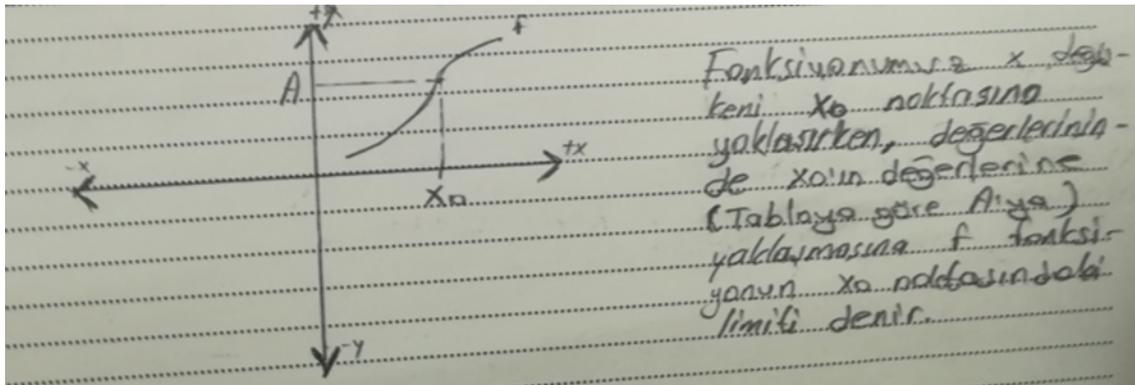


Figure 2. Example of a teacher candidate's response about limit definition

(English translation of Figure 2: When the variable x approaches the point x_0 , the value of the function is approached to A according to the table. In this case, the limit is A)

The teacher candidate made an informal definition for the limit of the function. On the other hand, the candidate did not address the concept of neighborhood while interpreting the limit of the function in the analytic plane. This suggests that the teacher candidate did not have sufficient conceptual knowledge of the concept of limit. Figures 2 show examples of teacher candidates placed in the Informal Partially Correct category.

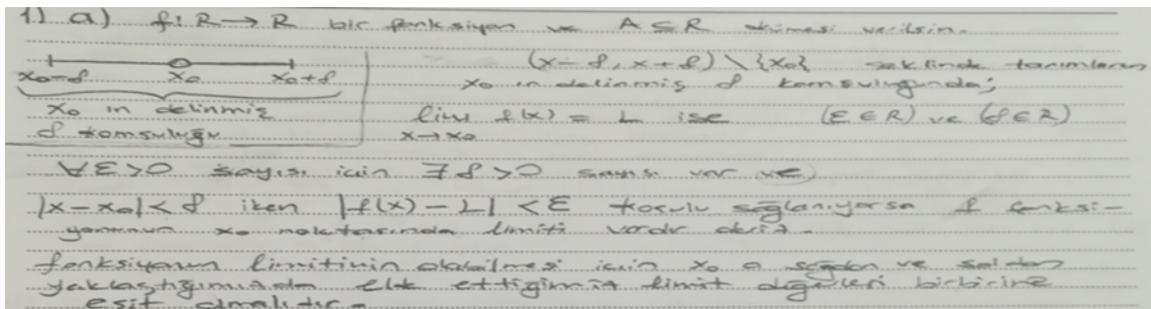


Figure 3. Example of a teacher candidate's expression of limit with the concept of deleted neighborhood

(English translation of Figure 3: If $\lim_{x \rightarrow x_0} f(x) = L$ in the perforated neighborhood defined in the form of point x_0 $(x - \delta, x + \delta) \setminus \{x_0\} \forall \epsilon > 0$, there exists $\exists \delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$. In addition, the right limit and the left limit must be equal for the limit of the function.)

Among the participants, 17% used the expression, “Let f be defined in the deleted neighborhood of x_0 .” The concept of limit becomes more explicable a relationship is established between the concept of neighborhood and the inequality $0 < |x - a| < \delta$ in the formal definition of limit (Kabael et al., 2015). When students do not make full sense of the concept of neighborhood, they tend to see the concept of limit as an abstract and difficult concept. This answer was considered in the Partially Correct category. The teacher candidate mentioned in the example shown in Figure 3 expressed the definition of a limit by means of both formal and informal and neighborhood concepts. This answer was considered in the Partially Correct category. This example clearly implies that the teacher candidate did not exactly know the formal definition and the meanings of quantifiers. That teacher candidates also incorrectly used the triangular inequality, which is an important property of absolute value. One of the reasons why the teacher candidates could not establish a relationship between the concept definition and the formal definition of the limit could be students’ inability to meaningfully use the quantifiers in the formal definition (Cottrill et al., 1996; Tall and Vinner, 1981).

During the interviews with teacher candidates, the participants were asked what they understood by the concept of limit. The majority of the candidates considered the limit with the concept of convergence, as follows:

“A: What do you understand by the concept of limit in single-variable functions?”

K1 : Investigating the limit of an unknown at that point.

K2 : The point at which a line can reach and converge is the limit.

K3: We use it to see the range of a function defined by a given variable when approaching a certain point.

K4: Assume that we want to take the limit of $f(x)$. There is only one variable here. Here, we need to find out where the result approaches based on where x approaches.

K5: It represents “convergence” for the limit of functions containing one variable.

A: Can you explain the definition of convergence in detail?

K5: I understand the result of this function converging a point.

K6: The function depends on only one parameter. We're talking about convergence the function for a certain point. By approaching the function from the left and right, we can learn about the presence and absence of a limit.”

In order to reveal how the teacher candidates established a relationship between the informal definition of a limit or the right-left limit equation theorem and the formal definition of a limit when they defined a limit, they were asked to explain their definitions of the concept of limit.

The purpose of this attempt was also to investigate how the teacher candidates perceived the concepts of neighborhood and approach, how they established connections between the concepts of neighborhood and approach and between the concept of neighborhood and the inequalities “ $|x - x_0| < \delta$ ” and “ $|f(x) - L| < \varepsilon$ ” in the formal definition, and how they used the quantifiers in the formal definition. The interviewed teacher candidates frequently used the term “approach” when explaining what they understood by the concept of limit. However, they avoided giving the formal definition a limit. In other words, they preferred the informal definition to the formal definition of the concept of limit. When explaining what he understood by the concept of limit, a teacher candidate said, “We use it to see the range of image clusters of the function defined by a given variable when going to a certain point”. We can conclude based on this statement that the teacher candidate, who attempted to define a limit informally, considered the limit as an approach. As a result, when the results of the interviews with the teacher candidates were considered as a whole, it was observed that the right-left limit equality theorem and the informal definition of a limit were more dominant in the teacher candidates’ minds.

The second question asked to the candidates involved writing a proof:

“Let $\lim_{x \rightarrow x_0} f(x) = L_1$ and $\lim_{x \rightarrow x_0} g(x) = L_2$. Prove that $\lim_{x \rightarrow x_0} (f - g)(x) = L_1 - L_2$.”

Table 5
Frequency and percentage distributions of responses to question 1/b

	Frequency	Percentage (%)
Correct	7	23
Partially correct	22	74
Incorrect	1	3
Total	30	%100

Among the teacher candidates, 23% gave correct answers to Part b of Question 1. This question addressed conceptual and procedural knowledge of the subject, and 74% of the candidates gave partially correct answers to the question. This result suggests that the candidates gave vague answers to the question.

Analysis of the teacher candidates' responses to Question 1/b shows that the participants used quantifiers incorrectly and incompletely. While the expression $\forall \varepsilon > 0$ should be used, the expression "arbitrary $\varepsilon > 0$ " was preferred. This result suggests that the teacher candidates did not exactly know the formal definition and meaning of quantifiers.

This question asked the participants to show the result $\lim_{x \rightarrow x_0} (f - g)(x) = L_1 - L_2$ for $\lim_{x \rightarrow x_0} f(x) = L_1$ and $\lim_{x \rightarrow x_0} g(x) = L_2$. The responses to the sub-problem were divided into sub-categories for in-depth analysis of the problem. These responses were categorized as those who chose δ_1 against $\forall \varepsilon > 0$ for L_1 , those who chose δ_2 against $\forall \varepsilon > 0$ for L_2 , and those who chose $\min(\delta_1, \delta_2)$ for $L_1 - L_2$.

Table 6.

Selection of values corresponding to δ_1 and δ_2

	Frequency	Percentage (%)
Those who chose $L_1 \rightarrow \delta_1$	24	80
Those who chose $L_2 \rightarrow \delta_2$	24	80
Those who chose $L_1 - L_2 \rightarrow$ $\min(\delta_1, \delta_2)$	22	73

The percentages of those who chose δ_1 against $\forall \varepsilon > 0$ for L_1 , those who chose δ_2 against $\forall \varepsilon > 0$ for L_2 , and those who chose $\min(\delta_1, \delta_2)$ for $L_1 - L_2$ were 80%, 80% and 73%, respectively. Among those who chose δ_1 and δ_2 , two participants did not choose $\min(\delta_1, \delta_2)$. Also, some of the participants who chose δ_1, δ_2 expressed the quantifiers incompletely.

The data obtained from the interviews with the K1, K2, K3, K4, K5, K6 coded students for the second question as follows:

"Assume that when we take $A: \varepsilon = 1$ we can take $\delta_1 = \frac{1}{2}$ for f and $\delta_2 = \frac{1}{3}$ for g .

What value can we assign to δ against $\varepsilon = 1$ for $f - g$ function?"

K1: "We can take δ value as $\delta_1 - \delta_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$."

R: "Why?"

K1: "This is what I remember by heart."

K2: "It is $\delta_1 - \delta_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$."

R: "Why?"

K2: "Because we assign the same value to ε ."

K3: "First of all, I would think I needed to know functions and come up with common solutions. But then I would think the formal definition would work for me and would say $\delta = \delta_1 - \delta_2 = \frac{1}{6}$."

K4: " $\delta = \delta_1 - \delta_2 = \frac{1}{6}$ "

R: "Why?"

K4: "Because I can say this by heart. Honestly, I don't know exactly why."

K5: " $\delta = \delta_1 - \delta_2 = \frac{1}{6}$ "

R: "Why $\frac{1}{6}$?"

K5: "Because the difference of the limits of individual functions is equal to the limit of the difference of functions."

K6: "I can break up $f-g$ function as $(f)-(g)$. If we assign $\delta_1 = \frac{1}{2}$ for f and $\delta_2 = \frac{1}{3}$ for g , we obtain $\delta = \frac{1}{6}$."

All the teacher candidates participating in the interviews estimated the value δ in the proof of the proposition as $\delta = \delta_1 - \delta_2$. Because the proposition was expressed as "If $\lim_{x \rightarrow x_0} f(x) = L_1$ and $\lim_{x \rightarrow x_0} g(x) = L_2$, show that $\lim_{x \rightarrow x_0} (f - g)(x) = L_1 - L_2$ ", they estimated the value δ as $\delta = \delta_1 - \delta_2$. The teacher candidates K1, K2, K3, K4, K5 and K6 explained the reason for this:

K1: "I remembered this situation by heart."

K2: "Because we give ϵ the same value."

K3: "I would firstly assume that I needed to know the functions and find a joint solution. But then I would think that the formal definition would work for me and would say $\delta = \delta_1 - \delta_2 = \frac{1}{6}$."

K4: "I say so because this is what I remember by heart. I do not know exactly why."

K5: "Because the difference of the limits of individual functions is equal to the limit of the difference of functions."

K6: "I can split the function $f-g$ as $(f)-(g)$. If I take $\delta_1 = \frac{1}{2}$ for f and $\delta_2 = \frac{1}{3}$ for g , the result will be $\delta = \frac{1}{6}$."

None of the teacher candidates who participated in the interviews determined the value δ as $\delta = \min(\delta_1, \delta_2) = \frac{1}{3}$. They estimated the value δ based on the theorem statement. This result suggests that the teacher candidates gave answers by heart as they clearly stated.

Results and Discussion

This study investigated teacher candidates' conceptual understanding of the concept of limit in single-variable functions. The teacher candidates were administered a test consisting of open-ended questions, and six focus students selected among the participants were interviewed. The following findings were obtained as a result of the test and interviews.

For the first question, the teacher candidates were asked to define the limit concept of a real-valued single-variable function. It was found that 20% of the candidates made a correct definition the concept of limit. The answers of the candidates were further sub-categorized in order to obtain in-depth results. For further analysis, the teacher candidates' responses about the definition of the concept of limit were examined in three categories: formal definitions, informal definitions, and definitions using neighborhood concept. It was determined that 70% of the candidates preferred to make a formal definition, but only 19% of the candidates who made a formal definition answered the question correctly. The candidates who made a definition of limit were divided into two sub-groups: those who used quantifiers and those who did not use quantifiers. It was found that 53% of the candidates who made use of quantifiers used them correctly. One of the noticeable errors of the candidates in making a formal definition of limit was that they preferred the expression "arbitrary $\varepsilon > 0$ " whereas the expression $\forall \varepsilon > 0$ should be used. It was also seen that the teacher candidates used the formal definition of limit and quantifiers without knowing what they actually meant. When teacher candidates were asked to define the concept of limit, they generally turned to the formal definition. When the data related to the sub-problems in the study were evaluated, it was found that the candidates used the quantifiers in the wrong place while using them in the formal definition, and during the interviews, they stated that they used quantifiers by memorization. The four different quantifier \forall, \exists, \ni definitions used in the formal definition of limit (ε - δ) make it difficult to understand (Todorov, 2001). This is consistent with the data obtained from this study.

In part b of Question 1, the participants were asked to show the result $\lim_{x \rightarrow x_0} (f - g)(x) = L_1 - L_2$ for $\lim_{x \rightarrow x_0} f(x) = L_1$ and $\lim_{x \rightarrow x_0} g(x) = L_2$. Among the participants, 3% answered the question incorrectly. The participants were found to be successful in applying the formal definition of the limit. The responses to the sub-problem were divided into sub-categories for in-depth analysis of the problem. These responses were categorized as those who chose δ_1 against $\forall \varepsilon$ for L_1 , those who chose δ_2 against $\forall \varepsilon$ for L_2 , and those who chose $\min(\delta_1, \delta_2)$ for $L_1 - L_2$. The percentages of those who chose δ_1 against $\forall \varepsilon'$ for L_1 and those who chose δ_2 against $\forall \varepsilon$ for L_2 were 80%, whereas the percentage of those who chose $\min(\delta_1, \delta_2)$ for $L_1 - L_2$ was 73%. When the candidates' responses were examined, it was determined that one of their noticeable errors was their use of quantifiers in the wrong place.

Research suggests that students have difficulty in formal conceptualization of limits (Tall and Vinner, 1981) and tend to interpret the formal definition of a limit as a formula (Przeniosla, 2004). Establishing the relationship between the concept of neighborhood and the inequality $0 < |x - a| < \delta$ in the formal definition of a limit facilitates the comprehension of the concept of limit (Kabael et al., 2015). When students cannot grasp the concept of neighborhood, they tend to regard the concept of limit as an abstract and challenging concept. This result is supported by the study of Kabael et al. (2015). The informal definition of a limit was dominant in the concept definitions of the teacher candidates in this study, and this result was consistent with the results reported by Williams (1991) and Szydlik (2000).

The results showed that the teacher candidates' concept definitions were generally based on the right-left limit equation theorem and the dynamic form of the limit. However, the results of the clinical interviews showed that they avoided giving the formal definition of a limit.

The teacher candidates' inability to use the quantifiers in the formal definition in a meaningful manner, as stated in the literature (Cottrill et al., 1996; Tall and Vinner, 1981), prevented them from establishing a relationship between the concept definition and the formal definition of a limit. Another reason for this result could be the teacher candidates' difficulty in formal conceptualization of limits due to their inability to make connections between the inequalities " $|x - x_0| < \delta$ " and " $|f(x) - L|$ " in their knowledge of the concepts of approach and neighborhood and the formal definition. In fact, as shown by evidence, students have difficulty in formal conceptualization of limits (Tall and Vinner, 1981) and tend to interpret the formal definition of a limit as a formula (Przeniosla, 2004). As Tall and Vinner (1981) and Cottrill et al. (1996) suggest, the reason why the teacher candidates had difficulty in formal conceptualization of limits could be the presence of "for every" and "there exists" in the formal definition of a limit.

When investigating the limit of a function at a point, the teacher candidates described the approach of x by taking a neighborhood of the point where the limit was investigated and taking the points close to the point where the limit was investigated. In addition, they described the approach of (x) by taking images of x as points close to the limit value within a neighborhood of the limit value.

In order to find out what the teacher candidates understood by the term “approach”, the researcher questioned what they meant when they used the word “approach” during the interviews. Most of the teacher candidates indicated approaching to a point correctly as a point. Limit, which various concepts such as continuity, derivative and integral are based on, is one of the key and fundamental concepts of mathematics. Teaching of this concept is of great importance.

Statements on ethics and conflict of interest

“I, as the Corresponding Author, declare and undertake that in the study titled as “*Mathematics Teacher Candidates’ Conceptual Knowledge of the Concept of Limit in Single-Variable Functions*”, scientific, ethical and citation rules were followed; Turkish Online Journal of Qualitative Inquiry Journal Editorial Board has no responsibility for all ethical violations to be encountered, that all responsibility belongs to the author/s and that this study has not been sent to any other academic publication platform for evaluation.”

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