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LEGENDRE POLYNOMIAL APPROXIMATION FOR NON-LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, it is concerned with the least squares method based on Legendre polynomials approximation for solving non-linear initial value problem. In particular, it is noted that such polynomials can be effective for the solution of non-linear equations if one needs to express products of Legendre polynomials as linear expansions of these functions. Besides, obtained results are compared with the least squares approximation based on Taylor series and the exact solutions. Furthermore, to show the performance of the method, some numerical examples and their figures of absolute errors are given.

Keywords: Least squares approximation, Legendre polynomials, Adomian polynomials, non-linear differential equations

DOĞRUSAL OLMAYAN DİFERANSİYEL DENKLEMLER İÇİN LEGENDRE POLİNOMLARI YAKLAŞIMI

ÖZET

Bu çalışmada, doğrusal olmayan başlangıç değer problemlerinin çözümü için Legendre polinom tabanlı en küçük kareler yöntemine yer verilmiştir. Bu tip polinomların doğrusal derlemeleri ile elde edilen fonksiyonların, doğrusal olmayan denklemlerin çözümleri için etkin olduğu vurgulanmıştır. Ayrıca, analitik ve Taylor serisi tabanlı en küçük kareler yaklaşımı kullanılarak elde edilen sonuçlar karşılaştırılmıştır. Daha sonra, sayısal örnekler ve grafikler ile sunulan yöntemin doğruluğu desteklenmiştir.

Anahtar Kelimeler: En küçük kareler yaklaşımı, Legendre polinomları, Adomian polinomları, doğrusal olmayan diferansiyel denklemler

1. INTRODUCTION

Studies have revealed the importance of differential equations in many fields. Especially, when we want to obtain mathematical models of physical or engineering science, generally we get non-linear differential equations or equation

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system [1]. To obtain analytical solution of such non-linear differential equation or equation system are not so easy. There are some methods to find analytic or approximate solution of these kinds of equations. Analytical solutions can not be obtained except for special types of non-linear differential equations. This type of differential equations, in general, semi-analytic or numerical solutions can be obtained. Previously, such equations, a serial method or the finite difference method and solved, recently tried to analyze the different methods [2-5]. For instance, some of these are differential transform method [6-10], spectral method [11-14], and Adomian decomposition methods [15-21], Adomian polynomials which are nothing else than combinations of elementary differentials of similar order, that play a fundamental role in determining order conditions in ordinary differential equation integrants.

In this study, a new method other than these methods will be given.

2. LEGENDRE POLYNOMIAL APPROXIMATION METHOD

The least squares problem is stated, for a finite interval [a, b], as follows: Assume that $\{\varphi_N(x) |\ N \geq 0\}$ is an orthonormal polynomials with weight function $w(x) \geq 0$, that is,

$$\langle \varphi_N, \varphi_M \rangle = \delta_{N,M} = \begin{cases} 1, & N = M \\ 0, & N \neq M \end{cases} \tag{2.1}$$

Then an arbitrary polynomial p(x) of degree less equal than N ($\leq$ N) can be written as

$$p(x) = a_0\, \varphi_0(x) + a_1\, \varphi_1(x) + \ldots + a_N\, \varphi_N(x) \tag{2.2}$$

then for a given nonnegative continuous function $g(x) \in C[a, b]$ [22-23];

$$\|g - p\|_2^2 = \int_a^b w(x)\left[g(x) - \sum_{k=0}^{N} a_k \varphi_k(x)\right]^2 dx \equiv G(a_0, a_1, \ldots, a_N) \tag{2.3}$$

for solving the least squares problem by minimizing G in Eq.(2.3), as the coefficients $\{a_i\}$ range over all real numbers. A necessary condition for a point (a_0, a_1, …, a_N) to be a minimizing point is

$$\frac{\partial G}{\partial a_i} = 0, \quad i = 0,1,\ldots, N \tag{2.4}$$

Explicit form of the Eq.(2.4) is written as follows:

$$\begin{aligned} \int_a^b w(x)\varphi_0(x)\left[\sum_{k=0}^{N} a_k \varphi_k(x)\right]dx &= \int_a^b w(x)\varphi_0(x) g(x)dx \\ \vdots \qquad\qquad & \qquad \vdots \\ \int_a^b w(x)\varphi_N(x)\left[\sum_{k=0}^{N} a_k \varphi_k(x)\right]dx &= \int_a^b w(x)\varphi_N(x) g(x)dx \end{aligned}$$

On the other hand, G is minimum if and only if

$b_j = \langle g, \varphi_j \rangle, \quad j = 0,1,\ldots, N$ (positive definite)

then the least squares approximation exists; it is unique (from the properties of inner product of functions) and given by

$$P_N^*(x) = \sum_{j=0}^{N} \langle g, \varphi_j \rangle \varphi_j(x) \tag{2.5}$$

In order to solve the least squares problem on a finite interval [a, b] with $w(x) \equiv 1$, we can convert it to a problem on [-1, 1]. The change of variable

$$x = \frac{b + a + (b - a)t}{2} \tag{2.6}$$

converts the interval $-1 \le t \le 1$ to $a \le x \le b$. For a given $g(x) \in C[a, b]$, it can be defined

$$G(t) = g\left(\frac{b + a + (b - a)t}{2}\right), \quad -1 \le t \le 1 \tag{2.7}$$

then

$$\int_a^b [g(x) - p_N(x)]^2 dx = \frac{b - a}{2} \int_{-1}^{1} [G(t) - P_N(t)]^2 dt \tag{2.8}$$

where $P_N(t)$ is obtained from $p_N(x)$ using Eq.(2.6). The change of variable Eq. (2.6) gives a one-to-one correspondence between polynomials of degree M on [a, b] and of degree M on [-1, 1], for every $M \ge 0$. Thus, minimizing $\|g - p_N\|_2$ on the interval [a, b] is equivalent to minimizing $\|G - P_N\|_2$ on the interval [-1, 1] [22] .The basic of this method is to expand the function g(x) as a finite series of very smooth basis functions as given below:

$$g(x) = \sum_{i=0}^{N} a_i \phi_i(x)$$

in which ϕ_i represents a family of polynomials which are orthogonal and complete over the interval [a, b] with respect to non-negative weight function w(x). If the weight function is $w(x) \equiv 1$, given $g(x) \in [-1, 1]$ the orthonormal family (if every member has length one, that is $\|g\|_2 = 1$) described in Gram-Schmidt theorem is

$$\varphi_0(x) = \frac{1}{\sqrt{2}}, \ \varphi_1(x) = \sqrt{\frac{3}{2}}x, \ \varphi_2(x) = \sqrt{\frac{5}{2}}\left(\frac{3x^2 - 1}{2}\right), \ldots$$

in which $\langle \varphi_N, \varphi_N \rangle = \int_{-1}^{1} w(x) L_N(x) L_N(x) dx = 1$ and further polynomials can be constructed by

$$\varphi_N(x) = \sqrt{\frac{2N+1}{2}} \left\{ \frac{(-1)^N}{2^N N!} \frac{d^N}{dx^N} \left[(1-x^2)^N \right] \right\}, \quad N \geq 1 \tag{2.9}$$

Eq.(2.2) defined general polynomial approximation is obtained by using the equation (2.5). Given in particular Legendre polynomials instead of the general polynomial approximation Eq. (2.10) is obtained similar to Eq.(2.5). The least squares approximation can be defined;

$$L_N^*(x) = \sum_{j=0}^{N} \langle g, \varphi_j \rangle \varphi_j(x) \tag{2.10}$$

where $\langle g, \varphi_j \rangle = \int_{-1}^{1} g(x) \varphi_j(x) dx$ the coefficients $\langle g, \varphi_j \rangle$ are called Legendre coefficients.

In this study, Legendre polynomials are indicated by Ln(x). Legendre polynomials are in a different form of the classical Taylor polynomial and trigonometric functions [17-18]. So, Legendre polynomials are eigen functions of Sturm-Liouville problem;

$$(1-x^2)[L_N(x)]'' - 2x[L_N(x)]' + N(N+1)L_N(x) = 0 \tag{2.11}$$

where it is generated form Rodrigue's formulas, in closed form

$$L_N(x) = \frac{(-1)^N}{2^N N!} \frac{d^N}{dx^N} \left[(1-x^2)^N \right] \tag{2.12a}$$

for N = 1 $\quad L_0(x) = 1$

for N = 2 $\quad L_1(x) = x$

for N = 3 $\quad L_2(x) = (3x^2 - 1) / 2$

for N = 4 $\quad L_3(x) = (5x^3 - 3x) / 2$

⋮

of which explicit expansion is [24]

$$L_N(x) = \frac{1}{2^N} \sum_{M}^{N/2} (-1)^M \binom{N}{M} \binom{2(N-M)}{N} x^{N-2M} \tag{2.12b}$$

if the expression is in the form of (cx-d) where c and d are constants; similar to Eq. (2.12a), the recurrence relation can be written as follows:

$L_0(cx - d) = 1$

$L_1(cx - d) = c\,(cx - d)$
$L_2(cx - d) = c^2\,[\,3\,(cx - d)^2 - 1]/\,2$
$\vdots$

and also triple recursion relation for Legendre polynomials is written [24]

$$L_{N+1}(x) = \frac{2N+1}{N+1} x L_N(x) - \frac{N}{N+1} L_{N-1}(x), \quad N \geq 1 \tag{2.13}$$

Now, let us consider the following general ordinary differential equation

$$Ly(x) + R\,y(x) + N\,y(x) = g(x) \tag{2.14}$$

where L is the highest-derivative operator, R is the linear term of which the degree is less than the degree of term L, N is non-linear term, L^{-1} is the inverse operator of L. Applying the inverse operator L^{-1} to both sides of Eq. (2.14), it is obtained as follows

$$L^{-1}\{L\,y(x)\} = L^{-1}\{g(x)\} - L^{-1}\{R\,y(x)\} - L^{-1}\{N\,y(x)\}$$
$$y(x) = y(x_0) + (x - x_0)\,y'(x_0) + L^{-1}\{g(x)\} - L^{-1}\{R\,y(x)\} - L^{-1}\{N\,y(x)\} \tag{2.15}$$

where $y(x_0) + (x - x_0)\,y'(x_0)$ comes from initial condition of problem. It is written $f(y) = Ny(x) = \sum_{i=0}^{N} A_i$ and $y(x) = \sum_{i=0}^{N} y_i$ where the components of A_i are called Adomian polynomials as follows[10-16]:

$$\begin{aligned}
A_0 &= f(y_0)\\
A_1 &= y_1\,f'(y_0)\\
A_2 &= y_2\,f'(y_0) + y_1^2\,f''(y_0)\,/\,2!\\
A_3 &= y_3\,f'(y_0) + y_1\,y_2\,f''(y_0) + y_1^3\,f'''(y_0)\,/\,3!\\
&\vdots
\end{aligned} \tag{2.16}$$

and, taking Eq. (2.15), it is constructed

$$y_0 = y(x_0) + (x - x_0)\,y'(x_0) + L^{-1}\{g(x)\} \tag{2.17}$$
$$y_k = -L^{-1}\{R\,y_{k-1}\} - L^{-1}\{N\,y_{k-1}\}\,,\ k \geq 1$$

Representation of function g(x) in terms of series expansion using orthogonal polynomials is a fundamental concept in approximation theory the basis of least squares approximation of solution of differential equations. The function g(x) is defined with Legendre polynomials which complete orthogonal sets of functions on the interval [a,b] for applying the method to non-homogeneous equations, as given below:

$$g(x) \cong \sum_{i=0}^{N} a_i L_i(x) \tag{2.18}$$

where N is arbitrary positive integer number and $L_i(x)$ denotes Legendre polynomials which is defined in Eq. (2.12), then the Adomian procedure can be defined

$$\begin{aligned}
y_0 &= y(x_0) + (x - x_0)\, y'(x_0) + L^{-1}\{a_0 L_0(x) + a_1 L_1(x) + \ldots + a_N L_N(x)\} \\
y_1 &= -L^{-1}\{R\, y_0\} - L^{-1}\{N\, y_0\} \\
y_2 &= -L^{-1}\{R\, y_1\} - L^{-1}\{N\, y_1\} \\
y_3 &= -L^{-1}\{R\, y_2\} - L^{-1}\{N\, y_2\} \\
&\vdots
\end{aligned} \tag{2.19}$$

or according to [20]

$$\begin{aligned}
y_0 &= y(x_0) + (x - x_0)\, y'(x_0) + L^{-1}\{a_0 L_0(x)\} \\
y_1 &= L^{-1}\{a_1 L_1(x)\} - L^{-1}\{R\, y_0\} - L^{-1}\{N\, y_0\} \\
y_2 &= L^{-1}\{a_2 L_2(x)\} - L^{-1}\{R\, y_1\} - L^{-1}\{N\, y_1\} \\
y_3 &= L^{-1}\{a_3 L_3(x)\} - L^{-1}\{R\, y_2\} - L^{-1}\{N\, y_2\} \\
&\vdots
\end{aligned} \tag{2.20}$$

or by converting to Eq. (2.19) into standard form;

$$\begin{aligned}
g(x) &\cong \sum_{i=0}^{N} b_i x^N \\
&= b_0 + b_1 x + b_2 x^2 + \ldots \\
&= 1\left[a_0 - \frac{1}{2}a_2 + \frac{3}{4}a_4 + \ldots\right] + x\left[a_1 - \frac{3}{2}a_3 + \frac{15}{84}a_5 + \ldots\right] + x^2\left[\frac{3}{2}a_2 - \frac{15}{4}a_4 + \frac{105}{16}a_6 + \ldots\right] + \ldots
\end{aligned} \tag{2.21}$$

it is obtained, then it is written in matrix form;

$$\begin{Bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -1/2 & 0 & 3/8 & 0 & -5/16 & 0 & 35/128 & \cdots \\ 0 & 1 & 0 & -3/2 & 0 & 15/8 & 0 & -35/16 & 0 & \cdots \\ 0 & 0 & 3/2 & 0 & -15/4 & 0 & 105/16 & 0 & -315/32 & \cdots \\ 0 & 0 & 0 & 0 & 35/8 & 0 & -315/16 & 0 & 3465/64 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 63/8 & 0 & -693/16 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 231/16 & 0 & -3003/32 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix} . \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \end{Bmatrix}$$

3. NUMERICAL EXAMPLES

In this section, the following non-linear differential equations are considered in order to support presented method.

Example 1: Let us consider the following non-linear initial value problem with variable coefficient for $0 \le x \le 1$:

$$y''(x) + x\, y'(x) + x^2 y^3(x) = (2 + 6x^2)\exp(x^2) + x^2 \exp(3x^2) \tag{3.1}$$

$y(0) = 1 \qquad y^{/}(0) = 0$ (3.2)

The exact solution of Eq. (3.1) under conditions Eq. (3.2) is $y_{exact}(x) = \exp(x^2)$ [14]. The operator form of Eq. (3.1) can be written as

$L\, y(x) + R\, y(x) + N\, y(x) = g(x)$ (3.3)

where $L = d^2 / dx^2$, $R = x\ d / dx$, $N\, y = x^2\, y^3$ and $g(x) = (2 + 6x^2) \exp(x^2) + x^2 \exp(3x^2)$. For non-linear term, the components of Adomian polynomials A_n are obtained from Eq. (2.16) as follows:

$A_0 = x^2\, y_0^3$

$A_1 = x^2\, (3y_0^2\, y_1)$ (3.4)

$A_2 = x^2\, (3y_0^2\, y_2 + 3\, y_0\, y_1^2)$

$A_3 = x^2\, (3y_0^2\, y_3 + 6\, y_0\, y_1\, y_2 + y_1^3)$

$\vdots$

and so on. Applying both sides of Eq. (3.1) by inverse operator $L^{-1}(*) = \int_0^x\int_0^x (*)(dx)^2$,

$L^{-1}\{y^{//}(x) + x\, y^{/}(x) + x^2\, y^3\, (x)\} = L^{-1}\{(2 + 6x^2) \exp(x^2) + x^2 \exp(3x^2)\}$ (3.5)

Now, the Taylor series of g(x) is obtained as follows

$g_T(x) \cong 2 + 9x^2 + 10x^4 + 47x^6/ 6 + O(x^7)$ (3.6)

from Eq. (3.5) under the initial conditions Eq. (3.2), it is written

$y(x) = y(0) + xy^{/}(0) + L^{-1}\{g_T(x)\} – L^{-1}\{ x\, y^{/}(x) + x^2\, y^3\,(x)\}$ (3.7)

$y_0 = 1 + L^{-1}\{2 + 9x^2 + 10x^4 + 47x^6/ 6\} = 1 + x^2 + 3x^4 /4 + x^6 /3 + 47x^8 / 336 + \ldots$

$y_1 = – L^{-1}\{ x\, y_0^{/} + A_0\} = – L^{-1}\{ x\, y_0^{/} + x^2\, y_0^3\} = -x^4 / 4 – x^6 / 5 - \ldots$

$y_2 = – L^{-1}\{ x\, y_1^{/} + A_1\} = – L^{-1}\{ x\, y_1^{/} + x^2\, (3y_0^2 y_1)\} = x^6 / 30 + 39x^8 / 1120 + \ldots$

$y_3 = – L^{-1}\{ x\, y_2^{/} + A_2\} = – L^{-1}\{ x\, y_2^{/} + x^2\, (3y_0^2 y_2 + 3\, y_0\, y_1^2)\} = -x^8 / 280-53x^{10} /12600 - \ldots$

$\vdots$

Hence, the solution is constructed by using Taylor series

$y_T(x) = \sum_{i=0}^{6} y_i = y_0 + y_1 + y_2 + \ldots + y_6$

$= 1 + x^2 + x^4 / 2 + x^6 / 6 + x^8 / 24 – 29x^{10} / 540 + \ldots$ (3.8)

Furthermore, to apply Legendre polynomial approximation for g(x); first of all, we can convert [0, 1] to a problem on [-1, 1] by using Eq. (2.6), after that for a given $g(x) \in C[0, 1]$ can be defined form Eq. (2.7):

$g(0.5x+0.5) = (2 + 6(0.5x+0.5)^2) \exp((0.5x+0.5)^2) + (0.5x+0.5)^2 \exp(3(0.5x+0.5)^2)$ (3.9) and, series expansion is

$$g(x)=\sum_{i=0}^{N} a_i L_i(2x-1), \quad 0 \le x \le 1 \tag{3.10}$$

where the Legendre polynomial coefficients are (from Eq. (2.10))

$$a_i=\sqrt{\frac{2i+1}{2}}\int_{-1}^{1} g(0.5x+0.5)L_i(x)dx, \quad i=0,1,2,... \tag{3.11}$$

for N = 6 , coefficients are obtained: a_0 = 13.203, a_1 = 11.451, a_2 = 6.38, a_3 = 2.652, a_4 = 1.024, a_5 = 0.353, a_6 = 0.114, then we have

$$\begin{aligned} g_L(x) &= \sum_{i=0}^{6} a_i L_i(2x-1) \\ &= a_0.1 + a_1 2(2x-1) + a_2 2^2(6x^2-6x+1)+... \\ &= 6.989 - 147.96x + 1683.3x^2 - 7800.7x^3 + 17012x^4 - 17378x^5 + 6741.5x^6 - ... \end{aligned} \tag{3.12}$$

by using Eq. (3.7)

$$\begin{aligned} y_0 &= 1 + L^{-1}\{g_L(x)\} = 1 + 3.4945x^2 - 24.66x^3 + 140.28x^4 - 390.04x^5 + 567.07x^6 - ... \\ y_1 &= -L^{-1}\{x\, y_0^{/} + A_0\} = -0.66575x^4 + 3.699x^5 - 19.053x^6 + 48.194x^7 - 68.926x^8 + \ldots \\ y_2 &= -L^{-1}\{x\, y_1^{/} + A_1\} = 0.088767x^6 - 0.44036x^7 + 2.0771x^8 - 4.8397x^9 + \ldots \\ y_3 &= -L^{-1}\{x\, y_2^{/} + A_2\} = -0.0095108x^8 + 0.042812x^9 - 0.18452x^{10} + 0.39537x^{11} + \ldots \\ &\vdots \end{aligned} \tag{3.13}$$

and so on. The other terms of series solution can be found by using *matcad7*. Therefore, Legendre polynomial solution of problem is constructed

$$y_L(x) \cong 1 + 3.4945\,x^2 - 24.66\,x^3 + 139\,x^4 - 386\,x^5 + 548\,x^6 - 366\,x^7 + 53.2\,x^8 + 58.9\,x^9 - \ldots \tag{3.14}$$

Example 2: Consider the following non-linear differential equation with constant coefficient:

$$y^{//}(x) + y(x) + y^2(x) = (2x^2 + 4x + 2)\, e^x + x^4 e^{2x} \tag{3.15}$$

under the initial conditions (for $0 \le x \le 1$):

$$y(0) = y^{/}(0) = 0 \tag{3.16}$$

where the exact solution is $y_{exact}(x) = x^2 e^x$ [14].

The operator form of Eq.(3.15) is written similar to Eq.(3.3) where $L = d^2/dx^2$, R = 1 , $Ny = y^2$ and $g(x) = (2x^2 + 4x + 2)\, e^x + x^4e^{2x}$. The components of A_n which is called Adomian polynomials can be obtained as given below:

$$\begin{aligned} A_0 &= y_0^2 \\ A_1 &= 2y_0\, y_1 \\ A_2 &= 2y_0\, y_2 + y_1^2 \\ &\vdots \end{aligned} \tag{3.17}$$

Applying both sides of Eq. (3.15) by inverse operator $L^{-1}(*) = \int_0^x\int_0^x (*)(dx)^2$, and using initial conditions Eq. (3.16), it is obtained

$$y(x) = y(0) + xy'(0) + L^{-1}\{g(x)\} - L^{-1}\{ y + y^2\} \tag{3.18}$$

Here, the Taylor series of g(x) is obtained as follows

$$g_T(x) \cong 2 + 6x + 7x^2 + 13x^3 /3 + 33x^4/ 12 + O(x^5) \tag{3.19}$$

using Eq. (3.18) and Eq. (3.19), it is written

$$y_0 = L^{-1}\{2 + 6x + 7x^2 + 13x^3 /3 + 33x^4/ 12\} = x^2 + x^3 + 7x^4/12 + 13x^5/60 + 11x^6/120 + \ldots$$
$$y_1 = - L^{-1}\{y_0 + y_0^2\} = -x^4/12 - x^5/20 - 19x^6/360 - 19x^7/360 - 271x^8/6720 - \ldots$$
$$y_2 = - L^{-1}\{y_1 + 2y_0y_1\} = x^6 /360 + x^7/840 + 79x^8/20160 + 23x^9/5184 + \ldots \tag{3.20}$$
$$\vdots$$

then, the solution is obtained by using Taylor series

$$y_T(x) = \sum_{i=0}^{N} y_i = y_0 + y_1 + y_2 + \ldots + y_N$$
$$= x^2 + x^3 + x^4 / 2 + x^5 / 6 + x^6/24 - 13x^7/252 - 367x^8/10080 + \ldots \tag{3.21}$$

On the other hand, using Legendre polynomial approximation for $g(x)\in C[0, 1]$ is written

$$g(0.5x+0.5) = (2(0.5x+0.5)^2 + 4(0.5x+0.5) + 2)\, e^{(0.5x+0.5)} + (0.5x+0.5)^4 e^{2(0.5x+0.5)} \tag{3.22}$$

for N = 7 in Eq. (3.10), coefficients are determined from Eq. (3.11), $a_0 = 14.1$, $a_1 = 9.632$, $a_2 = 3.229$, $a_3 = 0.887$, $a_4 = 0.223$, $a_5 = 0.046$, $a_6 = 0.00755$, $a_7 = 0.0009815$, the Legendre polynomial approximation of g(x) is written as follows;

$$g_L(x) = \sum_{i=0}^{7} a_i L_i(2x-1)$$
$$= 3.1096 + 5.7253x - 15.416x^2 + 182.59x^3 - 606.5x^4 + 1121x^5 - 1062.6x^6 + 431.16x^7 - \ldots \tag{3.23}$$

by using Eq. (3.18), non-linear term Adomian polynomials are used from Eq. (3.17) and similar to Eq. (3.20);

$$y_0 = L^{-1}\{g_L(x)\} = 1.5548x^2 + 0.95423x^3 - 1.2847x^4 + 9.129x^5 - 20.217x^6 + 26.69x^7 - \ldots$$
$$y_1 = -L^{-1}\{y_0 + A_0\} = -0.12956x^4 - 0.047712x^5 - 0.037757x^6 - 0.28801x^7 + \ldots$$
$$y_2 = -L^{-1}\{y_1 + A_1\} = 0.004318x^6 + 0.001136x^7 + 0.00787x^8 + 0.009494x^9 - \ldots$$
$$y_3 = -L^{-1}\{y_2 + A_2\} = -0.0000771x^8 - 0.0000159x^9 - 0.000423x^{10} - 0.000305x^{11} + \ldots$$
$$\vdots$$

Hence, Legendre approximation is obtained

$y_L(x) \cong 1.5548\, x^2 - 0.9542\, x^3 - 1.41\, x^4 + 9.08\, x^5 - 20.2\, x^6 + 26.4\, x^7 - 18.6\, x^8 + \ldots$ (3.24)

Example 3: Consider the second order non-linear initial value problem for $0 \le x \le 1$;

$y^{//}(x) - y^{/}(x) + 4y^2(x) = 2 - \mathrm{Sin}2x$ (3.25)

$y(0) = y^{/}(0) = 0$ (3.26)

The exact solution is $y_{exact}(x) = \mathrm{Sin}^2 x$ [14].

Eq. (3.25) is organized in operator form as;

$L^{-1}\{Ly(x)\} = L^{-1}\{2 - \mathrm{Sin}2x\} + L^{-1}\{y^{/}(x)\} - 4L^{-1}\{y^2(x)\}$ (3.27)

in which $L^{-1}(*) = \int_0^x \int_0^x (*)dxdx$ inverse operator of $L = d^2/dx^2$, for non-linear term from Eq. (2.16), the Adomian polynomials are

$A_0 = y_0^2$

$A_1 = 2y_0 y_1$ (3.28)

$A_2 = 2y_0 y_1 + y_1^2$

$\vdots$

Taylor series of g(x)

$g_T(x) = 2 - 2x + (2x)^3/3! - (2x)^5/5! + O(x^6)$ (3.29)

is found. From Eq. (3.27) under the initial conditions Eq. (3.26), we get

$y(x) = y(0) + xy^{/}(0) + L^{-1}\{g_T(x)\} + L^{-1}\{y^{/}(x)\} - 4L^{-1}\{y^2(x)\}$

$y_0 = x^2 - x^3/3 + x^5/15 - 2x^7/315 + \ldots$ (3.30)

$y_1 = x^3/3 - x^4/12 - 11x^6/90 + 4x^7/63 - 11x^8/1260 - x^9/135 + \ldots$

$y_2 = x^4/12 - x^5/60 - 17x^7/210 + x^8/28 - 23x^9/5670 + 11x^{10}/1350 - 1091x^{11}/155925 + \ldots$

$y_3 = x^5/60 - x^6/130 - 71x^8/5040 + 11x^9/1890 - 11x^{10}/113400 + 113x^{11}/34650 - \ldots$

$\vdots$

In this way, the Taylor solution of problem Eq. (3.25) is obtained as

$$y_T(x) = \sum_{i=0}^{N} y_i = x^2 + \frac{1}{15}x^5 - \frac{76}{585}x^6 - \frac{1}{42}x^7 + \frac{13}{1008}x^8 - \frac{16}{2835}x^9 + \ldots \quad (3.31)$$

Besides, using Eq. (2.6) and Eq. (2.7) Legendre polynomial form of $g(x) \in C[0, 1]$ which is converted and from Eq. (3.10), it can be written $g(0.5x+0.5) = 2 - \mathrm{Sin}(x + 1)$ from Eq. (3.11), Legendre polynomial coefficients are a_i, for N = 6, are obtained that; $a_0 = 1.827$, $a_1 = -0.399$, $a_2 = 0.165$, $a_3 = 0.018$, $a_4 = -0.0003609$, $a_5 = -0.0002346$, $a_6 = 0.00003071$ then we have

$g_L(x) = 3.2082 - 5.2906x + 7.239x^2 - 12.71x^3 + 14.963x^4 - 7.3399x^5 + 1.816x^6 - \ldots$ (3.32)

Substituting Eq. (3.32) which is the Legendre approximation of g(x) and conditions Eq. (3.26) in Eq. (3.30), it can be obtained that

$$y_0 = 1.6041x^2 - 0.88177x^3 + 0.60325x^4 - 0.6355x^5 + 0.49877x^6 - \ldots$$
$$y_1 = 0.5347x^3 - 0.22044x^4 + 0.12065x^5 - 0.449x^6 + 0.34067x^7 - 0.21563x^8 + \ldots \quad (3.33)$$
$$y_2 = 0.1337x^4 - 0.04408x^5 - 0.02012x^6 - 0.2276x^7 + 0.1605x^8 - 0.1029x^9 + \ldots$$
$$y_3 = 0.02674x^5 - 0.007347x^6 + 0.002874x^7 - 0.03866x^8 + 0.0257x^9 + \ldots$$
$$\vdots$$

Thus, the Legendre approximation solution can be obtained as follows

$$y_L(x) = \sum_{i=0}^{N} y_i = 1.6041x^2 - 0.34707x^3 + 0.51655\,x^4 - 0.5321x^5 + 0.06257x^6 - 0.05876x^7 - \ldots \quad (3.34)$$

We give absolute errors in Figures 1,2, and 3 to show how the series rapidly converge to the exact solution. Absolute errors are defined as $e_1 = |\, y(x) - y_L(x) \,|$ or $e_2 = |\, y(x) - y_T(x) \,|$ in Figures 1, 2, and 3 where y(x) is the exact solution, $y_T(x)$ and $y_L(x)$ are the least squares approximation solution based on Taylor series and Legendre polynomials, respectively.

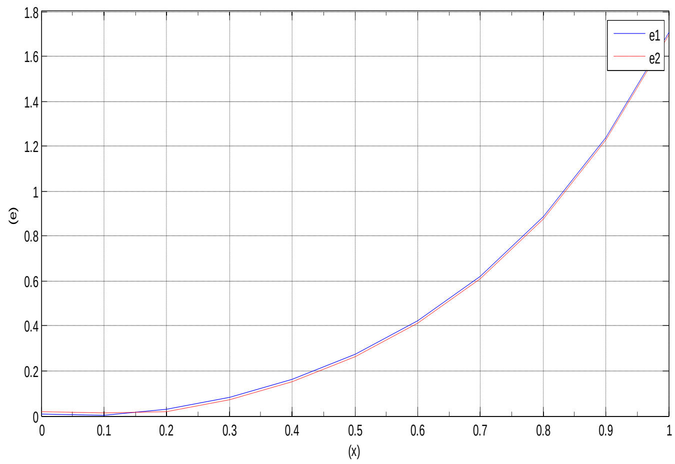

Figure 1: Absolute errors for the example 1.

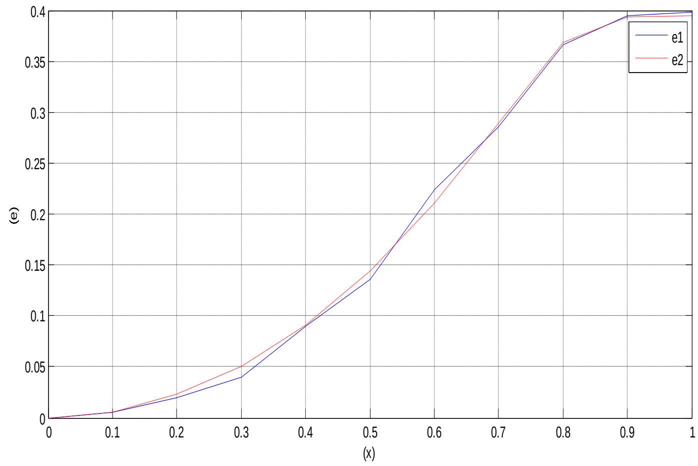

Figure 2: Absolute errors for the example 2.

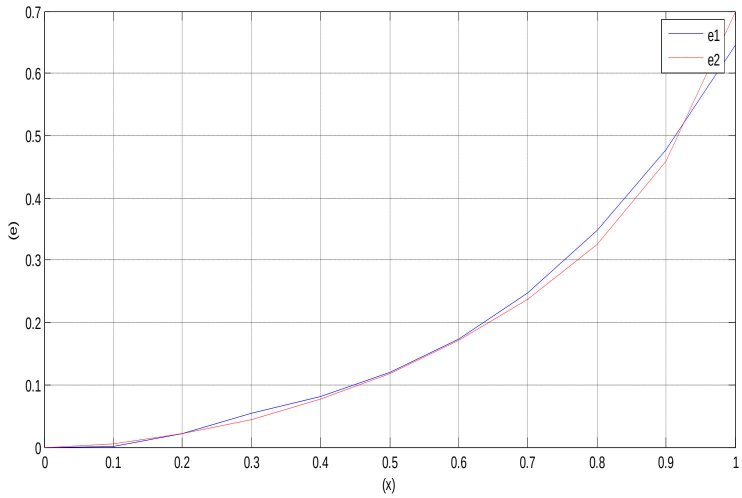

Figure 3: Absolute errors for the example 3.

4. CONCLUSIONS

The goal of this work has been to give an approximation for the solution of non-linear differential equations. We have achieved this goal by applying Legendre polynomial approximation method. The considered method which is called Legendre approximation method is defined in section 2, the examples are applied to the method to make it clear in section 3. In this work, for the Legendre approximation method, it is important thing that family of polynomials, called Jacobi polynomials, differs from each other according to the weight function with respect to which orthogonality holds. The existence of well-convergent expansions is guaranteed from the theory of orthogonal expansions, which gives the proof that any quadratically enterable function of bounded variation may be expanded into a complete orthogonal function system, such as the Legendre polynomials or Chebyshev polynomials. Therefore, in this paper Legendre polynomials are taken. Instead of Taylor polynomials for the solution of non-linear differential equations using Legendre polynomials is obtained a new approach in the least squares method. Obtained by the method of approach solved examples are presented. The results of the examples dealt with analytical solution, Taylor series solution and compared with Legendre polynomial approximation solutions, absolute errors are obtained. Graphs are plotted the absolute errors. As can be seen in the graphics, all the results are very close to each other trough was observed.

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