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Dynamics and Bifurcation of a Second Order Quadratic Rational Difference Equation

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Abstract

In this paper, we study the dynamics and bifurcation of

$$x_{n+1} = \frac{\alpha + \beta x^2_{n-1}}{A + Bx_n + Cx^2_{n-1}},\ n = 0,\ 1,\ 2,\ \ldots$$

with positive parameters α, β, A, B, C, and non-negative initial conditions.
Among others, we investigate local stability, invariant intervals, boundedness of the solutions, periodic solutions of prime period two and global stability of the positive fixed points.

1. Introduction

In this paper, we consider the second order, quadratic rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x^2_{n-1}}{A + Bx_n + Cx^2_{n-1}},\ n = 0,\ 1,\ 2,\ \ldots$$

with positive parameters α, β, A, B, C, and non-negative initial conditions.
We focus on local stability, invariant intervals, boundedness of the solutions, periodic solutions of prime period two and global stability of the positive fixed points.
Global asymptotic stability and Neimark-Sacker bifurcation of the difference equation

$$x_{n+1} = \frac{F}{bx_nx_{n-1} + Cx_{n-1}{}^2 + f},\ n = 0,\ 1,\ 2,$$

have been investigated by M. R. S. Kulenović et al. [1], with non-negative parameters and non-negative initial conditions such that the denominator is always positive.
Y. Kostrov and Z. Kudlak in [2] studied the boundedness character, local and global stability of solutions of the following second-order rational difference equation with quadratic denominator,

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{B + Dx_nx_{n-1} + x_{n-1}},\ n = 0,\ 1,\ 2,$$

where the coefficients are positive numbers, and the initial conditions are non-negative numbers such that the denominator is nonzero.
S. Moranjkić, and Z. Nurkanović [3] investigated local and global dynamics of difference equation

$$x_{n+1} = \frac{Bx_nx_{n-1} + Cx_{n-1}{}^2 + F}{bx_nx_{n-1} + cx_{n-1}{}^2 + f},\ n = 0,\ 1,\ 2,$$

with positive parameters and nonnegative initial conditions. Other higher ordered rational difference equations have been recently studied in [4], [5],[6],[7],[8],[9],[10],[11].

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2. Preliminaries

Before studying the behavior of solutions of this rational difference equation, we will review some definitions and basic results that will be used throughout this paper.
Consider the second order difference equation,

$$x(n+1) = f(x(n), x(n-1)),\ n = 0,\ 1,\ 2,\ \ldots \tag{2.1}$$

where $f : I \times I \to I$ is a continuously differentiable function, and I is an interval of real numbers. Then for every set of initial conditions $x_{-1},\ x_0 \in I$ the difference equation (2.1) has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

Definition 2.1. *[12] A point $\bar{x} \in I$ is an equilibrium point of equation (2.1) if $f(\bar{x}, \bar{x}) = \bar{x}$.*

Definition 2.2. *[12] Consider the difference equation (2.1). Then the linearized equation associated with this difference equation is*

$$y_{n+1} = ay_n + by_{n-1},\ n = 0,\ 1,\ 2,\ \ldots$$

where $a = \frac{\partial f}{\partial u}(\bar{x}, \bar{x})$, and $b = \frac{\partial f}{\partial v}(\bar{x}, \bar{x})$.
And the characteristic equation of (2.1) is

$$\lambda^2 - a\lambda - b = 0 \tag{2.2}$$

Theorem 2.3. *[13] (Linearized Stability)*
Consider the characteristic equation (2.2).

1. *If both characteristic roots of (2.2) lie inside the unit disk in the complex plane, then the equilibrium $\bar{x}$ of (2.1) is locally asymptotically stable.*
2. *If at least one characteristic root of (2.2) is outside the unit disk in the complex plane, the equilibrium point $\bar{x}$ is unstable.*
3. *If one characteristic root of (2.2) is on the unit disk and the other characteristic root is either inside or on the unit disk, then the equilibrium point $\bar{x}$ may be stable, unstable, or asymptotically stable.*
4. *A necessary and sufficient condition for both roots of (2.2) to lie inside the unit disk in the complex plane, is*

$$|a| < 1 - b < 2.$$

Let $A = Jf(\bar{x})$ be the Jacobian matrix of f at $\bar{x}$, where

$$Jf(\bar{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} |_{\bar{x}}$$

An important way to determine the stability of fixed points is given in the following result.

Theorem 2.4. *[14] Consider the map $f : H \subset \mathbb{R}^2 \to \mathbb{R}^2$, and let $A = Jf(\bar{x})$, with spectral norm $\rho(A)$. Then $\rho(A) < 1$, if and only if*

$$|tr(A)| - 1 < det(A) < 1$$

where $tr(A)$ is the trace of A, and $det(A)$ is the determinant of A.

The following theorem will be used to investigate global stability of fixed points.

Theorem 2.5. *[12] Let $[a,b]$ be an interval of real numbers and assume that $f : [a,b] \times [a,b] \to [a,b]$ is a continuous function satisfying the following properties:*

1. *$f(x,y)$ is non-increasing in $x \in [a,b]$ for each $y \in [a,b]$, $f(x,y)$ is non-decreasing in $y \in [a,b]$ for each $x \in [a,b]$.*
2. *The difference equation (2.1) has no solutions of prime period two in $[a,b]$.*
   Then (2.1) has a unique equilibrium $\bar{x} \in [a,b]$ and every solution of (2.1) converges to $\bar{x}$.

There are several types of bifurcation, the saddle-node bifurcation, period-doubling bifurcation, Neimark-Sacker bifurcation. For more information on types of bifurcation, the readers can refer to [15].

3. Dynamics of $x_{n+1} = \frac{\alpha + \beta x^2_{n-1}}{A + Bx_n + Cx^2_{n-1}}$

In this section, we consider the second order quadratic rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x^2_{n-1}}{A + Bx_n + Cx^2_{n-1}},\ n = 0,\ 1,\ 2,\ \ldots \tag{3.1}$$

with positive parameters $\alpha,\ \beta,\ A,\ B,\ C$, and non-negative initial conditions.

3.1. Change of variables

The change of variables

$$x_n = \frac{A}{B} y_n.$$

reduces equation (3.1) to the difference equation

$$y_{n+1} = \frac{p + qy^2{}_{n-1}}{1 + y_n + ry^2{}_{n-1}}, \; n = 0, \; 1, \; 2, \; \ldots$$

Where $p = \alpha \frac{B}{A^2}$, $q = \frac{\beta}{B}$, and $r = \frac{CA}{B^2}$.

3.2. Equilibrium points

To find the equilibrium point of

$$y_{n+1} = \frac{p + qy^2{}_{n-1}}{1 + y_n + ry^2{}_{n-1}}, \; n = 0, \; 1, \; 2, \; \ldots \tag{3.2}$$

with positive parameters $p, \; q, \; r$, and non-negative initial conditions. We solve the following equation

$$\bar{y} = \frac{p + q\bar{y}^2}{1 + \bar{y} + r\bar{y}^2.}$$

Hence,

$$r\bar{y}^3 + (1 - q)\bar{y}^2 + \bar{y} - p = 0. \tag{3.3}$$

can be considered as two curves with behavior

$$r\bar{y}^2 + (1 - q)\bar{y} = \frac{p}{\bar{y}} - 1.$$

Equation (3.2) has a unique positive equilibrium point $\bar{y}$, which can be obtained as an intersection point of these two curves. From Figure 3.1 and Figure 3.2 we obtain the required conclusion.

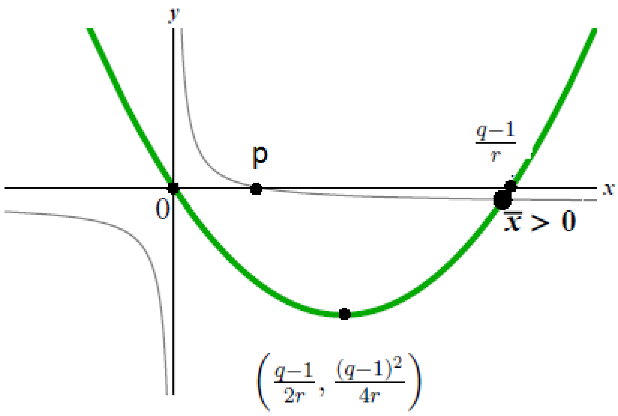

Figure 3.1: The equilibrium of (4.2.1), $q > 1$.

And then we choose the positive root to be $\bar{y}$.

3.3. Linearized equation

To find the linearized equation of (3.2) about the equilibrium point $\bar{y}$, let

$$f(x,y) = \frac{p + qy^2}{1 + x + ry^2}$$

We have

$$\frac{\partial f}{\partial x}(\bar{y},\bar{y}) = \frac{-\bar{y}}{1 + \bar{y} + r\bar{y}^2}.$$

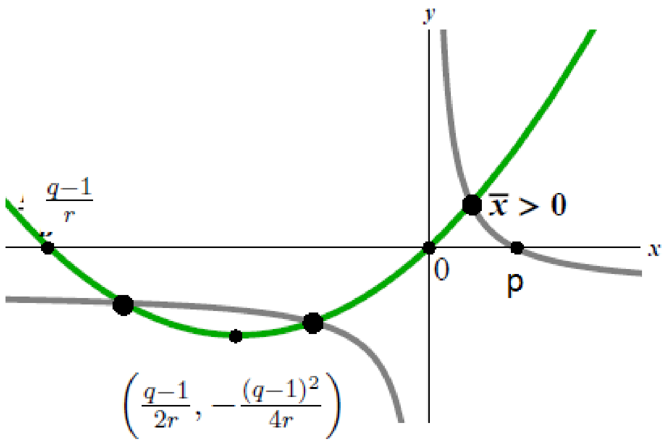

Figure 3.2: The equilibrium of (4.2.1), $0 < q < 1$.

And

$$\frac{\partial f}{\partial y}(\bar{y},\bar{y}) = \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}.$$

The linearized equation is

$$y_{n+1} = \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2}y_n + \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}y_{n-1}.$$

And the characteristic equation is

$$\lambda^2 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2}\lambda - \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = 0.$$

3.4. Local stability

To check when the unique positive equilibrium point $\bar{y}$ of equation (3.2) is locally asymptotically stable, let

$$a = \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2},\ b = \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}$$

Using Theorem 2.3 (4), a sufficient condition for asymptotic stability of $\bar{y}$ is $|a| < 1-b < 2$, which is equivalent to

$$-b < 1, \tag{3.4}$$

$$\text{and } |a| < 1-b. \tag{3.5}$$

(3.4) holds when

$$q > \frac{-1+r\bar{y}^2-\bar{y}}{2\bar{y}}.$$

And (3.5) is equivalent to

$$a > -1+b, \tag{3.6}$$

$$\text{and } a < 1-b. \tag{3.7}$$

(3.6) holds when

$$q < \frac{1+3r\bar{y}^2}{2\bar{y}}.$$

And (3.7) holds when

$$q < \frac{1+2\bar{y}+3r\bar{y}^2}{2\bar{y}}.$$

Hence a sufficient conditions for asymptotic stability of $\bar{y}$ is

$$q > \frac{-1+r\bar{y}^2-\bar{y}}{2\bar{y}}.$$

$$q < \frac{1+3r\bar{y}^2}{2\bar{y}}. \tag{3.8}$$

$$q < \frac{1+2\bar{y}+3r\bar{y}^2}{2\bar{y}}. \tag{3.9}$$

Note that if (3.8) holds, then (3.9) holds, thus $\frac{-1+r\bar{y}^2-\bar{y}}{2\bar{y}} < q < \frac{1+3r\bar{y}^2}{2\bar{y}}$ is a sufficient condition for asymptotic stability of $\bar{y}$.

3.5. Invariant intervals

Consider the difference equation (3.2), and $\{y_n\}_{n=-1}^{\infty}$ as a solution. Then $[0, \frac{q}{r}]$ when $pr \leq q$ is an invariant interval.

Proof. Assume that $pr \leq q$, and $y_{N-1},\ y_N \in [0, \frac{q}{r}]$ for some integer N.

$$\begin{aligned} y_{N+1} &= \frac{p+qy^2{}_{N-1}}{1+y_N+ry^2{}_{N-1}} \\ &= \frac{q(\frac{p}{q}+y^2{}_{N-1})}{r(\frac{1}{r}+\frac{1}{r}y_N+y^2{}_{N-1})} \\ &\leq \frac{q(\frac{1}{r}+y^2{}_{N-1})}{r(\frac{1}{r}+y^2{}_{N-1})} \\ &= \frac{q}{r} \end{aligned}$$

And working inductively we complete the proof. □

3.6. Boundedness

We will show that every solution of the difference equation (3.2) is bounded. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of (3.2). then we have for $n = 0,\ 1,\ 2,\ \ldots$

$$\begin{aligned} 0 < y_{n+1} &= \frac{p+qy^2{}_{n-1}}{1+y_n+ry^2{}_{n-1}} \\ &= \frac{p}{1+y_n+ry^2{}_{n-1}} + \frac{qy^2{}_{n-1}}{1+y_n+ry^2{}_{n-1}} \\ &\leq \frac{p}{1} + \frac{qy^2{}_{n-1}}{ry^2{}_{n-1}} \\ &= p+\frac{q}{r}. \end{aligned}$$

Hence, the solution is bounded, since it is bounded from below and from above.

3.7. Period two cycles

In general, we say that the solution $\{y_n\}_{n=-1}^{\infty}$ has a prime period two if the solution eventually takes the form:

$$\ldots,\ \phi,\ \psi,\ \phi,\ \psi,\ \ldots$$

where ϕ *and* ψ are positive, and $\phi \neq \psi$.

Theorem 3.1. *Assume that Equation (3.2) has a two periodic cycle $\{\phi,\ \psi\}$, where ϕ and ψ are positive, and $\phi \neq \psi$. Then q must satisfy the following condition:*

$$q > \frac{1+r(\phi^2+\psi^2)}{\phi+\psi}.$$

Proof. Assume $\{\phi,\ \psi\}$ is a prime period two solution of Equation (3.2), then $\phi,\ \psi$ satisfy:

$$\phi = \frac{p+q\phi^2}{1+\psi+r\phi^2} \tag{3.10}$$

and,

$$\psi = \frac{p+q\psi^2}{1+\phi+r\psi^2}. \tag{3.11}$$

From Equation (3.10), we have

$$\phi+\phi\psi+r\phi^3 = p+q\phi^2, \tag{3.12}$$

and from Equation (3.11), we have

$$\psi+\psi\phi+r\psi^3 = p+q\psi^2. \tag{3.13}$$

Subtracting Equation (3.13) from (3.12), we get:

$$(\phi-\psi)+r(\phi^3-\psi^3) = q(\phi^2-\psi^2).$$

Since, $\phi \neq \psi$, the last equation can be divided by $(\phi - \psi)$, and we get

$$1 + r(\phi^2 + \phi\psi + \psi^2) = q(\phi + \psi). \tag{3.14}$$

So,

$$\phi\psi = \frac{-1 - r(\phi^2 + \psi^2) + q(\phi + \psi)}{r}.$$

But, $\psi\phi \geq 0$, so

$$-1 - r(\phi^2 + \psi^2) + q(\phi + \psi) \geq 0.$$

Hence,

$$q > \frac{1 + r(\phi^2 + \psi^2)}{\phi + \psi}$$

which complete the proof. Note that from (3.14), we get:

$$\phi + \psi = \frac{r(\phi^2 + \phi\psi + \psi^2) + 1}{q}$$

which is always positive. □

3.8. Global stability

Now, we will investigate a result about the global stability of the positive equilibrium point of (3.2) $\bar{y}$.

Theorem 3.2. *Assume $pr \leq q \leq \frac{\sqrt{r}}{\sqrt{2}}$. Then the positive equilibrium point $\bar{y}$ on the interval $S = [0, \frac{q}{r}]$ is globally asymptotically stable.*

Proof: This proof can be easily done depending on Theorem 2.5. Assume $pr \leq q$, and consider the function

$$f(x,y) = \frac{p + qy^2}{1 + x + ry^2}.$$

Note that S is an invariant interval and all non-negative solutions of Equation (3.2) lie in this interval. And $f(x,y)$ on S is non-increasing function in x, and non-decreasing in y.
Now, we need to show that the difference equation (3.2) has no solution of prime period two in S.
For seek of contradiction, assume that the difference equation (3.2) has a solution of prime period two $\{\phi, \psi\} \in S$. Then q must satisfy

$$q > \frac{1 + r(\phi^2 + \psi^2)}{\phi + \psi},$$

but since $\{\phi, \psi\} \in S$

$$\frac{1 + r(\phi^2 + \psi^2)}{\phi + \psi} \geq \frac{1 + 0}{\frac{q}{r} + \frac{q}{r}},$$

hence

$$q > \frac{r}{2q},$$

so,

$$q^2 > \frac{r}{2},$$

which is a contradiction, since $q \leq \frac{\sqrt{r}}{\sqrt{2}}$.
So, Equation (3.2) has no solution of prime period two in S. Then both conditions of Theorem 2.5 hold, so (3.2) has a unique positive equilibrium point $\bar{y} \in S$, and it is globally asymptotically stable. □

4. Bifurcation of $y_{n+1}=\frac{p+qy^2{}_{n-1}}{1+y_n+ry^2{}_{n-1}}$

In this section we study types of bifurcation that occur at $q=q^*$ as q is the bifurcation parameter.
In order to convert Equation (3.2) to a second dimensional system with three parameters p,q, and r, let

$$z_n=y_{n-1},$$

and

$$v_n=y_n.$$

We get the following system

$$z_{n+1}=v_n$$

$$v_{n+1}=\frac{p+qz^2{}_n}{1+v_n+rz^2{}_n},\ n=0,\ 1,\ 2,\ \dots$$

This system has the unique fixed point $(\bar{z},\bar{v})^T=(\bar{y},\bar{y})^T$. Convert this system into a second dimensional map

$$F\begin{pmatrix} z\\ v\end{pmatrix}=\begin{pmatrix} f_1(z,v)\\ f_2(z,v)\end{pmatrix}=\begin{pmatrix} v\\ \frac{p+qz^2}{1+v+rz^2}\end{pmatrix}. \tag{4.1}$$

So, the Jacobian matrix of $F(z,v)$ at $(\bar{y},\bar{y})$ is

$$JF(z,v)|_{(\bar{y},\bar{y})}=\begin{pmatrix} 0 & 1\\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2}\end{pmatrix}$$

So,

$$\det(JF(\bar{y},\bar{y}))=-\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2},$$

and,

$$tr(JF(\bar{y},\bar{y}))=\frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2}.$$

Theorem 4.1. *The fixed point $(\bar{y},\bar{y})$ of the system (4.1) undergoes a saddle-node bifurcation, when $q=\frac{3r\bar{y}^2+2\bar{y}+1}{2\bar{y}}$.*

Proof: Saddle-node bifurcation happens when,

$$-\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}=\frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2}-1$$

thus,

$$q=\frac{3r\bar{y}^2+2\bar{y}+1}{2\bar{y}}.$$

So, saddle-node bifurcation happens if $q=\frac{3r\bar{y}^2+2\bar{y}+1}{2\bar{y}}$. □

Theorem 4.2. *The fixed point $(\bar{y},\bar{y})$ of the system (4.1) undergoes a period-doubling bifurcation, when $q=\frac{3r\bar{y}^2+1}{2\bar{y}}$.*

Proof: Period-doubling bifurcation happens when,

$$\det(J)=-tr(J)-1.$$

So, the fixed point $(\bar{y},\bar{y})$ of the system (4.1) undergoes a period-doubling bifurcation if

$$-\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}=\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2}-1$$

thus,

$$q=\frac{3r\bar{y}^2+1}{2\bar{y}}.$$

So, period-doubling bifurcation happens if $q=\frac{3r\bar{y}^2+1}{2\bar{y}}$. □

Theorem 4.3. *The fixed point $(\bar{y},\bar{y})$ of the system (4.1) undergoes Neimark-Sacker bifurcation when, $q=\frac{r\bar{y}^2-\bar{y}-1}{2\bar{y}}$, if $r>\frac{1+\bar{y}}{\bar{y}^2}$.*

Proof: Assume $r > \frac{1+\bar{y}}{\bar{y}^2}$. Neimark-Sacker bifurcation which happens when,

$$\det(J) = 1$$

and,

$$-2 < tr(J) < 2.$$

So, the system (4.1) undergoes Neimark-Sacker bifurcation when,

$$\det(JF(\bar{y},\bar{y})) = 1 \tag{4.2}$$

and,

$$-2 < tr(JF(\bar{y},\bar{y})) < 2.$$

The last inequality always holds, since it is equivalent to

$$-2 < \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} < 2,$$

which can be splitted into two inequalities, namely

$$-2 < \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2},$$

and,

$$\frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} < 2,$$

The first inequality

$$-2-2\bar{y}-2r\bar{y}^2 < -\bar{y},$$

implies

$$-2-\bar{y}-2r\bar{y}^2 < 0,$$

which always holds. And $\frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} < 2$ implies

$$2+3\bar{y}+2r\bar{y}^2 > 0.$$

which also always holds.
Now, Equation (4.2) holds if

$$-\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = 1$$

so,

$$-2\bar{y}(q-r\bar{y}) = 1+\bar{y}+r\bar{y}^2$$

thus,

$$q = \frac{r\bar{y}^2-\bar{y}-1}{2\bar{y}}.$$

Which is positive since $r > \frac{1+\bar{y}}{\bar{y}^2}$. So the system (4.1) undergoes Neimark-sacker bifurcation at $(\bar{y},\bar{y})$ when $q = \frac{r\bar{y}^2-\bar{y}-1}{2\bar{y}}$. □

4.1. Direction of the period-doubling (flip) bifurcation

In this subsection, we will find the direction of Flip bifurcation of system (4.1) at $q = \frac{3r\bar{y}^2+1}{2\bar{y}}$.
We need at first to shift the fixed point $(\bar{y},\bar{y})$ to the origin. Let

$$w_n = z_n - \bar{y},\ u_n = v_n - \bar{y}.$$

System (4.1) will be

$$w_{n+1} = u_n$$

$$u_{n+1} = \frac{p+q(w_n+\bar{y})^2}{1+(u_n+\bar{y})+r(w_n+\bar{y})^2},\ n = 0,\ 1,\ 2,\ \ldots$$

Or,

$$Y_{n+1} = AY_n + G(Y_n),$$

where,

$$A = \begin{pmatrix} 0 & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix}, \; Y_n = \begin{pmatrix} w_n \\ u_n \end{pmatrix},$$

and,

$$G(Y) = \frac{1}{2}B(Y,Y) + \frac{1}{6}C(Y,Y,Y) + O(\|Y\|^4)$$

$$B(Y,Y) = \begin{pmatrix} B_1(Y,Y) \\ B_2(Y,Y) \end{pmatrix} \; and, \; C(Y,Y,Y) = \begin{pmatrix} C_1(Y,Y,Y) \\ C_2(Y,Y,Y) \end{pmatrix}$$

where,

$$B_i(x,y) = \sum_{k,j=1}^{n} \frac{\partial^2 Y_i(\eta)}{\partial \eta_k \partial \eta_j}|_{\eta=0}(x_k y_j)$$

and,

$$C_i(x,y,z) = \sum_{l,k,j=1}^{n} \frac{\partial^3 Y_i(\eta)}{\partial \eta_l \partial \eta_k \partial \eta_j}|_{\eta=0}(x_l y_k z_j).$$

So, $B_1(\psi,\phi) = 0$ and $C_1(\psi,\phi,\xi) = 0$,

$$B_2(\psi,\phi) = \frac{2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_1\phi_1) - \frac{2\bar{y}(2r\bar{y}+q)}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_1\phi_2+\psi_2\phi_1) + \frac{2\bar{y}}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_2\phi_2),$$

and

$$C_2(\psi,\phi,\xi) = \frac{12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_1\phi_1\xi_1) + \frac{-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_1\phi_1\xi_2+\psi_1\phi_2\xi_1+\psi_2\phi_1\xi_1) + \frac{4\bar{y}(q-3r\bar{y})}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_2\phi_2\xi_1+\psi_2\phi_1\xi_2+\psi_1\phi_2\xi_2) + \frac{-6\bar{y}}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_2\phi_2\xi_2).$$

Now, we find the eigenvectors of A and A^T corresponding to the eigenvalue $\lambda = -1$ at the bifurcation point $q = \frac{3r\bar{y}^2+1}{2\bar{y}}$. Let $\hat{q}$ and p^* be the eigenvectors of A and A^T corresponding to the eigenvalue $\lambda = -1$ respectively. So, we have

$$A\hat{q} = -\hat{q}, \text{ and } A^T p^* = -p^*.$$

Or,

$$(A+I)\hat{q} = 0 \tag{4.3}$$

$$(A^T+I)p^* = 0. \tag{4.4}$$

From Equation (4.3), we get $\hat{q} \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

And from equation (4.4), we get $p^* \sim \begin{pmatrix} \frac{-2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} \\ 1 \end{pmatrix}$.

Now, we normalize p^* and $\hat{q}$,

$$\langle p^*, \hat{q} \rangle = \frac{-2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} - 1.$$

Take $\hat{p} = \eta \begin{pmatrix} \frac{-2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} \\ 1 \end{pmatrix}$, $\eta = -\frac{1+\bar{y}+r\bar{y}^2}{1+(2q+1)\bar{y}-r\bar{y}^2}$.

The critical eigenspace T^c corresponding to $\lambda = -1$ is a one-dimensional and spanned by an eigenvector $\hat{q}$. Let T^{su} denote a one-dimensional linear eigenspace of A corresponding to all eigenvalues other than λ. Note that the matrix $(A-\lambda I_n)$ has common invariant spaces with the matrix A, so we conclude that $y \in T^{su}$ if and only if $\langle \hat{p}, y \rangle = 0$.

So, to find $c(0)$ which is given by the following invariant formula:

$$c(0) = \frac{1}{6}\langle \hat{p}, C(\hat{q},\hat{q},\hat{q}) \rangle - \frac{1}{2}\langle \hat{p}, B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q})) \rangle.$$

We evaluate

$$B(\hat{q},\hat{q}) = \begin{pmatrix} 0 \\ \frac{2\bar{y}(3q+1+4r\bar{y})+2q-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2} \end{pmatrix}.$$

$$C(\hat{q},\hat{q},\hat{q}) = \begin{pmatrix} 0 \\ \frac{12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))}{(1+\bar{y}+r\bar{y}^2)^3} - 3\frac{-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^3} + \frac{12\bar{y}(q-3r\bar{y})}{(1+\bar{y}+r\bar{y}^2)^3} + \frac{6\bar{y}}{(1+\bar{y}+r\bar{y}^2)^3} \end{pmatrix}.$$

$$\langle \hat{p}, C(\hat{q},\hat{q},\hat{q})\rangle = -\left(\frac{1+\bar{y}+r\bar{y}^2}{1+(2q+1)\bar{y}-r\bar{y}^2}\right)\left[\frac{12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))}{(1+\bar{y}+r\bar{y}^2)^3} - 3\frac{-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^3} + \frac{12\bar{y}(q-3r\bar{y})}{(1+\bar{y}+r\bar{y}^2)^3} + \frac{6\bar{y}}{(1+\bar{y}+r\bar{y}^2)^3}\right].$$

$$(A-I)^{-1} = \begin{pmatrix} -1 & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & -1+\frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix}^{-1} = \frac{1+\bar{y}+r\bar{y}^2}{2\bar{y}+r\bar{y}^2}\begin{pmatrix} -1+\frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} & -1 \\ -\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & -1 \end{pmatrix}.$$

$$(A-I)^{-1}B(\hat{q},\hat{q}) = \frac{1+\bar{y}+r\bar{y}^2}{2\bar{y}+r\bar{y}^2}\begin{pmatrix} \frac{-2\bar{y}(3q+1+4r\bar{y})-2q+2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2} \\ \frac{-2\bar{y}(3q+1+4r\bar{y})-2q+2r(p+2\bar{y}(2q\bar{y}+2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2} \end{pmatrix}.$$

$$B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q})) = \frac{1+\bar{y}+r\bar{y}^2}{2\bar{y}+r\bar{y}^2}\begin{pmatrix} 0 \\ m \end{pmatrix},$$

where

$$m = \left(\frac{-2\bar{y}(3q+1+4r\bar{y})-2q+2r(p+2\bar{y}(2q\bar{y}+2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2}\right)\times \left(\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-2\bar{y}}{(1+\bar{y}+r\bar{y}^2)^2}\right).$$

$$\langle \hat{p}, B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q}))\rangle = \left(\left[\frac{2\bar{y}(3q+1+4r\bar{y})+2q-2r(p+2\bar{y}(2q\bar{y}+2r\bar{y}^2))}{(2\bar{y}+r\bar{y}^2)(1+(2q+1)\bar{y}-r\bar{y}^2)}\right]\left[\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-2\bar{y}}{(1+\bar{y}+r\bar{y}^2)^2}\right]\right).$$

If $c(0)>0$, then a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $q=\frac{3r\bar{y}^2+1}{2\bar{y}}$.

4.2. Direction and stability of Neimark-Sacker bifurcation

We will first show when the Neimark-Sacker bifurcation conditions are satisfied.

Theorem 4.4. *If $q=q^*=\frac{r\bar{y}^2-1-\bar{y}}{2\bar{y}}$, and $r>\frac{1+\bar{y}}{\bar{y}^2}$, then the characteristic equation of (3.2) has two complex conjugate roots that lie on the unit circle. Moreover, the Neimark-Sacker bifurcation conditions are satisfied.*

Proof: At the beginning, we will show that the characteristic equation of (3.2)

$$\lambda^2+\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2}\lambda-\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}=0. \tag{4.5}$$

has two complex roots. The roots of (4.5) are

$$\lambda_{1,2}=\frac{-\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2}\pm\sqrt{\Delta}}{2},$$

where,

$$\Delta=\frac{\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^2}+4\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}.$$

Substituting $q=q^*$, we get

$$\Delta=\frac{\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^2}+4\frac{-1-\bar{y}-r\bar{y}^2}{1+\bar{y}+r\bar{y}^2}.$$

So,

$$\Delta = \frac{\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^2} - 4.$$

Thus, (4.5) has two complex roots if $\Delta < 0$, which is equivalent to

$$\frac{\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^2} - 4 < 0,$$

which implies

$$\bar{y}^2 < 4(1+\bar{y}+r\bar{y}^2)^2,$$

so,

$$4(1+2\bar{y}+\bar{y}^2+2(1+\bar{y})r\bar{y}^2+r^2\bar{y}^4) - \bar{y}^2 > 0,$$

thus, $\Delta(q^*) < 0$ if

$$4+8\bar{y}+3\bar{y}^2+8(1+\bar{y})r\bar{y}^2+4r^2\bar{y}^4 > 0,$$

which always holds.
Next, we show that (4.5) has two conjugate complex roots on the unit circle when $q = q^*$.
Since, $\lambda_{1,2}$ are the roots of (4.5), we have

$$\lambda_1\lambda_2 = -\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}.$$

Substituting $q = q^*$ we get

$$\lambda_1\lambda_2 = 1.$$

But, $\lambda_1\lambda_2 = |\lambda_{1,2}|^2 = 1$. Thus, the two complex roots are on the unit circle.
Assume the roots of (4.5) at $q = q^*$ are $e^{\pm i\theta}$. So, we have

$$e^{i\theta}+e^{-i\theta} = -\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2},$$

but, $e^{i\theta}+e^{-i\theta} = 2\cos(\theta)$. Thus,

$$\cos(\theta) = -\frac{\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}.$$

Note that $\frac{-1}{2} < \cos(\theta) < 0$. So, there exists $\theta_0 \in (\frac{\pi}{2}, \pi)$ such that

$$\theta_0 = \cos^{-1}\left(-\frac{\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}\right).$$

And, $e^{ik\theta_0} \neq 1$ for $k = 1,\ 2,\ 3,\ 4$.
Next, we will show that $\frac{d|\lambda|^2}{dq}|_{q=q^*} \neq 0$.

$$|\lambda|^2 = -\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2},$$

differentiate with respect to q, we get

$$\frac{d|\lambda|^2}{dq} = -\frac{(1+\bar{y}+r\bar{y}^2)(2\bar{y}(1-r\frac{d\bar{y}}{dq})+(q-r\bar{y})2\frac{d\bar{y}}{dq}) - (2\bar{y}(q-r\bar{y}))(\frac{d\bar{y}}{dq}+2r\bar{y}\frac{d\bar{y}}{dq})}{(1+\bar{y}+r\bar{y}^2)^2}.$$

To find $\frac{d\bar{y}}{dq}$, we differentiate equation (3.3) with respect to q

$$\frac{d}{dq}(r\bar{y}^3+(1-q)\bar{y}^2+\bar{y}-p) = 0,$$

so,

$$3r\bar{y}^2\frac{d\bar{y}}{dq}+(1-q)2\bar{y}\frac{d\bar{y}}{dq}+\bar{y}^2(-1)+\frac{d\bar{y}}{dq} = 0,$$

thus,

$$\frac{d\bar{y}}{dq} = \frac{\bar{y}^2}{3r\bar{y}^2+(1-q)2\bar{y}+1}.$$

Substituting $q = q^*$, we get

$$\frac{d\bar{y}}{dq}|_{q=q^*} = \frac{\bar{y}^2}{2r\bar{y}^2 + 3\bar{y} + 2}.$$

So,

$$\frac{d|\lambda|^2}{dq}|_{q=q^*} = -\frac{(3r\bar{y}^3 + 6\bar{y}^2 + 3\bar{y})(r\bar{y}^2 + \bar{y} + 1)}{(2r\bar{y}^2 + 3\bar{y} + 2)(1 + \bar{y} + r\bar{y}^2)^2} < 0,$$

So the Neimark-Sacker bifurcation conditions are satisfied.

□

As in the previous subsection, we shift the fixed point $(\bar{y}, \bar{y})$ to the origin. We get

$$Y_{n+1} = AY_n + G(Y_n), \tag{4.6}$$

where,

$$A = \begin{pmatrix} 0 & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix}, \; Y_n = \begin{pmatrix} w_n \\ u_n \end{pmatrix},$$

and,

$$G(Y) = \frac{1}{2}B(Y,Y) + \frac{1}{6}C(Y,Y,Y) + O(\|Y\|^4)$$

$$B(Y,Y) = \begin{pmatrix} B_1(Y,Y) \\ B_2(Y,Y) \end{pmatrix} \; and, \; C(Y,Y,Y) = \begin{pmatrix} C_1(Y,Y,Y) \\ C_2(Y,Y,Y) \end{pmatrix}$$

where,

$$B_i(x,y) = \sum_{k,j=1}^{n} \frac{\partial^2 Y_i(\eta)}{\partial \eta_k \partial \eta_j}|_{\eta=0}(x_k y_j)$$

and,

$$C_i(x,y,z) = \sum_{l,k,j=1}^{n} \frac{\partial^3 Y_i(\eta)}{\partial \eta_l \partial \eta_k \partial \eta_j}|_{\eta=0}(x_l y_k z_j).$$

So $B_1(\psi,\phi) = 0$ and $C_1(\psi,\phi,\xi) = 0$,

$$B_2(\psi,\phi) = \frac{2q(1+\bar{y}) - 2r(p + 2\bar{y}(2q\bar{y} - 2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_1\phi_1) - \frac{2\bar{y}(2r\bar{y}+q)}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_1\phi_2 + \psi_2\phi_1) + \frac{2\bar{y}}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_2\phi_2),$$

and,

$$C_2(\psi,\phi,\xi) = \frac{12r\bar{y}(-3(q(1+\bar{y}) - rp) + 4\bar{y}^2(q - r\bar{y}))}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_1\phi_1\xi_1) + \frac{-2q(1+\bar{y}) + 24(q - r\bar{y}) - 6rq\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_1\phi_1\xi_2 + \psi_1\phi_2\xi_1 + \psi_2\phi_1\xi_1) + \frac{4\bar{y}(q - 3r\bar{y})}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_2\phi_2\xi_1 + \psi_2\phi_1\xi_2 + \psi_1\phi_2\xi_2) + \frac{-6\bar{y}}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_2\phi_2\xi_2).$$

Now, we find the eigenvectors of A and A^T corresponding to the eigenvalue $e^{\pm i\theta_0}$ at the bifurcation point $q = \frac{r\bar{y}^2 - \bar{y} - 1}{2\bar{y}}$. Let $\hat{q}$ and p^* be the eigenvectors of A and A^T corresponding to the eigenvalue $e^{\pm i\theta_0}$ respectively. So, we have

$$A\hat{q} = e^{i\theta_0}\hat{q}, \;\text{ and }\; A^T p^* = e^{-i\theta_0}p^*.$$

Or,

$$(A - e^{i\theta_0}I)\hat{q} = 0 \tag{4.7}$$

$$(A^T - e^{-i\theta_0}I)p^* = 0. \tag{4.8}$$

From Equation (4.7), we get $\hat{q} \sim \begin{pmatrix} 1 \\ e^{i\theta_0} \end{pmatrix}$.

And Equation (4.8) gives $p^* \sim \begin{pmatrix} 1 + e^{i\theta_0}\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} \\ e^{i\theta_0} \end{pmatrix}$.

Now, we normalize p^* and $\hat{q}$, take $\hat{p} = \eta \begin{pmatrix} 1 + e^{i\theta_0} \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} \\ e^{i\theta_0} \end{pmatrix}$, $\eta = \frac{1}{2+\frac{e^{-i\theta_0}\bar{y}}{1+\bar{y}+r\bar{y}^2}}$.
So, to determine the direction of the Neimark-sacker bifurcation, we compute $a(0)$ as given in ([15]).
where,

$$g_{20} = \frac{e^{i\theta_0}(2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2)) - 4\bar{y}(2r\bar{y}+q)e^{i\theta_0} + 2\bar{y}e^{2i\theta_0})}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}.$$

$$g_{11} = \frac{e^{i\theta_0}[2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2)) + 2\bar{y} - 4\bar{y}(2r\bar{y}+q)\cos(\theta_0)]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}.$$

$$g_{02} = \frac{e^{i\theta_0}[2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2)) - 4\bar{y}(2r\bar{y}+q)e^{-i\theta_0} + 2\bar{y}e^{-2i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}.$$

And,

$$g_{21} = \langle \hat{p}, C(\hat{q},\hat{q},\bar{\hat{q}})\rangle + 2\langle \hat{p}, B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}}))\rangle + \langle \hat{p}, B(\bar{\hat{q}},(e^{2i\theta_0}I-A)^{-1}B(\hat{q},\hat{q}))\rangle + \frac{e^{-i\theta_0}(1-2e^{i\theta_0})}{1-e^{-i\theta_0}}\langle \hat{p},B(\hat{q},\hat{q})\rangle\langle \hat{p},B(\hat{q},\bar{\hat{q}})\rangle - \frac{2}{1-e^{-i\theta_0}}|\langle \hat{p},B(\hat{q},\bar{\hat{q}})\rangle|^2 - \frac{e^{i\theta_0}}{e^{3i\theta_0}-1}|\langle \hat{p},B(\bar{\hat{q}},\bar{\hat{q}})\rangle|^2.$$

where,

$$\langle \hat{p}, C(\hat{q},\hat{q},\bar{\hat{q}})\rangle = \frac{e^{i\theta_0}[12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)^2} + \frac{e^{i\theta_0}[(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)(\cos(\theta_0)+e^{i\theta_0})+4\bar{y}(q-3r\bar{y})(2+e^{2i\theta_0})-6\bar{y}e^{i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)^2}.$$

And,

$$\langle \hat{p}, B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}}))\rangle = \frac{e^{i\theta_0}M(1+\bar{y}+r\bar{y}^2)}{2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y}}.$$

$$s = \frac{2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2)) + 2\bar{y} - 4\bar{y}(2r\bar{y}+q)\cos(\theta_0)}{(1+2\bar{y}-2q\bar{y}+3r\bar{y}^2)(1+\bar{y}+r\bar{y}^2)}.$$

$$M = s\frac{2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2)) - 2\bar{y}(2r\bar{y}+q)(1+e^{i\theta_0}) + 2\bar{y}e^{i\theta_0}}{(1+\bar{y}+r\bar{y}^2)^2}.$$

Finally,

$$\langle \hat{p}, B(\bar{\hat{q}},(e^{2i\theta_0}I-A)^{-1}B(\hat{q},\hat{q}))\rangle = \frac{e^{i\theta_0}L[2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)} + \frac{e^{i\theta_0}L[-2\bar{y}(2r\bar{y}+q)(e^{2i\theta_0}+e^{-i\theta_0})+2\bar{y}e^{2i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}.$$

where,

$$L = \frac{2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2)) - 4\bar{y}(2r\bar{y}+q)e^{i\theta_0} + 2\bar{y}e^{2i\theta_0}}{(e^{4i\theta_0}(1+\bar{y}+r\bar{y}^2)+e^{2i\theta_0}\bar{y}-2\bar{y}(q-r\bar{y}))(1+\bar{y}+r\bar{y}^2)}.$$

So,

$$a(0) = \frac{1}{2}Re\left(e^{-i\theta_0}\langle \hat{p}, C(\hat{q},\hat{q},\bar{\hat{q}})\rangle\right) + Re\left(e^{-i\theta_0}\langle \hat{p}, B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}}))\rangle\right) + \frac{1}{2}Re\left(e^{-i\theta_0}\langle \hat{p}, B(\bar{\hat{q}},(e^{2i\theta_0}I-A)^{-1}B(\hat{q},\hat{q}))\rangle\right).$$

Let

$$B_1 = Re\left(e^{-i\theta_0}\langle \hat{p}, C(\hat{q},\hat{q},\bar{\hat{q}})\rangle\right),\ B_2 = Re\left(e^{-i\theta_0}\langle \hat{p}, B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}}))\rangle\right),$$

and

$$B_3 = Re\left(e^{-i\theta_0}\langle \hat{p}, B(\bar{\hat{q}},(e^{2i\theta_0}I-A)^{-1}B(\hat{q},\hat{q}))\rangle\right).$$

To find B_1,

$$B_1 = Re\left(\frac{[12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)^2}+\right.$$
$$\left.\frac{e^{i\theta_0}[(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)(\cos(\theta_0)+e^{i\theta_0})+4\bar{y}(q-3r\bar{y})(2+e^{2i\theta_0})-6\bar{y}e^{i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)^2}\right).$$

Multiplying and dividing by the conjugate of the complex part of the denominator, the denominator becomes,

$$A_1 = (4(1+\bar{y}+r\bar{y}^2)^2+4(1+\bar{y}+r\bar{y}^2)\bar{y}\cos(\theta_0)+\bar{y}^2)(1+\bar{y}+r\bar{y}^2)^2.$$

Multiplying the numerator by the conjugate of the complex part of the denominator and taking the real part of the numerator, we get,

$$\begin{aligned} A_2 = {} & 2(1+\bar{y}+r\bar{y}^2)[12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))]+ \\ & \bar{y}[12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))]\cos(\theta_0) \\ & +4(1+\bar{y}+r\bar{y}^2)(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)(\cos(\theta_0))+ \\ & \bar{y}(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)(\cos^2(\theta_0))+ \\ & \bar{y}(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)\cos(2\theta_0)+ \\ & 16(1+\bar{y}+r\bar{y}^2)\bar{y}(q-3r\bar{y})+8(1+\bar{y}+r\bar{y}^2)\bar{y}(q-3r\bar{y})(\cos(2\theta_0))+ \\ & 8\bar{y}^2(q-3r\bar{y})(\cos(\theta_0))+4\bar{y}^2(q-3r\bar{y})(\cos(3\theta_0))-12\bar{y}(1+\bar{y}+r\bar{y}^2)\cos(\theta_0) \\ & -6\bar{y}^2\cos(2\theta_0). \end{aligned}$$

So, we have $B_1 = \frac{A_2}{A_1}$.
To find B_2:

$$B_2 = Re\left(s\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-2\bar{y}(2r\bar{y}+q)(1+e^{i\theta_0})}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}\right.$$
$$\left.+\frac{2\bar{y}e^{i\theta_0}}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}\right).$$

Multiplying and dividing by the conjugate of the complex part of the denominator, the denominator becomes,

$$A_3 = (4(1+\bar{y}+r\bar{y}^2)^2+4(1+\bar{y}+r\bar{y}^2)\bar{y}\cos(\theta_0)+\bar{y}^2)(1+\bar{y}+r\bar{y}^2).$$

Multiplying the numerator by the conjugate of the complex part of the denominator and taking the real part of the numerator, we get,

$$\begin{aligned} A_4 = {} & s[2(1+\bar{y}+r\bar{y}^2)(2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))) \\ & +\bar{y}(2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2)))\cos(\theta_0)-4(1+\bar{y}+r\bar{y}^2)\bar{y}(2r\bar{y}+q) \\ & -4(1+\bar{y}+r\bar{y}^2)\bar{y}(2r\bar{y}+q)\cos(\theta_0)-2\bar{y}^2(2r\bar{y}+q)\cos(\theta_0) \\ & -2\bar{y}^2(2r\bar{y}+q)\cos(2\theta_0)+4\bar{y}(1+\bar{y}+r\bar{y}^2)\cos(\theta_0)+2\bar{y}^2\cos(2\theta_0)]. \end{aligned}$$

We have $B_2 = \frac{A_4}{A_3}$.
To find B_3

$$B_3 = Re\left(\frac{[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}+\right.$$
$$\frac{[-2\bar{y}(2r\bar{y}+q)(e^{2i\theta_0}+e^{-i\theta_0})+2\bar{y}e^{2i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}$$
$$\left.\times\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{i\theta_0}+2\bar{y}e^{2i\theta_0}}{(e^{4i\theta_0}(1+\bar{y}+r\bar{y}^2)+e^{2i\theta_0}\bar{y}-2\bar{y}(q-r\bar{y}))(1+\bar{y}+r\bar{y}^2)}\right).$$

Multiplying and dividing by the conjugate of the complex part of the denominator, the denominator becomes,

$$\begin{aligned} A_5 = {} & (4(1+\bar{y}+r\bar{y}^2)^2+4(1+\bar{y}+r\bar{y}^2)\bar{y}\cos(\theta_0)+\bar{y}^2)(1+\bar{y}+r\bar{y}^2)^2[(1+\bar{y}+r\bar{y}^2)^2+\bar{y}^2 \\ & +4\bar{y}^2(q-r\bar{y})^2+2(1+\bar{y}+r\bar{y}^2)\bar{y}\cos(2\theta_0) \\ & -4\bar{y}(q-r\bar{y})(1+\bar{y}+r\bar{y}^2)\cos(4\theta_0)-4\bar{y}(q-r\bar{y})\cos(2\theta_0)]. \end{aligned}$$

Multiplying the numerator by the conjugate of the complex part of the denominator and taking the real part of the numerator, we get,

$$\begin{aligned} A_6 = {} & (a_8b_6+a_9b_1)\cos(5\theta_0)+((a_1+a_6+a_{12})b_1+a_9b_4+a_8b_3+a_5b_6)\cos(4\theta_0)+ \\ & ((a_1+a_6+a_{12})b_4+(a_2+a_7+a_{10})b_1+(a_3+a_4+a_{11})b_6+a_5b_3+a_8b_5+a_9b_2)\cos(3\theta_0)+ \\ & ((a_1+a_6+a_{12})b_2+(a_2+a_7+a_{10})b_4+(a_2+a_7+a_{10})b_6+(a_3+a_4+a_{11})b_1 \\ & +(a_3+a_4+a_{11})b_3+a_8b_2+a_9b_5)\cos(2\theta_0)+((a_1+a_6+a_{12})b_5 \\ & +(a_1+a_6+a_{12})b_6+(a_2+a_7+a_{10})b_2+(a_2+a_7+a_{10})b_3+(a_3+a_4+a_{11})b_4 \\ & +(a_3+a_4+a_{11})b_5+a_5b_1+a_5b_2+a_5b_5+a_8b_4+a_9b_3)\cos(\theta_0) \\ & +(a_1+a_6+a_{12})b_3+(a_2+a_7+a_{10})b_5+(a_3+a_4+a_{11})b_2+a_5b_4+a_8b_1+a_9b_6, \end{aligned}$$

where,

$$a_1 = [2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]^2.$$
$$a_2 = -4\bar{y}(2r\bar{y}+q)[2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))].$$

$$a_3 = 2[2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]\bar{y}.$$
$$a_4 = -2\bar{y}(2r\bar{y}+q)[2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))].$$

$$a_5 = a_6 = 8\bar{y}^2(2r\bar{y}+q)^2, \;\; a_7 = a_8 = -4\bar{y}^2(2r\bar{y}+q).$$

$$a_9 = -2[2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]\bar{y}(2r\bar{y}+q).$$
$$a_{10} = 2\bar{y}[2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))].$$

$$a_{11} = -8\bar{y}^2(2r\bar{y}+q), \;\; a_{12} = 4\bar{y}^2.$$

$$b_1 = 2(1+\bar{y}+r\bar{y}^2)^2, \;\; b_2 = 2(1+\bar{y}+r\bar{y}^2)\bar{y}.$$

$$b_3 = -4(1+\bar{y}+r\bar{y}^2)\bar{y}(q-r\bar{y}), \;\; b_4 = (1+\bar{y}+r\bar{y}^2)\bar{y}.$$

$$b_5 = \bar{y}^2, \;\; b_6 = -2\bar{y}^2(q-r\bar{y}).$$

And,

$$\cos(\theta_0) = -\frac{\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}.$$
$$\cos(2\theta_0) = 2\cos^2(\theta_0) - 1 = \frac{\bar{y}^2}{2(1+\bar{y}+r\bar{y}^2)^2} - 1.$$

$$\cos(3\theta_0) = 4\cos^3(\theta_0) - 3\cos(\theta_0) = -\frac{\bar{y}^3}{2(1+\bar{y}+r\bar{y}^2)^3} + \frac{3\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}.$$
$$\cos(4\theta_0) = 2\cos^2(2\theta_0) - 1 = 2(\frac{\bar{y}^2}{2(1+\bar{y}+r\bar{y}^2)^2} - 1)^2 - 1.$$
$$\cos(5\theta_0) = 2\cos(2\theta_0)\cos(3\theta_0) - \cos(\theta_0)$$
$$= 2(\frac{\bar{y}^2}{2(1+\bar{y}+r\bar{y}^2)^2} - 1)(-\frac{\bar{y}^3}{2(1+\bar{y}+r\bar{y}^2)^3} + \frac{3\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}) + \frac{\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}.$$

So, $B_3 = \frac{A_6}{A_5}$. And

$$a(0) = \frac{1}{2}B_1 + B_2 + \frac{1}{2}B_3.$$

Theorem 4.5. *If $a(0) < 0$ (respectively, > 0), then Neimark-Saker bifurcation of system (4.6) at $q = q^*$ is supercritical (respectively, subcritical) and there exists a unique invariant closed curve bifurcates from the positive fixed point $\bar{y}$ which is asymptotically stable (respectively, unstable).*

5. Numerical discussions

In this section some numerical examples which support our results are given.

Example 5.1. *Consider the difference equation (3.2). Fix p, r, and consider q as bifurcation parameter. Take $p = 0.5$, $r = 1.8$, and $0 < q \leq 10$. Equation (3.2) becomes*

$$y_{n+1} = \frac{0.5 + q{y^2}_{n-1}}{1 + y_n + 1.8{y^2}_{n-1}}, \; n = 0, \; 1, \; 2, \; \ldots \tag{5.1}$$

Which is equivalent to

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} y_2(n) \\ \frac{0.5+qy_1(n)^2}{1+y_2(n)+1.8y_1(n)^2} \end{pmatrix}.$$

The positive equilibrium point $\bar{y}$ of (5.1) satisfies

$$1.8\bar{y}^3 + (1-q)\bar{y}^2 + \bar{y} - 0.5 = 0. \tag{5.2}$$

Theorem 4.2 shows that the fixed point undergoes a period-doubling bifurcation at $q^* = \frac{3\times1.8\bar{y}^2+1}{2\bar{y}}$. *So Equation (5.2) at* q^* *becomes*

$$-0.9\bar{y}^3 + \bar{y}^2 + 0.5\bar{y} - 0.5 = 0.$$

Which has two positive roots, so we have two values of q^*.
Thus the first value of q^* *gives the following fixed point of (5.1)*

$$\bar{y} = 0.6495.$$

Substituting the value of $\bar{y}$ *in* q^* *we get*

$$q^* = 2.5235.$$

Now to determine the direction of period-doubling bifurcation we find $c(0)$.

$$c(0) = 0.9539 > 0$$

So this shows that a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $q^* = 2.5235$.
The second value of q^* *gives the following fixed point of (5.1)*

$$\bar{y} = 1.1840.$$

Substituting the value of $\bar{y}$ *in* q^* *we get*

$$q^* = 3.6192.$$

Now to determine the direction of period-doubling bifurcation we find $c(0)$.

$$c(0) = -0.4132$$

So this shows that no stable period-two cycle bifurcates from the fixed point at the bifurcation point $q^* = 3.6192$.
Figure 5.1 shows the stable period-two cycle.

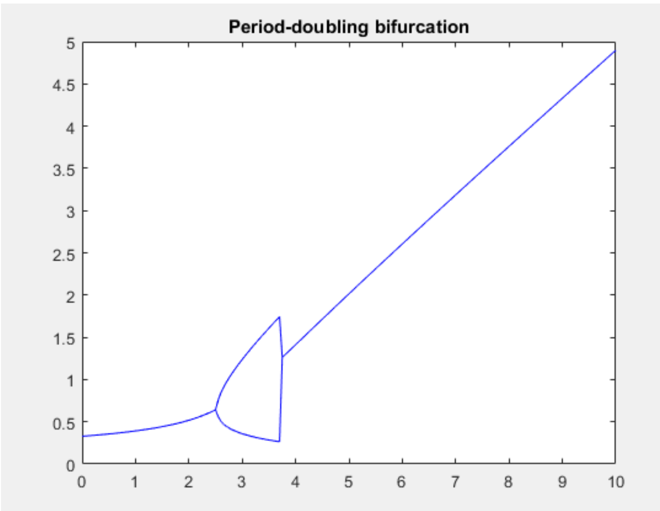

Figure 5.1: Period-doubling bifurcation of $y_{n+1} = \frac{0.5+qy^2_{n-1}}{1+y_n+1.8y^2_{n-1}}$.

Example 5.2. *Consider the difference equation (3.2). Fix* p, r, *and consider* q *as bifurcation parameter. Take* $p = 2$, $r = 9$, *and* $0 < q \leq 10$. *Equation (3.2) becomes*

$$y_{n+1} = \frac{2+qy^2_{n-1}}{1+y_n+9y^2_{n-1}}, \; n = 0,\; 1,\; 2,\; \ldots \tag{5.3}$$

Which is equivalent to

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} y_2(n) \\ \frac{2+qy_1(n)^2}{1+y_2(n)+9y_1(n)^2} \end{pmatrix}.$$

The positive equilibrium point $\bar{y}$ *of (5.3) satisfies*

$$9\bar{y}^3 + (1-q)\bar{y}^2 + \bar{y} - 2 = 0. \tag{5.4}$$

Theorem5.3 shows that the fixed point undergoes a Neimark-Sacker bifurcation at $q^* = \frac{9\bar{y}^2-\bar{y}-1}{2\bar{y}}$. *So Equation (5.4) at* q^* *becomes*

$$4.5\bar{y}^3 + 1.5\bar{y}^2 + 1.5\bar{y} - 2 = 0.$$

Which has one positive roots.

Thus the value of q^* *gives the following fixed point of (5.3)*

$$\bar{y} = 0.5462.$$

Substituting the value of $\bar{y}$ *in* q^* *we get*

$$q^* = 1.0424.$$

Now to determine the direction of period-doubling bifurcation we find $a(0)$.

$$a(0) = 11.7658 > 0$$

So this shows that the Neimark- Sacker bifurcation at $q^* = 1.0424$ *is subcritical.*

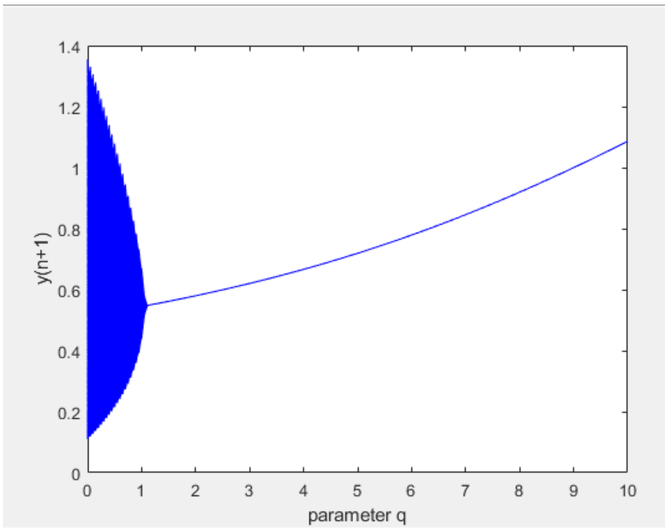

Figure 5.2: Neimark-Sacker bifurcation of $y_{n+1} = \frac{2+qy^2_{n-1}}{1+y_n+9y^2_{n-1}}$.

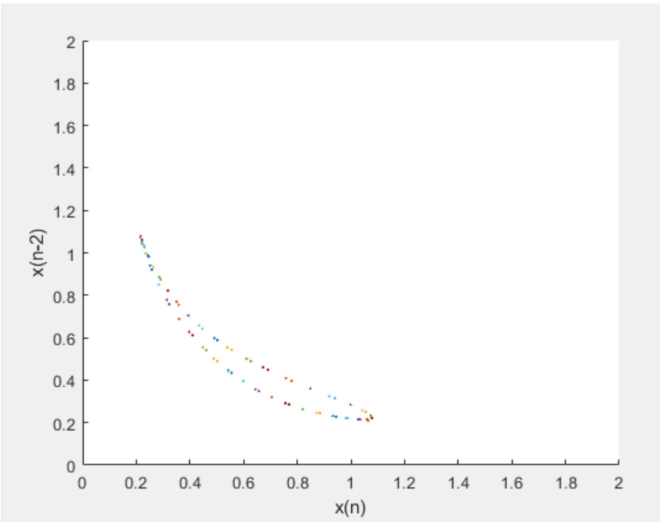

Figure 5.3: Phase portraits of the map $y_{n+1} = \frac{2+qy^2_{n-1}}{1+y_n+9y^2_{n-1}}$ at $q = 0.5$.

Figure 5.2 shows that the positive fixed point $\bar{y}$ is asymptotically stable for $q > q^*$ and change its stability at Neimark-Sacker bifurcation value q^*. Figure 5.3 shows phase portraits associated with Figure 5.2 at $q = 0.5$.

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