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Topological Properties of The Digital Line

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Abstract

The main purpose of Digital topology is the study of topological properties of discrete objects which are obtained digitizing continuous objects. Digital topology plays a very important role in computer vision, image processing and computer graphics. The ultimate aim of this article is to analyze the behavior of various general topological concepts in the Khalimsky topology. In this article, we provide some results and examples of topology on $\mathbb{Z}$, the set of all integers. Also, we explain the concepts of digital line and digital intervals with illustrative counterexamples.

1. Introduction

Digital topology is a term that has arisen in the study the digital images. Topological properties of images on a Cathode ray tube are essential in studying graphics, digital processing, pattern analysis and artificial intelligence. There are two fundamental approaches to the digital images. They are graph theoretic and topological approaches. The first approach was initiated by A. Rosenfeld [1]-[4] and the topological approach was originated by Kong, Kopperman, Meyer and Khalimsky et. al.[5] in the 1990s. For finite spaces, these two approaches are equivalent. The study commences with the Jordan Curve Theorem and elucidates that a simple closed curve separates the real plane $\mathbb{R}\times\mathbb{R}$ into exactly two connected components. Khalimsky et.al [5] utilized a connected topology on a finite ordered set in the context of computer graphics. One such a topology on $\mathbb{Z}$, (the set of all integers) is the topology generated by the triples $\{2m-1, 2m, 2m+1\}$ as a subbase. This topology was introduced by Khalimsky and so it is called the Khalimsky topology.

2. Preliminaries

Let $\mathscr{D}$ stand for the set of all triples $\{2m-1, 2m, 2m+1\}$ where $m\in\mathbb{Z}$. Then $\mathscr{D}$ is a subbase for some topology on $\mathbb{Z}$, symbolized by k. The set of all integers $\mathbb{Z}$ with this topology k that is $(\mathbb{Z},\mathsf{k})$ is called the digital line. Throughout, $\mathbb{O}$ and $\mathbb{E}$ denote the set of all odd and even integers respectively and $\check{D}$ denotes the set of all dense subsets of $\mathbb{Z}$. The closure and interior of a set A of a topological space (X,τ) is denoted by $cl_\tau(\mathsf{A})$ and $int_\tau(\mathsf{A})$. Similarly the interior and closure of $(\mathbb{Z},\mathsf{k})$ is denoted by $cl_{\mathsf{k}}(\mathsf{A})$ and $int_{\mathsf{k}}(\mathsf{A})$. Beyond doubt, a base for $(\mathbb{Z},\mathsf{k})$ is $\mathscr{D}\cup\mathscr{B}$ where $\mathscr{B}=\{\{m\}: m\in\mathbb{O}\}$. This follows from the fact that : Let $\mathscr{C}=\{2m-1, 2m, 2m+1\}$ and $\mathscr{D}=\{2n-1, 2n, 2n+1\}, m,n\in\mathbb{Z}$. Then

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$$\mathscr{C}\cap\mathscr{D}=\begin{cases}\mathscr{C} & \text{if } n=m\\ (2m+1) & \text{if } m=n-1\\ (2m-1) & \text{if } m=n+1\\ \emptyset & \text{elsewhere}\end{cases}$$

Also, Let $\mathscr{D}\subseteq\mathbb{Z}$. $\mathscr{D}$ is open (resp. closed) $\Leftrightarrow$ for every $d\in\mathscr{D}$, (d is odd)(resp. d is even) or (d is even with $d-1,d+1\in\mathscr{D}$) (resp. d is odd with $d-1,d+1\in\mathscr{D}$). Let $\mathscr{N}(n)$ (resp. $\mathscr{N}[n]$) denote the smallest neighbourhood (resp. closed neighbourhood) of n in $(\mathbb{Z},\mathsf{k})$. Now $\mathscr{N}(n)=\begin{cases}\{n\} & \text{if } n\in\mathbb{O}\\ \{n-1,n,n+1\} & \text{if } n\in\mathbb{E}\end{cases}$ and $\mathscr{N}[n]=\begin{cases}\{n\} & \text{if } n\in\mathbb{E}\\ \{n-1,n,n+1\} & \text{if } n\in\mathbb{O}\end{cases}$. Also pay attention to $cl_{\mathsf{k}}(\{2n+1\})=\{2n,2n+1,2n+2\}$, $int_{\mathsf{k}}(cl_{\mathsf{k}}(\{2n+1\}))=\{2n+1\}$, $cl_{\mathsf{k}}(int_{\mathsf{k}}(\{2n+1\}))=\{2n,2n+1,2n+2\}$, $int_{\mathsf{k}}(cl_{\mathsf{k}}(int_{\mathsf{k}}(\{2n+1\})))=\{2n+1\}$, $cl_{\mathsf{k}}(int_{\mathsf{k}}(cl_{\mathsf{k}}(int_{\mathsf{k}}(\{2n+1\}))))=\{2n,2n+1,2n+2\}$.

Definition 2.1. *Let (X,τ) be a topological space and $\mathsf{S}\subseteq\mathsf{X}$. S is semi-open [6] if $\mathsf{S}\subseteq cl_\tau(int_\tau(\mathsf{S}))$, semi-closed if $int_\tau(cl_\tau(\mathsf{S}))\subseteq\mathsf{S}$, p-set [7] if $cl_\tau(int_\tau(\mathsf{S}))\subseteq int_\tau(cl_\tau(\mathsf{S}))$, q-set if $int_\tau(cl_\tau(\mathsf{S}))\subseteq cl_\tau(int_\tau(\mathsf{S}))$, G_δ-set if it equals the countable intersection of open sets of X, F_σ-set if it equals the countable union of closed sets of X, pointwise dense [8] if $\Upsilon_{x\in\mathscr{B}}cl_\tau(\{x\}):\{x\}$ is open $=\mathsf{X}$ and g-closed [9] if $cl_\tau(\mathsf{S})\subseteq\mathsf{U}$ whenever $S\subseteq\mathsf{U}$ and U is open in X.*

Definition 2.2. *A topological space (X,τ) is $\mathsf{T}_{\frac{1}{2}}$ [9] if every g-closed set is closed, semi-T_0 [10] if for any two distinct points x and y of X, there exists a semi-open set S such that $(x\in\mathsf{S}$ and $y\notin\mathsf{S})$ or $(y\in\mathsf{S}$ and $x\notin\mathsf{S})$, semi-T_1 [10] if for $x\neq y\in\mathsf{X}$, there exist semi-open sets S_1 and S_2 such that $x\in\mathsf{S}_1$ but $y\notin\mathsf{S}_1$ and $y\in\mathsf{S}_2$ but $x\notin\mathsf{S}_2$, semi-R_0 [10] if for each semi-open set $\mathsf{S},x\in\mathsf{S}$ implies $scl(\{x\})\subseteq\mathsf{S}$, where $scl(\{x\})$ is the set of all semi-closed sets that containing $\{x\}$, semi-R_1 [11] if for $x,y\in\mathsf{X}$ such that $scl(\{x\})\neq scl(\{y\})$, there are disjoint semi-open sets U and V such that $scl(\{x\})\subseteq\mathsf{U}$ and $scl(\{y\})\subseteq\mathsf{V}$, Urysohn [12] if whenever $x\neq y$ in X, there are neighbourhoods S of x and T of y with $cl(\mathsf{S})\cap cl(\mathsf{T})\neq\emptyset$, door [13] if every subset is either open or closed, extremally disconnected [12] if the closure of every open set is open, Alexandroff [14] if every intersection of open sets is open locally finite [8] if each point lies in a finite open set and in a finite closed set.*

3. Properties

In this section, we investigate the properties of the topological space $(\mathbb{Z},\mathsf{k})$ and discuss the subspaces of the digital intervals of the digital line.

Proposition 3.1.

(i) $cl_{\mathsf{k}}(\mathbb{O})=\mathbb{Z}$.
(ii) *$\mathbb{O}$ is pointwise dense in $\mathbb{Z}$.*
(iii) *Every dense subset of $\mathbb{Z}$ is open.*
(iv) *$(\mathbb{Z},\mathsf{k})$ is second countable, Lindelof and separable.*
(v) *$(\mathbb{Z},\mathsf{k})$ is $\mathsf{T}_{\frac{1}{2}},\mathsf{T}_0$ and neither T_1 nor R_0.*
(vi) *$(\mathbb{Z},\mathsf{k})$ is semi-T_0, semi-T_1 and semi-R_0.*
(vii) *$(\mathbb{Z},\mathsf{k})$ is locally finite, connected, Alexandroff and neither door nor extremally disconnected.*
(viii) *Every F_σ set is closed and G_δ set is open in $(\mathbb{Z},\mathsf{k})$*

Proof.

(i) Let $n\in\mathbb{Z}$ and $\mathscr{N}$ be neighbourhood of n in $(\mathbb{Z},\mathsf{k})$. Since $\mathscr{N}$ contains an odd integer, $cl_{\mathsf{k}}(\mathbb{O})=\mathbb{Z}$.
(ii) Clearly $\mathbb{O}$ is open in $\mathbb{Z}$, and for each $n\in\mathbb{O}$, $cl_{\mathsf{k}}(n)=\{n-1,n,n+1\}$ and hence $\mathbb{Z}=\Upsilon_{n\in\mathbb{O}}cl_{\mathsf{k}}(n)$.
(iii) Let $\mathscr{D}\in\check{D}$. Then $\mathscr{D}=\mathsf{A}\cup\mathsf{B}$ where $\mathsf{A}\in\mathbb{O}$ and $\mathsf{B}\in\mathbb{E}$. Take $\mathsf{A}=\{n\}$, and n is even implies $\mathscr{D}=\mathsf{A}\cup\{n-1,n,n+1\}$ and hence $\mathscr{D}$ is open.
(iv) Follows from the fact that $\mathbb{Z}$ is countable.
(v) Since any neighbourhood of $2n$ contains $2n-1$, $(\mathbb{Z},\mathsf{k})$ is not T_1. We can easily verify that $(\mathbb{Z},\mathsf{k})$ is T_0. Since $cl_{\mathsf{k}}(2n-1)=\{2n,2n-1,2n+1\}$ and $\{2n-1\}$ is open implies that $(\mathbb{Z},\mathsf{k})$ is not R_0. $\mathsf{T}_{\frac{1}{2}}$ follows from every singleton is either open or closed in $\mathbb{Z}$.
(vi) $(\mathbb{Z},\mathsf{k})$ is semi-T_0, semi-T_1 and semi-R_0 follows from the fact that every singleton is semi-closed.
(vii) Let $n\in\mathbb{Z}$. If $n\in\mathbb{O}$, then $\{n\}$ is a finite open set such that $n\in\{n\}$ and $\{n-1,n,n+1\}$ is a finite closed set and $n\in\{n-1,n,n+1\}$. This implies $(\mathbb{Z},\mathsf{k})$ is locally finite. A locally finite space is Alexandroff implies $(\mathbb{Z},\mathsf{k})$ is an Alexandroff space. $\{2n,2n-1\}$ is neither open nor closed implies $(\mathbb{Z},\mathsf{k})$ is not door. Also $\{2n+1\}$ is open and $cl_{\mathsf{k}}(\{2n+1\})=\{2n+2n+1,2n+2\}$ is not open. Therefore $(\mathbb{Z},\mathsf{k})$ is not extremally disconnected. Let A be a non-empty clopen subset of $(\mathbb{Z},\mathsf{k})$. Fix $n\in\mathsf{A}$. If $n\in\mathbb{O}$ and A is closed, $n-1,n+1\in\mathsf{A}$. Thus $\{n-1,n,n+1\}\subseteq\mathsf{A}$. Since $n-1$ and $n+1$ are even and A is open, $\{n-2,n-1,n,n+1,n+2\}\subseteq\mathsf{A}$. Continuing we get $\mathbb{Z}=\mathsf{A}$. If $n\in\mathbb{E}$ and A is open, $n-1,n+1\in\mathsf{A}$. Thus $\{n-1,n,n+1\}\subseteq\mathsf{A}$. Since $n-1$ and $n+1$ are odd and A is closed, $\{n-2,n-1,n,n+1,n+2\}\subseteq\mathsf{A}$. Continuing, we get τ equals A. That is $(\mathbb{Z},\mathsf{k})$ is connected.

(viii) Let $\mathsf{A} \in (\mathbb{Z},\mathsf{k})$ where $\mathsf{A} = \cup \mathsf{A}_n$ each A_n is closed if x is in A, then x is in some A_n. If x is even, it is evident. If x is odd, then $\{x-1,x,x+1\} \subseteq \mathsf{A}_n \subseteq \mathsf{A}$. That is A is closed.

(ix) Let $\mathsf{A} \in (\mathbb{Z},\mathsf{k})$ where $\mathsf{A} = \cap \mathsf{A}_n$ each A_n is open. If x is in A, then x is in every A_n. If x is even, then the open set $\{x-1,x,x+1\} \subseteq \mathsf{A}_n \subseteq \mathsf{A}$. That is A is open.

□

Levine's Property $\mathscr{Q}$

Levine defined that a set S has the property $\mathscr{Q}$ if the interior and the closure operators commute on S and characterized the sets having the property $\mathscr{Q}$. That is a set A in a topological space (X,τ) has the property $\mathscr{Q}$ [15] if $int_\tau(cl_\tau(\mathsf{S})) = cl_\tau(int_\tau(\mathsf{S}))$.

Example 3.2.

(i) Let X be an infinite set with co-finite topology. Then every non-empty open subset is infinite and hence every finite subset of X has the property $\mathscr{Q}$.

(ii) Let X be an uncountable set with co-countable topology. Then every non-empty open subset is uncountable and hence every countable subset of X has the property $\mathscr{Q}$.

(iii) Let X be an non-empty set with $x \in \mathsf{X}$. Assign X with x-inclusion topology, then X does not have the property $\mathscr{Q}$.

(iv) In (X,τ), $\mathsf{X} = \mathsf{W}$, the set of all whole numbers and $\tau = \{\emptyset,\{0\},\mathsf{X}\}$. Then every subset of X has the property $\mathscr{Q}$.

4. Digital subspaces

From now on we consider subspaces of $(\mathbb{Z},\mathsf{k})$, and investigate the behaviour of cardinalities of some kind of subspace topologies. We will now prove that the cardinalities of topologies on the intervals $\{1\},\{1,2\},\{1,2,3\},....,\{1,2,3,..,n\},....$ form a subsequence of the well known Fibonacci sequence. Also observe that

(i) If $\mathsf{S} = \{1\}, \tau_1 = \{\emptyset,\mathsf{S}\}$, then $|\tau_1| = 2$.

(ii) If $\mathsf{S} = \{1,2\}, \tau_2 = \{\emptyset,\{1\},\mathsf{S}\}$, then $|\tau_2| = 3$.

(iii) If $\mathsf{S} = \{1,2,3\}$, subbase = $\{\{1\},\{1,2,3\},\{3\}\}$, Base = $\{\{1\},\{3\},\mathsf{S}\}$ and $\tau_3 = \{\emptyset,\{1\},\{3\},\{1,3\},\mathsf{S}\}$ then $|\tau_2| = 5$.

(iv) Let $\mathsf{S} = \{1,2,3,4\}$, subbase = $\{\{1\},\{1,2,3\},\{3,4\}\}$, Base = $\{1\},\{3\},\{1,2,3\},\{3,4\},\mathsf{S}\}$ and $\tau_4 = \{\emptyset,\{1\},\{3\},\{1,3\},\{3,4\},\{1,2,3\},\{1,3,4\},\mathsf{S}\}$ then $|\tau_4| = 8$.

(v) Let $\mathsf{S} = \{1,2,3,4,5\}$ with $\mathsf{S}_\mathsf{k} = \tau_5$, the subspace topology generated by $\{\{1\},\{5\},\{1,2,3\},\{3,4,5\}\}$. Here $\tau_5 = \{\emptyset,\{1\},\{3\},\{5\},\{1,3\},\{1,5\},\{3,5\},\{1,2,3\},\{1,3,5\},\{3,4,5\},\{1,2,3,5\},\{1,3,4,5\},\mathsf{S}\}$. Then (S,τ_5) is a subspace of $(\mathbb{Z},\mathsf{k})$. Also from this we observe that $q(\tau_5)$ and $p(\tau_5)$ are discrete topology on S. Also, $\mathscr{Q}(\tau_5) = \{\emptyset,\{4\},\{2,4\},\{1,3,5\},\{1,2,3,5\},\{1,3,4,5\},\mathsf{S}\}$. Then $|\tau_5| = 13$.

(vi) Let $\mathsf{S} = \{1,2,3,4,5,6\}$, then subbase = $\{\{1\},\{1,2,3\},\{3,4,5\},\{5,6\}\}$ and Base = $\{\{1\},\{3\},\{1,2,3\},\{3,4,5\},\{5,6\}\}$ and $\tau_6 = \{\emptyset,\{1\},\{3\},\{5\},\{1,3\},\{1,5\},\{3,5\},\{5,6\},\{1,2,3\},\{1,3,5\},\{3,4,5\},\{1,5,6\},\{3,5,6\},\{1,2,3,5\},\{1,3,4,5\},\{1,3,5,6\},\{3,4,5,6\},\{1,2,3,5,6\},\{1,2,3,4,5\},\{1,3,4,5,6\},\mathsf{S}\}$. Then $|\tau_6| = 21$.

(vii) Let $\mathsf{S} = \{1,2,3,4,5,6,7\}$, then subbase = $\{\{1\},\{7\},\{1,2,3\},\{3,4,5\},\{5,6,7\}\}$ and Base = $\{\{1\},\{3\},\{5\},\{7\},\{1,2,3\},\{3,4,5\},\{5,6,7\}\}$ and $\tau_7 = \{\emptyset,\{1\},\{3\},\{5\},\{7\},\{1,3\},\{1,5\},\{1,7\},\{3,5\},\{3,7\},\{5,6\},\{5,7\},\{1,2,3\},\{3,4,5\},\{5,6,7\},\{1,3,5\},\{1,3,7\},\{1,5,7\},\{3,5,7\},\{1,2,3,5\},\{1,2,3,7\},\{1,3,4,5\},\{1,3,5,7\},\{1,5,6,7\},\{3,4,5,7\},\{3,5,6,7\},\{1,2,3,4,5\},\{3,4,5,6,7\},\{1,2,3,5,7\},\{1,3,5,6,7\},\{1,3,4,5,7\},\{1,2,3,4,5,7\},\{1,2,3,5,6,7\},\{1,3,4,5,6,7\},\mathsf{S}\}$. Then $|\tau_7| = 34$.

Lemma 4.1. *Let $\mathsf{S} = \{1,2,3,4,5\}$. Then*

(i) $q(\tau_5)$ is the discrete topology on S,

(ii) $p(\tau_5)$ is the discrete topology on S,

(iii) $\mathscr{Q}(\tau_5)$ is a topology on S other than discrete topology and the indiscrete topology on S, where $q(\tau_5)$, $p(\tau_5)$ and $\mathscr{Q}(\tau_5)$ respectively denote the collection of all q-sets, p-sets and collection of all subsets of S having the property Q in (S,τ_5).

Lemma 4.2. *If $\tau_{m-1},\tau_m,\tau_{m+1}$ are the topologies on the digital intervals $\mathbb{Z}\cap[1,m-1]$, $\mathbb{Z}\cap[1,m]$, $\mathbb{Z}\cap[1,m+1]$ respectively inherited from the Khalimsky topology k on $\mathbb{Z}$, then $|\tau_{m-1}| = |\tau_{m+1}| = |\tau_m|$.*

Proof.

Case (a) Since $[1,m-1] \subseteq [1,m+1]$ and $m-1 \in \mathbb{O}$. Then $\tau_{m+1} \subseteq \tau_{m+2}$. Now $\{m+1\}$ is the basic open set in $[1,m+1]$. $\tau = \{\{m+1\}\cup \mathsf{A} : \mathsf{A} \in \tau_m\}$. Then $\tau_{m-1} \cup \tau = \tau_{m+1}$ and $\tau_{m-1} \cap \tau = \emptyset$.

Case (b) Since $[1,m] \subseteq [1,m+1]$ and $m \in \mathbb{O}$. Then $\tau_m \subseteq \tau_{m+1}$. Now $\{m,m+1\}$ is the base open set in $[1,m+1]$. $\tau = \{\{m,m+1\}\cup \mathsf{A} : \mathsf{A} \in \tau_{m+1}\}$. Then $\tau_m \cup \tau = \tau_{m+1}$ and $\tau_m \cap \tau = \emptyset$.

In both cases $|\tau_{m+1}| = |\tau_{m-1}| = |\tau_m|$. □

5. Conclusion

In this work, we provide some results and examples of the topology on $\mathbb{Z}$, the set of all integers. Also we explain the concepts of digital line and digital intervals. We prove that the cardinalities of topologies on the digital intervals form a sub sequence of the Fibonacci sequence.

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