SOLUTION OF THE DIFFERENT TYPES OF PARTIAL DIFFERENTIAL EQUATIONS USING DIFFERENTIAL TRANSFORM AND ADOMIAN DECOMPOSITION METHODS

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ABSTRACT

In this study, the Differential Transform Method (DTM) and Adomian's Decomposition Method (ADM) are applied to certain linear and non-linear partial differential equations.

Keywords: linear and nonlinear partial differential equations, differential transform method, Adomian decomposition method.

DİFERANSİYEL DÖNÜŞÜM VE ADOMIAN AYRIŞTIRMA YÖNTEMLERİ İLE FARKLI TİPTE KISMİ DIFERANSİYEL DENKLEMLERİN ÇÖZÜMLERİ

ÖZET

Bu çalışmada, Diferansiyel Dönüşüm Metodu (DTM) ve Adomian Ayrıştırma Metodu (ADM) uygulanarak belirli doğrusal ve doğrusal olmayan kısmi diferansiyel denklemlerin nümerik-analitik çözümleri sunulmuştur.

Anahtar Kelimeler: doğrusal kısmi diferansiyel denklemler, doğrusal olmayan kısmi diferansiyel denklemler, diferansiyel dönüşüm yöntemi, Adomian ayrıştırma yöntemi.

1. INTRODUCTION

The Differential Transform method (DTM) is an approximate method for solving differential equations. The method is applied in [1] and for applications [2,3,4,5,6,7].

The Adomian decomposition method (ADM) is used widely to solve differential equations [8,9,10,11,12,13].

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2. DIFFERENTIAL TRANSFORM METHOD

2.1. One-dimensional differential transform

One-dimensional differential transformation of the function $f(x)$ is defined as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0}, \; k \geq 0. \tag{1}$$

The differential inverse transformation of $F(k)$ is defined as:

$$f(x) = \sum_{k=0}^{\infty} (x - x_0)^k F(k). \tag{2}$$

From (1) and (2) we get

$$f(x) = \sum_{k=0}^{\infty} \frac{(x - x_0)^k}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0}. \tag{3}$$

(3) implies that the concept of the differential transformation is derived from Taylor's series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by iterative procedure, which are described by the transformed equations of the original functions.

From the definitions of (1) and (2), it is easily proven that the transformed functions comply with the basic mathematical operations. In real applications, the function f(x) in (2) is expressed by a finite series and can be written as

$$f(x) = \sum_{k=0}^{N} (x - x_0)^k F(k). \tag{4}$$

(4) implies that $\sum_{k=N+1}^{\infty} (x - x_0)^k F(k)$ is negligibly small and N is decided by the convergence of the solution, where N is series size [14,15].

2.2. Two-dimensional differential transform

Consider a function of two variables $u(x, y)$, and suppose that it can be represented as a product of two single-variable functions, for example, $u(x,y) = f(x)g(y)$. Based on the properties of one-dimensional differential transform, function $u(x,y)$ with two variables can be represented as

$$u(x,y)=\sum_{i=0}^{\infty}F(i)x^{i}\sum_{i=0}^{\infty}G(j)y^{j}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U(i,j)x^{i}y^{j} \tag{5}$$

where $U(i,j)=F(i)G(j)$ is called the spectrum of $u(x,y)$.
If the function $u(x,y)$ with two variables is analytic and differentiable continuously with respect to time t in the domain of interest, then let

$$U(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^{k}\partial y^{h}}u(x,y)\right]_{\substack{x=x_0\\ y=y_0}},\ k\geq 0,\ h\geq 0. \tag{6}$$

where the spectrum $U(k,h)$ is the transformed function.
The differential inverse transform of $U(k,h)$ is defined as follows:

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^{k}y^{h} \tag{7}$$

Combining (6) and (7), it can be obtained that

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^{k}\partial y^{h}}u(x,y)\right]_{\substack{x=x_0\\ y=y_0}}x^{k}y^{h} \tag{8}$$

From these definitions, it can be found that the concept of the two-dimensional differential transform is derived from the two-dimensional Taylor series expansion. The fundamental mathematical operations of DTM are listed in Table 1, [14,15].

Table 1: Operations of differential transform

Original function	Transformed function
$au(x,y)+bv(x,y)$	$aU(k,h)+bV(k,h)$
$\dfrac{\partial u(x,y)}{\partial x}$	$(k+1)U(k+1,h)$
$\dfrac{\partial u(x,y)}{\partial y}$	$(h+1)U(k,h+1)$
$\dfrac{\partial^{r+s}u(x,y)}{\partial x^r \partial y^s}$	$(k+1)(k+2)\ldots(k+r)(h+1)(h+2)\ldots(h+s)U(k+r,h+s)$
$u(x,y)v(x,y)$	$\sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)V(k-r,s)$
$x^m y^n$	$\delta(k-m,h-n)=\begin{cases}1 & k=m, n=h\\ 0 & \text{otherwise}\end{cases},$
e^{ax}	$\dfrac{a^k}{k!}$
$\mathrm{Sin}(ax)$	$\dfrac{a^k}{k!}\mathrm{Sin}\left(\dfrac{k\pi}{2}\right)$
$\mathrm{Cos}(ax)$	$\dfrac{a^k}{k!}\mathrm{Cos}\left(\dfrac{k\pi}{2}\right)$

2.3. The Adomian decomposition method

For the Adomian Decomposition Method see [16,17,18,19]. One considers the following equation:

$$F\,u(x)=g(x) \tag{9a}$$

where F is a general nonlinear ordinary or partial differential equation operator including both linear and nonlinear terms. Thus one write this equation as

$$Lu+Ru+Nu=g(x) \tag{9b}$$

where N is a nonlinear operator, L is the highest-order derivative which is assumed to be invertible, R is the remains of linear differential operator and g is the source term. Thus,

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu \qquad (10)$$

For the initial value problem, L^{-1} can be an integral operator defined from t_0 to t, for the boundary value problems, undefined integration is used and constants of integration are found from the boundary conditions. Here, $Nu = \sum_{n=0}^{\infty} A_n$ where $u = \sum_{n=0}^{\infty} u_n$, and the components of A_n are called Adomian polynomials in [8,9,10]:

$$\begin{aligned}
&A_0 = F(u_0) \\
&A_1 = u_1 F'(u_0) \\
&A_2 = u_2 F'(u_0) + (u_1)^2 F''(u_0)/2 \\
&A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + (u_1)^3 F'''(u_0)/6 \\
&\vdots
\end{aligned} \qquad (11)$$

and so on. The other polynomials can be constructed in a similar way.

Numerical Examples

In this part, some different types of linear and non-linear partial differential equations are solved by using DTM and ADM and numerical results are compared.
Example 1: Consider the following Poisson’s Equation [20]:

$$u_{xx} + u_{yy} = x e^y \qquad (12)$$

with the boundary conditions;

$$u(0,y) = 0 \quad u(2,y) = 2e^y \quad \text{for } 0 \le y \le 1 \qquad (13a)$$

$$u(x,0) = x \quad u(x,1) = e\,x \quad \text{for } 0 \le x \le 2 \qquad (13b)$$

By applying DTM to (12) with given formulas from Table 1 one gets

$$(k+2)(k+1)\,U(k+2,h) + (h+2)(h+1)\,U(k,h+2) = \delta(0,1) \otimes F(0,y) \qquad (14)$$

where $F(0,y) = 1/h!$. From the boundary conditions (13b) and (12), it can be obtained that

$$U(0,h)=\begin{cases}1, & \text{for} \quad h=1\\ 0, & \text{otherwise}\end{cases} \tag{15}$$

and

$$U(k,h)=\begin{cases}\dfrac{1}{h!}, & \text{for} \quad k=1\\ 0, & \text{otherwise}\end{cases} \tag{16}$$

Putting $U(k,h)$ into (8), it can be written that the closed form solution is

$$u_{DTM}(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k y^h=(\sum_{k=0}^{\infty}U(k,h)x^k)(\sum_{h=0}^{\infty}U(k,h)y^h)=x(\sum_{h=0}^{\infty}\frac{1}{h!}y^h) \tag{17}$$

By applying ADM to (12) one gets

$$L_x.u+L_y.u=xe^y \tag{18}$$

where $L_x=\dfrac{\partial^2}{\partial x^2}, L_y=\dfrac{\partial^2}{\partial y^2}, L_y^{-1}(*)=\iint(*)(dy)^2$ and applying both sides of (12)

$$L_y^{-1}.L_y.u=L_y^{-1}(xe^y)-L_y^{-1}.(L_x.u) \tag{19}$$

$$u(x,y)=\iint xe^y(dy)^2-L_y^{-1}.(L_x.u)=xe^y+C_1y+C_2-L_y^{-1}.(L_x.u) \tag{20}$$

where $u(x,y)=\sum_{n=0}^{\infty}u_n$ substituting into (20), it can be obtained that

$$\sum_{n=0}^{\infty}u_n(x,y)=xe^y+C_1y+C_2-L_y^{-1}(L_x\sum_{n=0}^{\infty}u_n) \tag{21}$$

Hence, the solution can be written as follows:

$$u_0=xe^y+C_1y+C_2 \tag{22}$$

and

$$u_{n+1}=-L_y^{-1}(L_x\sum_{n=0}^{\infty}u_n), \quad n\geq 0 \tag{23}$$

In this way, from recurrence (23), it is obtained following terms; for $n=0$,

$u_1 = -L_y^{-1}(L_x u_0) = -L_y^{-1}(0) = 0$, $u_2 = 0, \ldots, u_n = 0$, it can be written the solution of Example 1:

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) = xe^y + C_1 y + C_2 \tag{24}$$

from (13a) and (13b) are found $C_1 = 0$ and $C_2 = 0$, then it is obtained as

$$u_{ADM}(x,y) = x\,e^y \tag{25}$$

The exact solution of the Example1 is,

$$u_{exact}(x,y) = x\,e^y \tag{26}$$

Table 2: Solutions and absolute errors for the Example 1

x	y	$u_{exact}(x,y)$	$u_{DTM}(x,y)$	$u_{ADM}(x,y)$	Error(DTM)	Error(ADM)
0.2	0.1	0.2210341836	0.2210341836	0.2210341836	0	0
0.4	0.2	0.4885611033	0.4885611033	0.4885611033	0	0
0.6	0.3	0.8099152845	0.8099152845	0.8099152845	$2.73114864*10^{-14}$	0
0.8	0.4	1.1934597581	1.1934597581	1.1934597581	$8.69748718*10^{-13}$	0
1.0	0.5	1.6487212707	1.6487212707	1.6487212707	$1.27626798*10^{-11}$	0
1.2	0.6	2.1865425605	2.1865425604	2.1865425605	$1.14781962*10^{-10}$	0
1.4	0.7	2.8192537905	2.8192537897	2.8192537905	$7.36252837*10^{-10}$	0
1.6	0.8	3.5608654856	3.5608654819	3.5608654856	0.0000000037	0
1.8	0.9	4.4272856001	4.4272855848	4.4272856001	0.0000000153	0
2.0	1.0	5.4365636569	5.4365636023	5.4365636569	0.0000000546	0

Example 2: Consider the following Wave Equation:

$$u_{yy} - u_{xx} = 0 \quad \text{for } 0 \le x \le 1 \quad \text{and} \quad y > 0 \tag{27}$$

with the initial and boundary conditions;

$$u(x,0) = \cos(\pi x/2) \qquad u_y(x,0) = 0 \tag{28a.b}$$

$$u_x(0,y) = 0 \tag{29}$$

By applying DTM to (27) with Table 1 together with (28) and (29) one gets

$$(h+2)(h+1)\,U(k,h+2) = (k+2)(k+1)\,U(k+2,h) \tag{30}$$

$$(h+1)\,U(k,h+1) = 0 \tag{31}$$

$$(k+1)\,U(k+1,h)=0 \tag{32}$$

from the condition (28a) $U(x,0)=\sum_{i=0}^{\infty} U(i,0)x^{i} = \cos(\frac{\pi x}{2})$. The corresponding spectra can be obtained as follows:

$$U(i,0) = \begin{cases} \left(\frac{\pi}{2}\right)^{i}\left(-\frac{1}{i!}\right), & \text{for } i = 2,6,10,\ldots,m \\ \left(\frac{\pi}{2}\right)^{i}\left(\frac{1}{i!}\right), & \text{for } i = 0,4,8,\ldots,m \\ 0, & \text{for } i = 1,3,5,\ldots,m \end{cases} \tag{33}$$

and from (28b), it can be obtained that $u_y(x,0)=\sum_{i=0}^{\infty} U(i,1)x^{i} = 0$ hence, we get;

$$U(i,0)=0 \qquad i=0,1,2,\ldots,m \tag{34}$$

from (29), it can be obtained

$$u_x(0,y)=\sum_{i=0}^{\infty} U(1,i)y^{i} = 0 \quad \text{where} \quad U(1,i)=0,\ i=0(1)n \tag{35}$$

Substituting (33) for k = h, in (30) is obtained as in bellows:

$$u(k,0)=u(0,h)=\begin{cases} \left(\frac{\pi}{2}\right)^{h}\left(-\frac{1}{h!}\right), & \text{for } k=h=2,6,10,\ldots,m \\ \left(\frac{\pi}{2}\right)^{h}\left(\frac{1}{h!}\right), & \text{for } k=h=0,4,8,\ldots,m \\ 0, & \text{for } k=h=1,3,5,\ldots,m \end{cases} \tag{36}$$

then,

$$U(k,h)=\left(\frac{\pi}{2}\right)^{2h}\left(\frac{1}{h!}\right)^{2}, \quad k=h=0,2,4,6,\ldots,m \tag{37}$$

is concluded. Therefore, the closed form of solution can be defined as;

$$u_{DTM}(x,y)=\left(1+\left(\frac{\pi}{2}\right)^2\left(-\frac{1}{2!}\right)x^2+\left(\frac{\pi}{2}\right)^4\left(-\frac{1}{4!}\right)x^4+\left(\frac{\pi}{2}\right)^6\left(-\frac{1}{6!}\right)x^6+\left(\frac{\pi}{2}\right)^8\left(-\frac{1}{8!}\right)x^8+\ldots\right)$$
$$.\left(1+\left(\frac{\pi}{2}\right)^2\left(-\frac{1}{2!}\right)y^2+\left(\frac{\pi}{2}\right)^4\left(-\frac{1}{4!}\right)y^4+\left(\frac{\pi}{2}\right)^6\left(-\frac{1}{6!}\right)y^6+\left(\frac{\pi}{2}\right)^8\left(-\frac{1}{8!}\right)y^8+\ldots\right) \quad (38)$$
$$=\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s)!}\left(\frac{\pi x}{2}\right)^{2s}\right)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s)!}\left(\frac{\pi y}{2}\right)^{2s}\right)=\cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right)$$

where the exact solution of given problem $u_{exact}(x,y)$:

$$u_{exact}(x,y)=\frac{1}{2}\left[\cos\frac{\pi}{2}(x+y)+\cos\frac{\pi}{2}(x-y)\right]=\cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right) \quad (39)$$

Now, by using ADM to

$$u_{yy}=u_{xx} \quad (40)$$

Operator form of this equation is defined as in bellows:

$$L_y.u=L_x.u \quad (41)$$

where $L_x=\dfrac{\partial^2}{\partial x^2}, L_y=\dfrac{\partial^2}{\partial y^2}, L_y^{-1}(*)=\iint(*)(dy)^2$ and applying both sides of (40)

$$u(x,y)-u(x,0)-(y-0)u(x,0)=L_y^{-1}.(L_x.u) \quad (42)$$

where $u(x,y)=\sum_{n=0}^{\infty}u_n(x,y)$ substituting (20), it can be written;

$$\sum_{n=0}^{\infty}u_n(x,y)=u(x,0)+y.u_y(x,0)+L_y^{-1}(L_x\sum_{n=0}^{\infty}u_n) \quad (43)$$

Hence, the solution can be arranged as follows:

$$u_0=u(x,0)+y.u_y(x,0) \quad \text{and} \quad u_{n+1}=L_y^{-1}(L_x\sum_{n=0}^{\infty}u_n), \quad n\geq 0 \quad (44)$$

In this way, from recurrence relation (44), following terms can be written as;

$$u_0 = \cos\left(\frac{\pi x}{2}\right)$$

$$u_1 = \int_0^y\int_0^y \left[\frac{\partial^2}{\partial x^2}(u_0)\right](dy)^2 = -\left(\frac{\pi}{2}\right)^2 \frac{y^2}{2!}\cos\left(\frac{\pi x}{2}\right)$$

$$u_2 = \int_0^y\int_0^y \left[\frac{\partial^2}{\partial x^2}(u_1)\right](dy)^2 = \left(\frac{\pi}{2}\right)^4 \frac{y^4}{4!}\cos\left(\frac{\pi x}{2}\right)$$

$$u_3 = \int_0^y\int_0^y \left[\frac{\partial^2}{\partial x^2}(u_2)\right](dy)^2 = -\left(\frac{\pi}{2}\right)^6 \frac{y^6}{6!}\cos\left(\frac{\pi x}{2}\right)$$

$$u_4 = \int_0^y\int_0^y \left[\frac{\partial^2}{\partial x^2}(u_3)\right](dy)^2 = \left(\frac{\pi}{2}\right)^8 \frac{y^8}{8!}\cos\left(\frac{\pi x}{2}\right)$$

$$u_5 = \int_0^y\int_0^y \left[\frac{\partial^2}{\partial x^2}(u_4)\right](dy)^2 = -\left(\frac{\pi}{2}\right)^{10} \frac{y^{10}}{10!}\cos\left(\frac{\pi x}{2}\right)$$

$$u_6 = \int_0^y\int_0^y \left[\frac{\partial^2}{\partial x^2}(u_5)\right](dy)^2 = \left(\frac{\pi}{2}\right)^{12} \frac{y^{12}}{12!}\cos\left(\frac{\pi x}{2}\right)$$

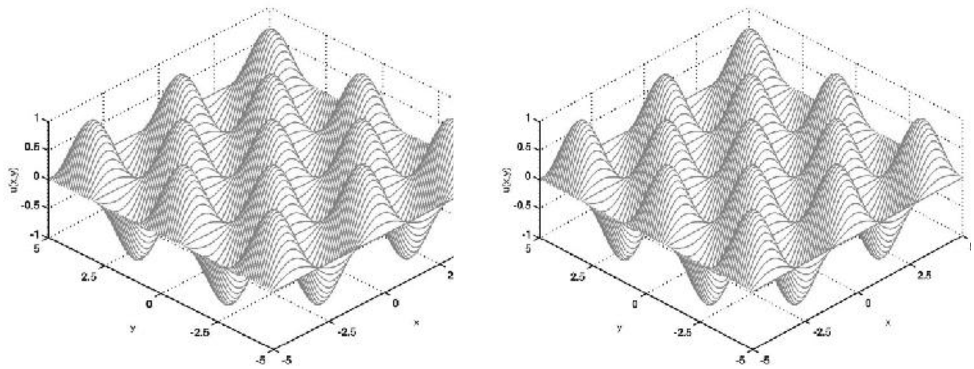
$$\vdots \qquad (45)$$

Thus, the approximate solution of the given equation is written as:

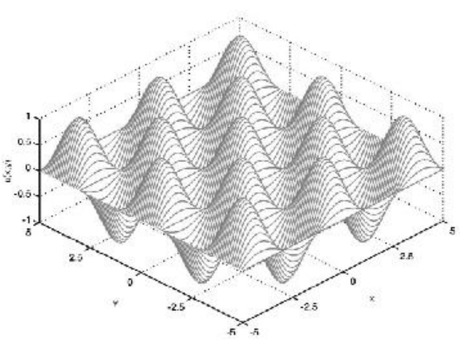
$$u_{ADM}(x,y) = \cos\left(\frac{\pi x}{2}\right)\left(1 - \left(\frac{\pi}{2}\right)^2 \frac{y^2}{2!} + \left(\frac{\pi}{2}\right)^4 \frac{y^4}{4!} - \left(\frac{\pi}{2}\right)^6 \frac{y^6}{6!} + \ldots\right) = \cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right) \quad (46)$$

Table 3: Solutions and absolute errors for the Example 2

x	y	$u_{exact}(x,y)$	$u_{DTM}(x,y)$	$u_{ADM}(x,y)$	Error(DTM)	Error(ADM)
0.0	0.0	1	1	1	0	0
0.1	0.1	0.9755282581	0.9755282581	0.9755282581	0	0
0.2	0.2	0.9045084972	0.9045084972	0.9045084972	$3.663736*10^{-15}$	0
0.3	0.3	0.7938926261	0.7938926261	0.7938926261	$4.455325*10^{-13}$	$1.887379*10^{-15}$
0.4	0.4	0.6545084972	0.6545084972	0.6545084972	$1.276057*10^{-11}$	$2.2271074*10^{-13}$
0.5	0.5	0.5	0.4999999998	0.4999999999	$1.621012*10^{-10}$	$6.3802297*10^{-12}$
0.6	0.6	0.3454915028	0.3454915016	0.3454915022	0.0000000012	$8.105056*10^{-11}$
0.7	0.7	0.2061073739	0.2061073680	0.2061073709	0.0000000059	$5.998164*10^{-10}$
0.8	0.8	0.0954915028	0.0954914830	0.0954914929	0.0000000198	0.0000000029
0.9	0.9	0.0244717419	0.0244717007	0.0244717213	0.0000000412	0.0000000099
1.0	1.0	0	$2.160074*10^{-13}$	0	$2.160074*10^{-13}$	0.0000000206



(a) $u_{exact}(x,y)$ solution of Example 2. (b) $u_{DTM}(x,y)$ solution of Example 2.



(c) $u_{ADM}(x,y)$ solution of Example 2.

Figure 1: $u_{exact}(x,y)$, $u_{DTM}(x,y)$, and $u_{ADM}(x,y)$ solutions of Example 2.

Example 3: Consider the following Heat Conduction Problem:

$$u_{xx} = 4u_t \;, 0 \le x \le 2 \quad t > 0 \tag{47}$$

with the boundary conditions;

$$u(0,t) = 0 \quad u(2,t) = 0 \quad \text{for } t > 0 \tag{48a}$$

$$u(x,0) = 2\sin(\pi x/2) \quad \text{for } 0 \le x \le 2 \tag{48b}$$

By applying DTM to (47) with given formulas from Table 1. One gets

$$(k+2)(k+1)\,U(k+2,h) = 4\,(h+1)\,U(k,h+1) \tag{49}$$

From the boundary condition (48b), the corresponding spectra can be obtained as follows:

$$U(k,0) = \begin{cases} 0, & \text{for } k = 0,2,4,... \\ 2\left(\frac{\pi}{2}\right)^k \frac{1}{k!}, & \text{for } k = 1,5,9,... \\ -2\left(\frac{\pi}{2}\right)^k \frac{1}{k!}, & \text{for } k = 3,7,11,... \end{cases} \tag{50}$$

In the (49), for $h = 0$ and $k = 0,1,2,\ldots$, we get

$$U(k,1) = \begin{cases} 0, & \text{for } k = 0,2,4,... \\ -\left(\frac{\pi}{2}\right)^{k+2} \frac{k!}{2}, & \text{for } k = 1,5,9,... \\ \left(\frac{\pi}{2}\right)^k \frac{k!}{2}, & \text{for } k = 3,7,11,... \end{cases} \tag{51}$$

In the (49), for $h = 1$ and $k = 0,1,2,\ldots$, we have

$$U(k,2) = \begin{cases} 0, & \text{for } k = 0,2,4,... \\ \frac{(k+2)(k+1)}{4.2}\left(\frac{\pi}{2}\right)^{k+4} \frac{(k+2)!}{2}, & \text{for } k = 1,5,9,... \\ -\frac{(k+2)(k+1)}{4.2}\left(\frac{\pi}{2}\right)^{k+4} \frac{(k+2)!}{2}, & \text{for } k = 3,7,11,... \end{cases} \tag{52}$$

In (49), for $h = 2$ and $k = 0,1,2,\ldots$, we have

$$U(k,3)=\begin{cases} 0, & \text{for } k=0,2,4,\ldots \\ \dfrac{(k+4)(k+3)(k+2)(k+1)}{4.3.4.2}\left(\dfrac{\pi}{2}\right)^{k+6}\dfrac{(k+4)!}{2}, & \text{for } k=1,5,9,\ldots \\ -\dfrac{(k+4)(k+3)(k+2)(k+1)}{4.3.4.2}\left(\dfrac{\pi}{2}\right)^{k+6}\dfrac{(k+4)!}{2}, & \text{for } k=3,7,11,\ldots \end{cases} \tag{53}$$

and so on. Putting $U(k,h)$ into (8), it can be obtained that the closed form solution is

$$\begin{aligned} u_{DTM}(x,y) &= x\left\{2\left(\frac{\pi}{2}\right)\frac{1}{1!}-\frac{1}{2}\left(\frac{\pi}{2}\right)^3 t+\frac{3.3!}{4.2}\left(\frac{\pi}{2}\right)^5 t^2-5^2.3\left(\frac{\pi}{2}\right)^7 t^3+\ldots\right\} \\ &+x^3\left\{-2\left(\frac{\pi}{2}\right)^3\frac{1}{3!}+\frac{3!}{2}\left(\frac{\pi}{2}\right)^5 t-\frac{5.2.5!}{4.2}\left(\frac{\pi}{2}\right)^7 t^2+\frac{5.7.7!}{4.2}\left(\frac{\pi}{2}\right)^9 t^3+\ldots\right\} \\ &+x^5\left\{2\left(\frac{\pi}{2}\right)^5\frac{1}{5!}-\frac{5!}{2}\left(\frac{\pi}{2}\right)^7 t-\frac{7.3.7!}{4.2}\left(\frac{\pi}{2}\right)^9 t^2-\frac{3^2.7.9!}{4}\left(\frac{\pi}{2}\right)^{11} t^3+\ldots\right\}+\ldots \end{aligned} \tag{54}$$

Now, by applying ADM to

$$u_{xx}=4u_t \tag{55}$$

Operator form of this equation is defined as in bellows:

$$L_{xx}.u=L_t.u \tag{56}$$

where $L_{xx}=\dfrac{\partial^2}{\partial x^2}$, $L_t=\dfrac{\partial}{\partial t}$, $L_t^{-1}(*)=\int(*)dt$ and applying both sides of (55):

$$u(x,t)-u(x,0)-t.u_t(x,0)=\frac{1}{4}L_t^{-1}.(L_{xx}.u) \tag{57}$$

where $u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)$ substituting (57), it can be written;

$$\sum_{n=0}^{\infty}u_n(x,t)=u(x,0)+t.u_t(x,0)+L_t^{-1}(L_{xx}\sum_{n=0}^{\infty}u_n) \tag{58}$$

Hence, solution can be arranged as follows:

$$u_0 = u(x,0) + t.u_y(x,0) \quad \text{and} \quad u_{n+1} = \frac{1}{4} L_t^{-1}(L_{xx} \sum_{n=0}^{\infty} u_n), \quad n \geq 0 \tag{59}$$

In this way, from recurrence relation (59), following terms can be written as;

$$\begin{aligned}
u_0 &= 2\sin\left(\frac{\pi x}{2}\right) \\
u_1 &= \frac{1}{4}\int_0^t \frac{\partial^2}{\partial x^2}\left\{2\sin\left(\frac{\pi x}{2}\right)\right\}dt = -\frac{\pi^2}{8}\sin\left(\frac{\pi x}{2}\right)t \\
u_2 &= \frac{1}{4}\int_0^t \frac{\partial^2}{\partial x^2}\left\{-\frac{\pi^2}{8}\sin\left(\frac{\pi x}{2}\right)t\right\}dt = \frac{\pi^4}{256}\sin\left(\frac{\pi x}{2}\right)t^2 \\
u_3 &= \frac{1}{4}\int_0^t \frac{\partial^2}{\partial x^2}\left\{\frac{\pi^4}{256}\sin\left(\frac{\pi x}{2}\right)t^2\right\}dt = -\frac{\pi^6}{12288}\sin\left(\frac{\pi x}{2}\right)t^3 \\
&\vdots
\end{aligned} \tag{60}$$

Thus, approximate solution of the given equation is written as:

$$u_{ADM}(x,t) = 2\sin\left(\frac{\pi x}{2}\right)\left(1 - \frac{\pi^2}{16}t + \frac{\pi^4}{512}t^2 - \frac{\pi^6}{24576}t^3 + \ldots\right) = 2\sin\left(\frac{\pi x}{2}\right)\exp\left(-\frac{\pi^2 t}{16}\right) \tag{61}$$

where the exact solution of given problem $u_{exact}(x,t)$ as same as in equation (61).

Example 4: Consider the following

$$u_y + u.u_x = 0 \tag{62}$$

with the initial conditions;

$$u(x,0) = x \tag{63a}$$

$$u(0,y) = 0 \tag{63b}$$

where the exact solution is $u_{exact}(x,y) = x/(1+y)$ (Jang vd., 2001). Taking the differential transform of (62), it can be obtained as;

$$(h+1)\,U(k,h+1) + U(k,h) \otimes [(k+1)\,U(k+1),h)] = 0 \tag{64}$$

from the initial condition (63a)

$$U(k,0) = \begin{cases} 1, & \text{for } k = 1 \\ 0, & \text{otherwise} \end{cases} \tag{65}$$

Substituting (65), the resulting spectra for all i and k can be obtained as

$$U(k,k) = \begin{cases} (-1)^h, & \text{for } k = 1 \text{ and } h = 0,1,... \\ 0, & \text{otherwise} \end{cases} \tag{66}$$

By (66), the solution of $u(x,y)$, can be concluded.

$$u_{DTM}(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k y^h = x\left(1 - y + y^2 - y^3 + ...\right) = x\frac{1}{1+y} \tag{67}$$

On the other hand, according to ADM, from (9) and (10),

$$L_y.u + u.L_x.u = 0 \tag{68}$$

can be written where $L_x = \frac{\partial}{\partial x}, L_y = \frac{\partial}{\partial y}, L_y^{-1}(*) = \int(*)dy$ and applying both sides of (62)

$$L_y^{-1}.(L_y.u) + L_y^{-1}.[u(L_x.u)] = 0 \tag{69}$$

$$u(x,y) = x - L_y^{-1}[u.(L_x.u)] \tag{70}$$

Now, by considering (70) we have

$$u_0 = x, \quad u_i = -y, \quad \text{for} \quad i = 1,2,3,\ldots \tag{71}$$

and from (11), Adomian polynomials A_n can be obtained as follows:

$$A_0 = x, \quad A_1 = -x\,y, \quad A_2 = x\,y^2, \quad A_3 = -x\,y^3 \tag{72}$$

etc. The desired result

$$u_{ADM}(x,y) = \sum_{n=0}^{\infty} A_n = x - xy + xy^2 - xy^3 + xy^4 - ... = x\frac{1}{1+y}. \tag{73}$$

Table 4: Solutions and absolute errors for the Example 4.

x	y	$u_{exact}(x,y)$	$u_{DTM}(x,y)$	$u_{ADM}(x,y)$	Error for DTM and ADM
0.0	0.0	0	0	0	0
0.1	0.1	0.0909090909	0.0909090909	0.0909090909	$9.091200015*10^{-13}$
0.2	0.2	0.1666666667	0.1666666701	0.1666666701	0.0000000034
0.3	0.3	0.2307692308	0.2307696396	0.2307696396	0.0000004088
0.4	0.4	0.2857142857	0.2857262694	0.2857262694	0.0000119837
0.5	0.5	0.3333333333	0.3334960938	0.3334960938	0.0001627604
0.6	0.6	0.375	0.3763604890	0.3763604890	0.0013604890
0.7	0.7	0.4117647059	0.4199066395	0.4199066395	0.0081419336
0.8	0.8	0.4444444444	0.4826219315	0.4826219315	0.0381774871
0.9	0.9	0.4736842105	0.622331335	0.622331335	0.1486471245
1.0	1.0	0.5	1	1	0.5

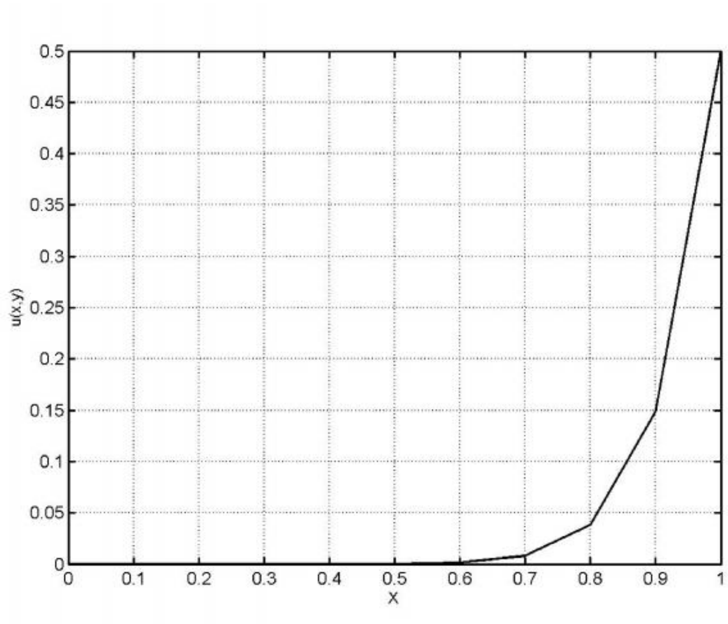


Figure 2: Absolute error for DTM and ADM for Example 4.

Example 5: Consider the following non-linear partial differential equation:

$$u_y (u_x)^2 - 1 = 0 \tag{74}$$

with the initial condition;

$$u(x,0) = x \tag{75}$$

and the exact solution is $u_{exact}(x,y) = x + y$.
Taking the differential transform of (74), we have;

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(k+1-r)^2\left[U(k+1-r,s)\right]^2(h+1-s)U(r,h-s+1)-\delta(k,h)=0 \tag{76}$$

where

$$\delta(k,h)=\delta(h)\delta(k-1)=\begin{cases}1, & \text{for } k=1, h=0\\ 0, & \text{for } k\neq 1, h\neq 0\end{cases}$$

from the initial conditions (75)

$$U(i,0)=0 \quad \text{for} \quad i=0,2,3,\ldots,m \tag{77}$$

$$U(1,0)=1 \quad \text{for} \quad i=1 \tag{78}$$

$$U(1,0)=U(0,1) \quad \text{for} \quad h=k=0 \tag{79}$$

Thus, investigated solution;

$$\begin{aligned} u_{DTM}(x,y) &= \left[U(0,h)+U(1,h)x+U(2,h)x^2+\ldots\right]\left[U(k,0)+U(k,1)y+U(k,2)y^2+\ldots\right] \\ &= U(1,0)x+U(0,1)y = x+y \end{aligned} \tag{80}$$

is found, because all the other terms are zero. By using ADM in (74) is in operator form, where $L_y=\frac{\partial}{\partial y}, L_x=\frac{\partial}{\partial x}, L_y^{-1}(*)=\int_0^y(*)dy$

$$L_y^{-1}(L_y.u)\left[L_x.u\right]^2=L_y^{-1}(1) \tag{81}$$

$$\left[u(x,y)-u(x,0)\right].\left[L_x u\right]^2=\int_0^y 1.dy$$

Substituting $u(x,y)=\sum_{n=0}^{\infty}u_n(x,y)$ into (81), it is obtained that

$$\left(\sum_{n=0}^{\infty}u_n(x,y)-x\right)\left[L_x\sum_{n=0}^{\infty}u_n\right]^2=y \tag{82}$$

Where

$$u_0 = x \quad \text{and} \quad u_{n+1} = \left[L_x \sum_{n=0}^{\infty} u_n \right]^2, n \geq 0 \tag{83}$$

Thus, from recurrence relation, the following equation can be obtained:

$$u_{n+1} = \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{otherwise} \end{cases} \tag{84}$$

Substituting (84) into (82);

$$\sum_{n=0}^{\infty} u_n(x, y) - x = y \tag{85}$$

is concluded. Hence, approximate solution of (74) can be written as

$$u(x, y) = x + y \tag{86}$$

As seen in (80) and (86), the solution functions are the same as the exact solution.

3. CONCLUSION

Partial differential equations in mathematics and physics, as well as in many areas of engineering are widely encountered. Many times these equations are not solved analytically. Therefore, it is important to know at least their approximating solutions. In this respect we use DTM and ADM used by many for solving of linear and especially nonlinear partial differential equations. Approximating solutions are compared with analytic solutions. Calculated absolute errors are arranged in Table 2, 3, and 4. Tables and figures show that these approximations are so compatible with real ones.

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