Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 1-7
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.623500

Numerical Oscillation Analysis for Gompertz Equation with One Delay

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Article Info

Keywords: *Asymptotic behavior, Gompertz equation, Oscillation, θ-method*
2010 AMS: *65L10, 65L30*
Received: *23 September 2019*
Accepted: *20 March 2020*
Available online: *10 June 2020*

Abstract

This paper concerns with the oscillation of numerical solutions of a kind of nonlinear delay differential equation proposed by Benjamin Gompertz, this equation usually be used to describe the population dynamics and tumour growth. We obtained some conditions under which the numerical solutions are oscillatory. The non-oscillatory behaviors of numerical solutions are also analyzed. Numerical examples are given to test our theoretical results.

1. Introduction

In recent years, the studies on oscillation of the solutions of delay differential equations (DDEs) are developing rapidly (see [1, 2]). This research has been applied to many fields including biology, physics, ecology and so on. Nonetheless there are few papers have been published on the oscillation of numerical solutions of DDEs (see [3]-[6]). So we will consider numerical oscillation for Gompertz equation with one delay in this paper. In the past few years, Gompertz equation has been generally used to describe the population dynamics and tumour growth (see [7, 8]). In 1825, Benjamin Gompertz proposed the classical Gompertz model[9]

$$\dot{V}(t) = -rV(t)\ln\frac{V(t)}{K}, \quad V(0) = V_0 > 0.$$

In 1932, Winsor analyzed some analytical properties of a modified Gompertz model and pointed that it can be used to describe empirically the deceleration of tumour growth[10]. In 2000, Ferrante et al. considered a stochastic version of the Gompertz model to describe vivo tumor growth [11]. While to study the investigated phenomena better, some researchers prefer to incorporate various equations with the time delays in different ways. In [12], four kinds of models were derived by introducing the discrete delays into the classical Gompertz model. One of them, which occurs in the following form will be discussed in the rest paper

$$\dot{V}(t) = -rV(t)\ln\frac{V(t-\tau)}{K}, \quad t \geq 0, \tag{1.1}$$

with $r, K \in (0,\infty)$, where r is the growth rate, V is the number of individuals or cells and K is the plateau number of individuals or cells. The time delay figures maturation period of the individuals in the context of population growth. While it may figure the time lag during the course of tumor growth (or degradation) owing to the time which is required for the cells to identify and accommodate to changes in the environment. The existence, uniqueness and asymptotic properties of the solutions of (1.1)

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were studied in [12]. In [13], the occurrence of period orbits owing to the Hopf bifurcation was analyzed. Meanwhile, the existence of periodic solutions was confirmed and some results for the asymptotically stability of the periodic solutions were given. Later, for the Gompetrz model with one delay, the stability and Hopf bifurcation were studied in [14]. However, to the best of our knowledge, until now very few results dealing with the oscillation of solutions of (1.1) were found. Therefore, from the viewpoint of analytically and numerically, our objective in this paper is to acquire some sufficient conditions for oscillation of all positive solutions of (1.1) about the equilibrium. We also prove that every non-oscillatory solution will tend to the equilibrium when the time approaches to infinity.

In the rest paper, we only study the solutions of (1.1) with initial condition of the form

$$V(t) = \phi(t), \quad -\tau \leq t \leq 0,$$

where $\phi \in C([-\tau,0],[0,\infty))$ and $\phi(0) > 0$. By the method of steps one can prove that (1.1) has positive solutions for all $t \geq 0$.

From [15], we know the difference equation

$$a_{n+1} - a_n + \sum_{j=-k}^{l} q_j a_{n+j} = 0 \tag{1.2}$$

is oscillatory if and only if the characteristic equation of (1.2) has no positive roots. So we introduce a useful theorem.

Theorem 1.1. *[15] Consider the difference equation*

$$a_{n+1} - a_n + p a_{n-k} = 0, \tag{1.3}$$

where $p \in \mathbb{R}$, $k \in \mathbb{Z}$. Then every solution of (1.3) oscillates if and only if one of the following conditions holds:

1. *$k = -1$ and $p \leq -1$;*
2. *$k = 0$ and $p \geq 1$;*
3. *$k \in \{\ldots,-3,-2\} \cup \{1,2,\ldots\}$ and $p\frac{(k+1)^{k+1}}{k^k} > 1$.*

2. The oscillation of solutions

In this section, we will illustrate some sufficient conditions for oscillation of (1.1) about the equilibrium K analytically and numerically.

Theorem 2.1. *Every positive solution of (1.1) oscillates about K if*

$$r\tau > \frac{1}{e}. \tag{2.1}$$

Proof. Set $V(t) = Ke^{y(t)}$, then $V(t)$ oscillates about K if and only if $y(t)$ oscillates about zero. So from (1.1) we find that

$$\dot{y}(t) = -ry(t-\tau). \tag{2.2}$$

Then by Theorem 2.2.3 in [15], we know that every solution of (1.1) oscillates if and only if (2.1) holds. □

Next, we transfer to discuss the numerical case. Applying the linear θ-method to (2.2), one has

$$y_{n+1} = y_n - h\theta r y_{n+1-m} - h(1-\theta) r y_{n-m}, \tag{2.3}$$

where $0 \leq \theta \leq 1$, $h = \tau/m$ is stepsize and m is a positive integer. y_{n+1} and y_{n+1-m} are approximations to $y(t)$ and $y(t-\tau)$ at t_{n+1}, respectively. Let $y_n = \ln(V_n/K)$, then (2.3) reads

$$\ln\frac{V_{n+1}}{K} = \ln\frac{V_n}{K} - h\theta r \ln\frac{V_{n+1-m}}{K} - h(1-\theta) r \ln\frac{V_{n-m}}{K} = \ln\left[\frac{V_n}{K}\left(\frac{K}{V_{n+1-m}}\right)^{h\theta r}\left(\frac{K}{V_{n-m}}\right)^{h(1-\theta)r}\right],$$

that is

$$V_{n+1} = V_n K^{hr} \frac{1}{V_{n+1-m}^{h\theta r}} \frac{1}{V_{n-m}^{h(1-\theta)r}}. \tag{2.4}$$

It is obvious that V_n is oscillatory about K if and only if y_n is oscillatory. In the following we seek the conditions under which (2.4) is oscillatory.

Lemma 2.2. *The characteristic equation of (2.3) is given by*

$$\lambda = R(-hr\lambda^{-m}), \tag{2.5}$$

where $R(x) = (1+(1-\theta)x)/(1-\theta x)$ is the stability function of the linear θ-method.

The proof of this Lemma can be given directly and we omit it.

Lemma 2.3. *Under the condition (2.1), (2.5) has no positive roots for $0 \le \theta \le 1/2$.*

Proof. Let $P(\lambda) = \lambda - R(-hr\lambda^{-m})$. From [16], we have

$$R(-hr\lambda^{-m}) \le \exp(-hr\lambda^{-m}), \quad \lambda > 0, \quad 0 \le \theta \le 1/2.$$

Further, we will prove $Q(\lambda) = \lambda - \exp(-hr\lambda^{-m}) > 0$ for $\lambda > 0$. Assume there is a $\lambda_0 > 0$ such that $Q(\lambda_0) \le 0$, then $\lambda_0 \le \exp(-hr\lambda_0^{-m})$, and $\lambda_0^m \le \exp(-r\tau\lambda_0^{-m})$. Thus

$$r\tau e \le r\tau\lambda_0^{-m}\exp(1 - r\tau\lambda_0^{-m}).$$

So we have

- If $1 - r\tau\lambda_0^{-m} = 0$, then $r\tau e \le 1$, which contradicts to (2.1).
- If $1 - r\tau\lambda_0^{-m} \ne 0$, since $e^x < 1/(1-x)$ for $x < 1$ and $x \ne 0$, we get $r\tau e \le 1$, which also contradicts to (2.1).

Therefore, for $\lambda > 0$, $P(\lambda) = \lambda - R(-hr\lambda^{-m}) \ge \lambda - \exp(-hr\lambda^{-m}) = Q(\lambda) > 0$, which suggests that (2.5) has no positive roots. $\square$

Next we consider the case $1/2 < \theta \le 1$ under the assumption $m > 1$.

Lemma 2.4. *Under the conditions (2.1) and $1/2 < \theta \le 1$, (2.5) has no positive roots for $h < h_0$, where*

$$h_0 = \begin{cases} \infty, & r\tau \ge 1, \\ \tau(1+\ln r\tau), & r\tau < 1. \end{cases} \tag{2.6}$$

Proof. It can be noted that $R(-hr\lambda^{-m})$ is an increasing function for θ when $\lambda > 0$, then

$$R(-hr\lambda^{-m}) = \frac{1-h(1-\theta)r\lambda^{-m}}{1+h\theta r\lambda^{-m}} \le \frac{1}{1+hr\lambda^{-m}}.$$

Next, we will illustrate that $\lambda - 1/(1+hr\lambda^{-m})$ is positive under some conditions. Actually

$$\lambda - \frac{1}{1+hr\lambda^{-m}} = \frac{\lambda^{-m+1}}{1+hr\lambda^{-m}}S(\lambda),$$

we need to prove $S(\lambda) = \lambda^m - \lambda^{m-1} + hr > 0$ for each $\lambda > 0$. Obviously, $S(\lambda)$ is the characteristic polynomial of the difference equation

$$w_{n+1} = w_n - hrw_{n-m+1}.$$

According to Theorem 1.1, $S(\lambda)$ has no positive roots if and only if

$$hr\frac{m^m}{(m-1)^{m-1}} > 1,$$

equivalently

$$\ln r\tau + (m-1)\ln\left(1+\frac{1}{m-1}\right) > 0. \tag{2.7}$$

If $r\tau \ge 1$, then (2.7) holds. If $r\tau < 1$ and $h < \tau(1+\ln r\tau)$, from the fact "$\ln(1+x) > x/(1+x)$ holds for $x > -1$ and $x \ne 0$" we have

$$\ln r\tau + (m-1)\ln\left(1+\frac{1}{m-1}\right) > \ln r\tau + \frac{m-1}{m} > 0.$$

Thus we find

$$P(\lambda) = \lambda - R(-hr\lambda^{-m}) \ge \lambda - 1/(1+hr\lambda^{-m}) > 0,$$

which implies that (2.5) has no positive roots. $\square$

In view of Lemmas 2.3, 2.4 and Theorem 2.1, we get the following theorem.

Theorem 2.5. *Under the condition* (2.1), (2.4) *is oscillatory for*

$$h < \begin{cases} \infty, & for \quad 0 \leq \theta \leq 1/2, \\ h_0, & for \quad 1/2 < \theta \leq 1, \end{cases}$$

where h_0 *is defined in* (2.6).

3. Non-oscillatory solutions

In this section, we study the asymptotic behavior of non-oscillatory solutions of (1.1) and (2.4).

Theorem 3.1. *Let* $V(t)$ *be a positive solution of* (1.1)*, which does not oscillate about* K*, then* $\lim_{t\to\infty} V(t) = K$.

Proof. Since $V(t) = Ke^{y(t)}$ we only need to prove that $\lim_{t\to\infty} y(t) = 0$. Assume that $y(t) \geq 0$ for sufficiently large t (the case $y(t) < 0$ is similar and will be omitted). Then from (2.2) we have $\dot{y}(t) \leq 0$. So $y(t)$ is decreasing and

$$\lim_{t\to\infty} y(t) = Y \in [0, \infty), \tag{3.1}$$

we prove $Y = 0$ by contradiction. Assume $Y > 0$ and (2.2) produces

$$\lim_{t\to\infty} \dot{y}(t) = -r \lim_{t\to\infty} y(t-\tau) = -rY < 0.$$

Then $\lim_{t\to\infty} y(t) = -\infty$, which is a contradiction to (3.1). □

In the following, we will prove that the numerical solution V_n can inherit this property.

Theorem 3.2. *Let* y_n *be a solution of* (2.3)*, which does not oscillate, then* $\lim_{t\to\infty} y_n = 0$.

Proof. Assume that $y_n > 0$ for n sufficiently large (the case $y_n < 0$ is similar and will be omitted). From (2.3) we know

$$y_{n+1} - y_n = -(h\theta r y_{n+1-m} + h(1-\theta) r y_{n-m}) < 0, \tag{3.2}$$

then y_n is decreasing. So there exists a constant Z such that

$$\lim_{n\to\infty} y_n = Z \in [0, \infty). \tag{3.3}$$

We argue $Z = 0$ by contradiction. Suppose $Z > 0$, then there is $N \in \mathbb{N}$ and $\varepsilon > 0$ such that $0 < Z - \varepsilon < y_n < Z + \varepsilon$ for $n - m > N$, hence $y_{n-m} > Z - \varepsilon$ and $y_{n-m+1} > Z - \varepsilon$. So (3.2) gives

$$y_{n+1} - y_n < -(h\theta r Z + h(1-\theta) r Z),$$

which indicates that $y_{n+1} - y_n < A$, where $A = -(h\theta r Z + h(1-\theta) r Z) < 0$. Thus $y_n \to -\infty$ as $n \to \infty$, which contradicts to (3.3). □

Theorem 3.3. *Let* V_n *be a positive solution of* (2.4)*, which does not oscillate about* K *, then* $\lim_{n\to\infty} V_n = K$.

4. Numerical examples

In this section we give two numerical examples to verify the previous results.

Firstly, in order to test Theorems 2.1 and 2.5, we consider the following equation

$$\dot{V}(t) = -\frac{1}{15} V(t) \ln \frac{V(t-13)}{2}, \quad t \geq 0 \tag{4.1}$$

with the initial condition

$$V(t) = 7, \quad -13 \leq t \leq 0.$$

In (4.1), we have $1/e < r\tau = 13/15 < 1$, which implies that the solutions of (4.1) are oscillatory according to Theorem 2.1. In Figure 4.1, we draw the figures of the analytic solutions and the numerical solutions with $\theta = 0.1 \leq 1/2$ and $h = \tau/m =$

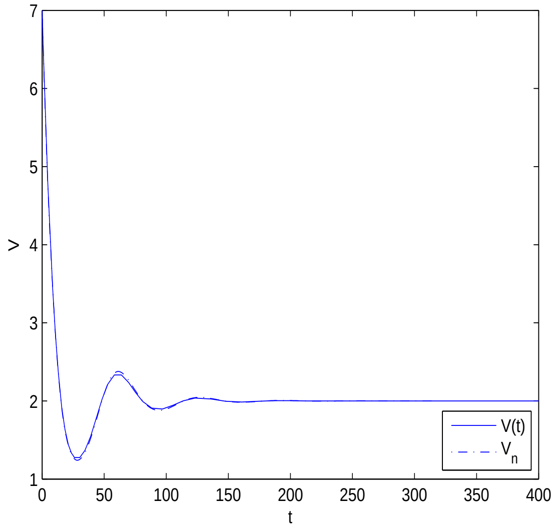

Figure 4.1: The analytic solutions and the numerical solutions of (4.1) with $h = 0.52$, $\theta = 0.1$.

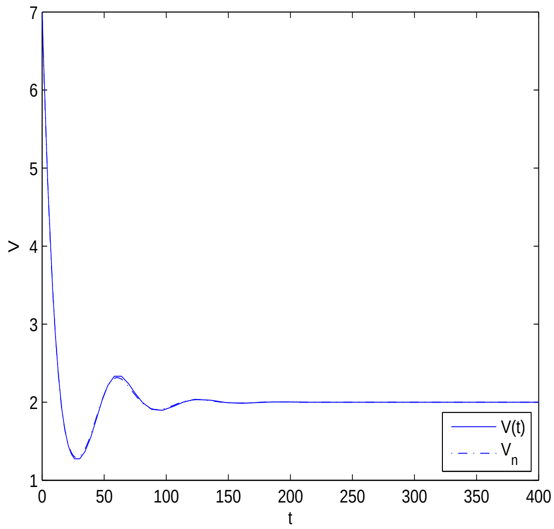

Figure 4.2: The analytic solutions and the numerical solutions of (4.1) with $h = 0.65$, $\theta = 0.9$.

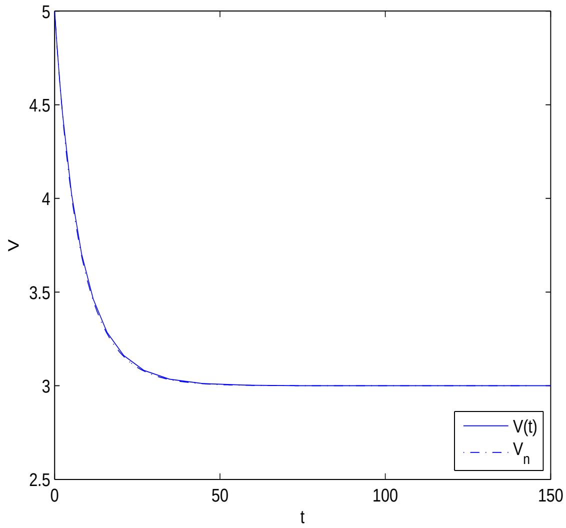

Figure 4.3: The analytic solutions and the numerical solutions of (4.2) with $h = 0.4$ and $\theta = 0.2$.

$13/25 = 0.52 < +\infty$. On the other hand, we set $1/2 < \theta = 0.9 \leq 1$ and $m = 20$ in Figure 4.2. Then $h_0 = \tau(1+\ln r\tau) \approx 8.1140$ and $h = \tau/m = 13/20 = 0.65 < h_0$. Therefore, according to Theorem 2.5, the numerical solutions of (4.1) are also oscillatory for these two cases, which are all the same with Figures 4.1 and 4.2.

Next, we illustrate the validity of Theorems 3.1 and 3.2 in the second example. Consider the equation

$$\dot{V}(t) = -\frac{1}{10}V(t)\ln\frac{V(t-\frac{4}{5})}{3}, \quad t \geq 0 \tag{4.2}$$

with the initial condition

$$V(t) = 5, \quad -\frac{4}{5} \leq t \leq 0.$$

In (4.2), we have $r\tau = 0.08 < 1/e$, which does not satisfy Theorem 2.1. So the analytic solutions and the numerical solutions of (4.2) are non-oscillatory. In Figure 4.3, we draw the figures of the analytic solutions and the numerical solutions of (4.2). From this figure, we can see that $V(t) \to K = 3$ as $t \to \infty$ and $V_n \to K = 3$ as $n \to \infty$. That is, the linear $\theta-$method preserves the asymptotic behavior of non-oscillatory solutions, which coincides with Theorems 3.1 and 3.2.

5. Conclusion

In this paper, numerical oscillation and asymptotic behavior for Gompertz equation with one delay are studied. Some sufficient conditions are proposed. Numerical examples are provided to illustrate the validity of our results. In the future, we will consider the multidimensional and stochastic case.

Acknowledgement

This work is supported by Natural Science Foundation of Guangdong Province (No. 2017A030313031).

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