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EVALUATING FIRST PASSAGE TIMES IN MARKOV CHAINS FROM THE PERSPECTIVE OF ASYMPTOTIC AND EMPIRICAL INFORMATION

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ABSTRACT

First passage times underlie many stochastic processes in which the event, such as a chemical reaction, the firing of a neuron, or the triggering of a stock option, relies on a variable reaching a specified value for the first time. In this study, a transition matrix was estimated by taking into account the closing values of Istanbul Stock Exchange (ISE -100) index from 20.01.2009 to 18.01.2013 for discrete Markov model. Empirical information regarding the first passage time was obtained by writing a computer program. Later, the first passage time which calculated by using WinQSB software was accepted as asymptotic information. Under the assumption that the frequency distribution of the first passage time fits with the geometric distribution, the fittings of the first passage time obtained from empirical information and the first passage time obtained from asymptotic information to the geometric distribution was compared with a chi-square analysis. It is found that, in some cases of the first passage time of asymptotic information gives better results regarding the fitting with the geometric distribution. Graphics of continuous distribution which comply with frequency of first passage time were also provided by using Easy-fit software. From these graphs first passage time distribution was seen to be a positively skewed or reverse j-shaped distribution.

Keywords: Empirical information, Asymptotic information, First passage time, Markov chain, Markov chain simulation

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MARKOV ZİNCİRİNDE İLK GEÇİŞ ZAMANLARININ ASİMTOTİK VE AMPİRİK BİLGİ AÇISINDAN DEĞERLENDİRİLMESİ

ÖZET

İlk geçiş zamanları bir değişkenin ilk kez belli bir değere ulaşmasına dayanan olaylarda örneğin kimyasal bir reaksiyon, sinir sistemindeki nöronun uyarılması, stok seçiminin başlaması gibi pek çok stokastik süreci vurgular. Bu çalışmamızda kesikli Markov modeli için 20.01.2009-18.01.2013 tarihleri arasındaki IMKB-100 endeksinin kapanış değerleri dikkate alınarak geçiş matrisi tahmin edildi. Yazılan bir bilgisayar programı aracılığıyla ilk geçiş zamanına ilişkin ampirik bilgi elde edildi. Win-QSB yazılımı kullanılarak hesaplanan ilk geçiş zamanı ise Asimtotik bilgi olarak kabul edildi. İlk geçiş zamanının frekans dağılımının Geometrik dağılıma uyduğu varsayımı altında Ampirik bilgiden elde edilen ilk geçiş zamanı ile Asimtotik bilgiden elde edilen

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ilk geçiş zamanın Geometrik dağılıma uyumu hesaplanan Ki-Kare değeriyle karşılaştırılmıştır. İlk geçiş zamanının bazı durumları için asimptotik bilginin geometrik dağılıma uyumu bakımından daha iyi sonuç verdiği bulunmuştur. Ayrıca Easy-Fit yazılımı ile İlk geçiş zamanının frekansına uyan sürekli dağılımların grafikleri verilmiştir. Bu grafiklerden ilk geçiş zamanının dağılımının sağa çarpık ve ters J şeklinde olduğu görülmüştür.

Anahtar Kelimeler: Ampirik bilgi, Asimtotik bilgi, İlk geçiş zamanı, Markov zinciri, Markov zinciri simülasyonu

1. INTRODUCTION

The first-passage concept used in many areas from controlled reactions in physical and chemical processes to chromatographic, stochastic processes play an important role. Therefore, you need to know the first passage characteristics to understand the movements of real systems. Once this connection is established, it is quite simple to obtain the dynamical properties of the system in terms of well-known first passage. The problems associated with these systems can be found in [1,2,3,4,5].

First passage times underlie many stochastic processes in which the event, such as a chemical reaction, the firing of a neuron, or the triggering of a stock option, relies on a variable reaching a specified value for the first time. The behavior of normal or abnormal can be revealed by looking at the mean first passage times of processes. Although two processes are very different microscopically, their long-time properties-including first passage characteristics are essentially the same. Books are devoted merely to first passage process [6,7,8,9] or in books on stochastic process that discuss first passage processes as a subtopic [10,11,12,13,14].

Passage times and their applications have been investigated since early days of probability theory. The best known example is the first entrance time to a set, which embraces waiting times, absorption problems, extinction phonomena, busy periods and other applications. Probability of the first passage to obtain in stochastic models such as diffusion, random walk, the mathematical tools as Green's function, Fourier and Laplace transform are used. Thus all first passage characteristics can be expressed in terms of the first passage probability. In this sense, [9] introduces passage times, a concept, which includes both the first passage/entrance time and last exit time from a set and stresses on the role of sample functions behaviour in the determination of passage probabilities. (Ross and Schechner, 1985) considered a discrete time Markov process for estimating first passage time and used the estimators based on observed hazard and then extended their results to continuous time by uniformizing [15]. Thus, they have used gamma function to estimate the distribution of the first

passage time. Although passage times are in fact examples of stopping times, they enjoy important position in theoretical and practical applications. In analyzing and using Markov chain, first passage times are fundamental to understanding the long-run behavior of a Markov chain [16].

2. METHODOLOGY

In this section, the techniques that shall be used for the analysis will be given. Accordingly, we will summarize the methodology used for calculating first passage times for Markov chain.

2.1. Markov Chain

Modern probability theory studies random (stochastic) processes for which the knowledge of previous outcomes influences predictions for future experiments. In this principle, it is thought when we observe a sequence of chance experiments, all of the past outcomes could influence our predictions for the next experiment [17]. In 1907, A. A. Markov began the study of an important new type of chance process. In this process, the outcome of a given experiment can affect the outcome of the next experiment [17]. In other words, Markov chains are the stochastic processes whose futures are conditionally independent of their pasts provided that their present values are known [18].

Let $\{X_n : n = 0,1,2,...\}$ be a stochastic process that has a finite or countable infinite state space S. When $X_n = i$ we say that the process is in state i at time *n*. The probability that the process is in state *j* in the next time provided that its present state is *i*, is denoted by P_{ij}.

Let $i_0, i_1, ..., i_{n-1}, i, j$ be the states of the process and $n \geq 0$. The stochastic process $\{X_n : n = 0,1,2,...\}$ is called a Markov chain provided that

$$\begin{aligned} P\{X_{n+1} = j \mid \mathrm{X}_0 = i_0, X_1 = i_1, ..., X_{n-1} = i_{n-1}, X_n = i\} &= P\{X_{n+1} = j \mid X_n = i\} \\ &= P_{ij,n} \end{aligned} \tag{2.1}$$

for all *j*'s and *i*'s, and $n \geq 0$. By this definition, a Markov chain is a sequence of random variables such that for any *n*, the "next" state of the process X_{n+1} is independent of the "past" states $X_0, X_1, ..., X_{n-1}$ given that the present X_n is known; that is, the strong Markov property is to hold at randomly chosen times [18]. The probability P_{ij} is called (*one step*) *transition probability* from state *i* to state *j*. When the transition probabilities satisfy the condition, $P_{ij,n} = P_{ij}$ for all $n \geq 0$, i.e., they are independent of the time parameter *n*, then the Markov chain $\{X_n : n = 0,1,2,...\}$ is said to be *time-homogeneous*, or *stationary* [17].

$$P\{X_{n+1} \mid X_n = i\} = P(i, j) = P_{ij,n} = \mathrm{P}_{ij} \text{ ; i,j} \in \mathrm{S}$$

For the Markov chains, the transition probabilities are arranged in a matrix form and the resulting matrix is called the *transition matrix* of the chain. The elements of a transition matrix hold the following conditions:

a) for any two states $\mathrm{i,j} \in \mathrm{S}$, $P_{ij} \geq 0$; and
b) for all $i \in S$, $\sum_j P_{ij} = 1$.

As it can be easily seen from the next theorem and following corollary, the joint distribution $X_0, X_1, ..., X_m$ can be completely specified for every *m* once the initial distribution and the transition matrix P are known [17].

Theorem 2.1.1. Let $X = \{X_n : n \in N\}$ be a Markov chain. For any $m,n \in N$; $m \geq 1$ and $i_0,\ i_1, i_2, ..., i_m \in S$,

$$P\{X_{n+1} = i_1, X_{n+2} = i_2, ...X_{n+m} = i_m \ / \ X_n = i_0\} = P_{i_0 i_1} P_{i_1 i_2} ... P_{i_{m-1} i_m} \qquad (2.2)$$

Corollary 2.1.1 For the Markov chain, let the initial probability distribution $®_0$ be given on the state space S; i.e., let $P\{X_0 = i\} = ®_0(i)$ be for all $i \in S$ given. Then for any $m \in N$ and $i_0, i_1, i_2, ..., i_m \in S$

we have

$$P\{X_0 = i_0, X_1 = i_1, \ldots, X_m = i_m\} = ®_0(i_0)\, p_{i_0 i_1} p_{i_1 i_2} \ldots p_{i_{m-1} i_m} \qquad (2.3)$$

In some cases, it is needed to calculate the probabilities for the transitions between distant times for Markov chain. Thus, the following definition is given.

Definition 2.1.1. For any $m \in N$ *n-step transition probability* from state i to state *j* is given by

$$P\{X_{n+m} = j \mid X_m = i\} = P_{ij}^{(n)};\ \text{i,j} \in \text{S},\ n \in N\,. \qquad (2.4)$$

It is often desirable to make also probability statements about the number of transitions followed by the process in going from state *i* to state *j for the first time.* This length of time is called the *first passage time* in going from state *i* to state *j*

To illustrate these definitions, reconsider the inventory example where X_t is the number of cameras on hand at the end of week *t*, where we start with X_0. Suppose that it turns out that

$$X_0 = 3,\ X_1 = 2,\ X_2 = 1,\ X_3 = 0,\ X_4 = 3,\ X_5 = 1$$

In this case, the first passage time in going from state 3 to state 1 is 2 weeks, the first passage time in going from state 3 to state 0 is 3 weeks, and the recurrence time for state 3 is 4 weeks.

In general, the first passage times are random variables. The probability distributions associated with them depend upon the transition probabilities of the process. In particular, let $f_{ij}^{(n)}$ denote the probability that the first passage time from state *i* to *j* is equal to *n.* For $n > 1$, this first passage time is *n* if the first transition is from state *i* to some state *k* ($k \neq j$) and then the first passage time from state *k* to state j is *n* - 1. Therefore, these probabilities satisfy the following recursive relationships (Hillier, 2001).

$$f_{ij}^{(1)} = p_{ij}^{(1)} = p_{ij}\,, \qquad (2.5)$$

$$f_{ij}^{(2)} = \sum_{k \neq j} p_{ik} f_{kj}^{(1)}\,, \qquad (2.6)$$

$$f_{ij}^{(n)} = \sum_{k \neq j} p_{ik} f_{kj}^{(n-1)} \qquad (2.7)$$

Among the Markov chain characteristics, the first passage times play an important role. For any two states, the first passage time probability in *n* steps is defined as follows and this probability is related to the ever reaching probability.

Definition 2.1.2. For any two states *i* and *j*, the *first passage time probability* from *i* to *j* in n steps, $f_{ij}^{(n)}$ is defined as

$$f_{ij}^{(n)} = \begin{cases} p_{ij} & ; n = 1 \\ \sum_{k \neq j} p_{ik} \, f_{kj}^{(n-1)} & ; n = 2,3,... \end{cases} \qquad (2.8)$$

Definition 2.1.3. The value $f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$ is called *ever reaching probability*, or *reaching probability in every step* from state *i* to state *j* [18].

Unfortunately, this sum may be strictly less than 1, which implies that a process initially in state *i* may never reach state *j*. When the sum does equal 1, $f_{ij}^{(n)}$, (for n = 1, 2, . . .) can be considered as a probability distribution for the random variable, the first passage time.

The following theorem reflects how to calculate the steady state probabilities for the process.

Theorem 2.1.2. If $\{X_n : n = 0,1,2,...\}$ is an irreducible aperiodic finite state Markov chain, the system of equations

$$®'.P = ®' \qquad (2.9)$$
$$®'.\underline{1} = \underline{1} \qquad (2.10)$$

has a unique positive solution. This solution is called the *limit distribution* of Markov chain.

Definition 2.1.4. An important indicator of the first passage times is the *mean first passage time* and for an irreducible recurrent Markov chain, this quantity is calculated as [18].

$$«_{ij} = 1 + \sum_{k \neq j} p_{ik} «_{kj} \text{ or } «_{ii} = \frac{1}{®_i}.$$

This equation is recognized as that the first transition from state *i* can be to either state *j* or to some other state *k*. If it is to state *j*, the first passage time is 1. Given that the first transition is to some state $k(k \neq j)$ instead, which occurs with probability p_{ik}, the conditional expected ,

first passage time from state *i* to state *j* is $1 + «_{kj}$. Combining these facts, and summing over all the possibilities for the first transition, leads directly to this equation.

For the case of $«_{ij}$ where *j* = *i*, $«_{ii}$ is the expected number of transitions until the process returns to the initial state *i* [19].

Application to Istanbul Stock Exchange (Ise)

Aim and Content of the Application

In this section, the increasing and the decreasing values that is observed in Istanbul Stock Exchange's closing index was modelled with transition matrix by using Markov chain analysis. With this modeling, first passage time between states was calculated with initially WinQSB software and then by a written a computer program code.

Application Data

Transition matrix was estimated by taking the closing values of Istanbul Stock Exchange (ISE -100) index from 20.01.2009 to 18.01.2013 into account for discrete Markov model. 998 data was used to estimate for this transition matrix.

According to the increments and decrements in Stock Exchange's closing index value, states of Markov chain was determined.

Let the rates be percent values, in the form of

-1 and more decrements constitute state 1
-1 with 0 (including 0) decrements constitute state 2
0 with 1(including 1) increments constitute state 3
1 with more increments constitute state 4

of Markov chain. The transition matrix is estimated by the method of maximum likelihood as follows.

$$P = \begin{bmatrix} 0,212 & 0,190 & 0,275 & 0,323 \\ 0,204 & 0,272 & 0,276 & 0,248 \\ 0,185 & 0,263 & 0,315 & 0,237 \\ 0,163 & 0,259 & 0,359 & 0,219 \end{bmatrix}$$

First passage time and limit distribution obtained via WinQSB-Process by using this transition matrix

Table 1. Mean First Passage Times.

Cases	Starting State	Entering State	Mean First Passage Time
1	1	4	3,5573
2	1	3	3,3500
3	1	2	4,3262
4	1	1	5,2816
5	2	4	3,8483
6	2	3	3,3671
7	2	2	3,9932
8	2	1	5,3077
9	3	4	3,8981
10	3	3	3,2391
11	3	2	4,0236
12	3	1	5,4101
13	4	4	3,9760
14	4	3	3,0962
15	4	2	4,0328
16	4	1	5,5274

Table 2. Limit Distribution.

State	Probability
1	0,1893
2	0,2504
3	0,3087
4	0,2516

Markov Chain Analysis

After the estimation of transition matrix, we determine whether or not the observed transition matrix is appropriate for the analysis and then the first passage time distribution has been studied to be found. Finally, we give the stages of analyzing and interpreting the information that is obtained from the first passage times.

Chi-Square Analysis

In order to perform a chi-square goodness of fit test, the estimated frequencies need to be found. Therefore, based on Markov Chain simulations the expected frequencies were calculated by a computer program. Here, Markov chain simulation is summarized briefly.

For a Markov chain with transition matrix $P = (P_{ij})\,, i, j \in S$ let Y_i denote a generic random variable distributed as the i^{th} row of the matrix, that is, having the distribution

$$P(Y_i = j) = p_{ij}\,, j \in S. \tag{3.1}$$

This distribution is used for the inverse transform to generate such a Y_i. For example, if $S = \{0,1,2,3,...\}$, then Y_i is generated via deriving a $U \sim U(0,1)$ and setting $Y_i = 0$, if $U \le p_{i0}$; $Y_i = 1$, if $p_{i0} < U \le p_{i0} + p_{i1}$, and in general $Y_i = j$, if $\sum_{k=0}^{j-1} p_{ik} < U \le \sum_{k=0}^{j} p_{ik}$. Steps to generate Y_i by using this inverse transform method and an independent uniform is provided in the following algorithm [20].

Algorithm for simulating a Markov chain up to N steps

1. Generate a U~U(0,1)
2. Choose an initial value, $X_0 = i_0$. Set n=1
3. Generate Y_{i_0}, and set $X_1 = Y_{i_0}$
4. If $n < N$, then set $i = X_n$, generate Y_i, set n = n + 1 and set $X_n = Y_i$; otherwise stop.
5. Go back to step 4.

This simulation study was performed in order to obtain the same total frequency. Expected frequencies and observed frequencies are as follows

Observed Frequencies

$$\begin{bmatrix} 40 & 36 & 52 & 61 \\ 51 & 68 & 69 & 62 \\ 57 & 81 & 97 & 73 \\ 41 & 65 & 90 & 55 \end{bmatrix}$$

Expected Frequencies

$$\begin{bmatrix} 34 & 39 & 49 & 65 \\ 42 & 75 & 73 & 60 \\ 67 & 69 & 80 & 82 \\ 43 & 67 & 96 & 57 \end{bmatrix}$$

H_0 : Estimated transition matrix fits the data

H_1 : Estimated transition matrix does not fit the data

$\chi^2_{Cal} =$	13,365
$\chi^2_{0.05} =$	24,996
Conclusion:	H_0 Accept

Some of the first passage distributions observed are given below and their goodness of fit to geometric distribution is tested by chi-square analysis.

H_0 : Transitions have a geometric distribution with success probability *p*.

H_1 : Transitions do not have a geometric distribution with success probability *p*.

Table 3. The observed distributions of the first passage time from state i to state j .

From state $i=1$ to state $j=4$

Period	Frequency
1	61
2	41
3	28
4	25
5	11
6	7
7	5
8	3
9	1
10	4
12	2
15	1

Mean =	3,0212
p =	0,3310
χ^2_{Cal} =	2,6454
$\chi_{6,0.05}$ =	12,592
Conclusion:	H_0 Accept

From state $i=3$ to state $j=2$

Period	Frequency
1	80
2	57
3	49
4	32
5	20
6	14
7	16
8	5
9	12
10	6
11	5
12	3
13	3
15	1
17	1

Mean =	3,7336
p =	0,2678
χ^2_{Cal} =	8,8780
$\chi^2_{8,0.05}$ =	15,507
Conclusion:	H_0 Accept

Bar graphs of the geometric distribution are shown below.

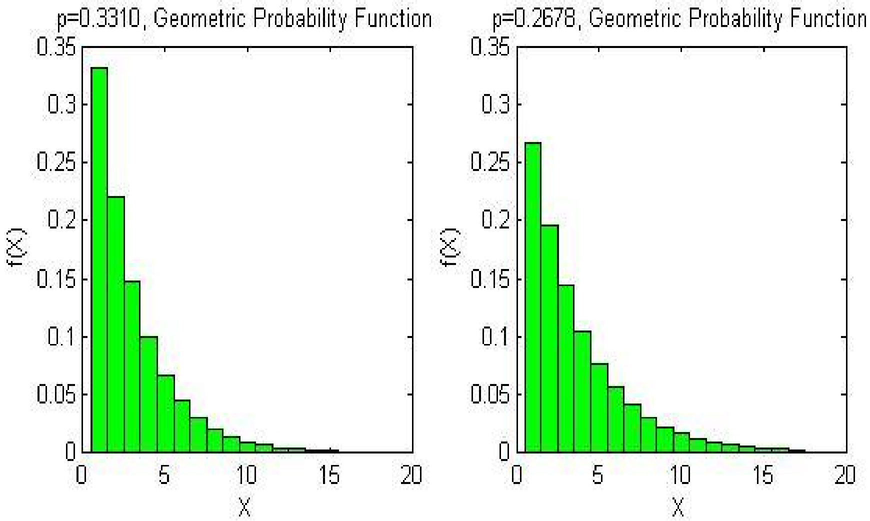

Figure 1 : Graphs of the geometric distribution

In the following Table 3.4, *p* success probability of geometric distribution and mean first passage time, the calculated chi-square values for goodness of fit are shown. As a result, when asimptotic information is used instead of empirical information, better results and increasing fitting with the geometric distribution has been found in some cases.

Table 4. p and mean first passage time, calculated chi-square.

Starting State	***Entering State***	***p and mean***		***Calculated Chi-Square***		***Is there a Recovery?***	***Ho Accept/Reject***	
		Empirical	***Asymptotic***	***Empirical***	***Asymptotic***		***Empirical***	***Asymptotic***
1	4	0,3310 3,0212	0,2811 3,5573	2,6454	8,4598	No	Accept	Accept
1	3	0,2688 3,7196	0,2985 3,3500	1,5677	5,4141	No	Accept	Accept
1	2	0,2436 4,1058	0,2311 4,3262	18,0558	17,9924	Yes	Reject	Reject
1	1	0,1983 5,0426	0,1893 5,2816	6,2998	6,9287	No	Accept	Accept
2	4	0,2753 3,6320	0,2599 3,8483	8,4457	8,5979	No	Accept	Accept
2	3	0,2662 3,7560	0,2970 3,3671	3,7202	8,9447	No	Accept	Accept
2	2	0,2513 3,9799	0,2504 3,9932	11,0709	10,9860	Yes	Accept	Accept
2	1	0,2010 4,9746	0,1884 5,3077	2,3666	4,3596	No	Accept	Accept

Table 4. (Continued) p and mean first passage time, calculated chi-square.

Starting State	*Entering State*	*p and mean*		*Calculated Chi-Square*		*Is there a Recovery?*	*Ho Accept/Reject*	
		Empirical	*Asymptotic*	*Empirical*	*Asymptotic*		*Empirical*	*Asymptotic*
3	4	0,2598 3,8497	0,2565 3,8981	7,0480	6,9785	Yes	Accept	Accept
3	3	0,3086 3,2403	0,3087 3,2391	8,2947	8,2933	Yes	Accept	Accept
3	2	0,2678 3,7336	0,2485 4,0236	8,8780	9,4969	No	Accept	Accept
3	1	0,1740 5,7456	0,1848 5,4101	5,9229	7,7020	No	Accept	Accept
4	4	0,2518 3,9720	0,2515 3,9760	10,1737	10,1541	Yes	Accept	Accept
4	3	0,3134 3,1912	0,3230 3,0962	5,7475	6,4780	No	Accept	Accept
4	2	0,2580 3,8755	0,2480 4,0328	3,4301	3,1158	Yes	Accept	Accept
4	1	0,1992 5,0207	0,1809 5,5274	4,5537	6,3915	No	Accept	Accept

Using Easy Fit software, fitting distributions for first passage time are shown below.

Table 5. First Passage Time Distribution

Cases	Starting State	Entering State	First Passage Time Distribution	Kolmogorov-Smirnov Goodness of Fit p value
1	1	4	Beta	0,91583
2	1	3	Weibull	0,90204
3	1	2	Chi-Squared	0,98186
4	1	1	Johnson SB	0,92867
5	2	4	Johnson SB	0,96254
6	2	3	Johnson SB	0,81151
7	2	2	Beta	0,97800
8	2	1	Gen.Gamma	0,85360
9	3	4	Weibull	0,91920
10	3	3	Kumaraswamy	0,70628
11	3	2	Gen.Gamma	0,89069
12	3	1	Burr	0,91058
13	4	4	Gamma	0,96357
14	4	3	Weibull	0,67077
15	4	2	Gamma	0,92335
16	4	1	Johnson SB	0,97891

Distributions given in the column of first passage time distribution are fitting with distributions corresponding to cases because *p* values are greater than 5% (all of them having values at least over 67%.). Graphs of fitting distribution for first passage time and their parameter values are shown below.

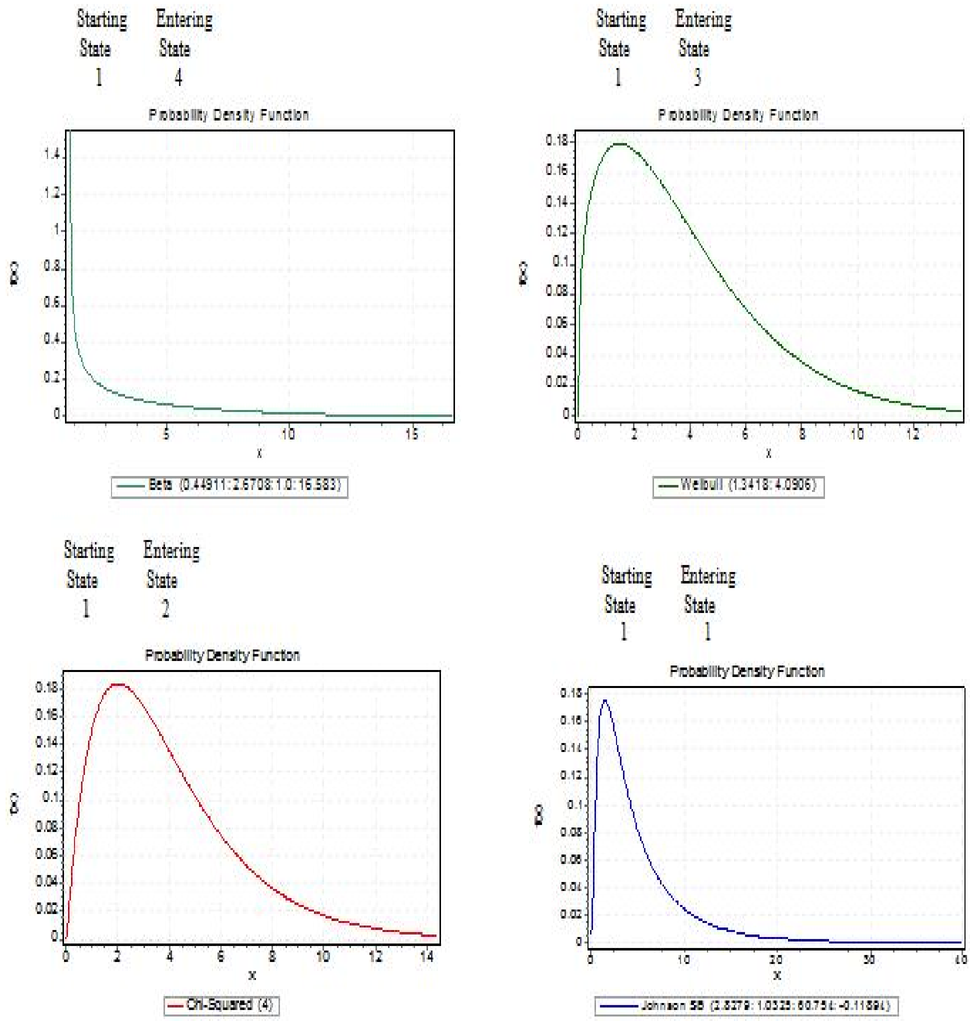

Figure 2. Graphs of fitting distribution for first passage time starting state 1

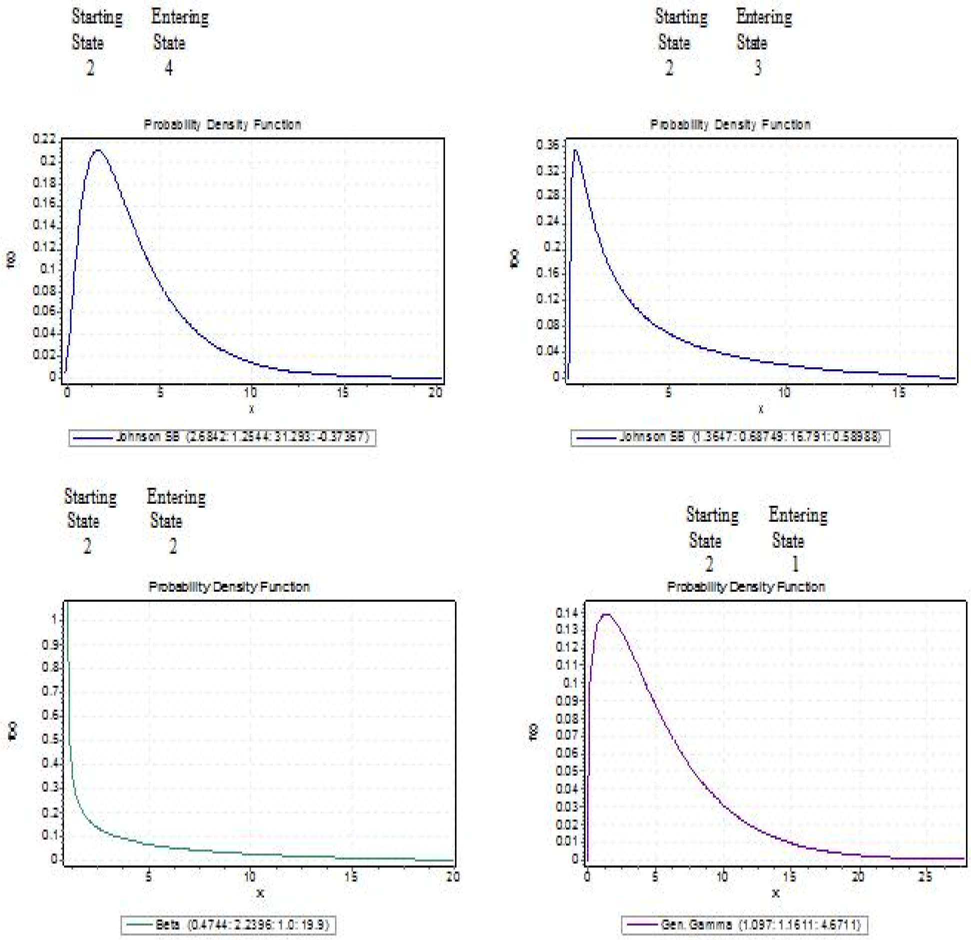

Figure 3. Graphs of fitting distribution for first passage time starting state 2

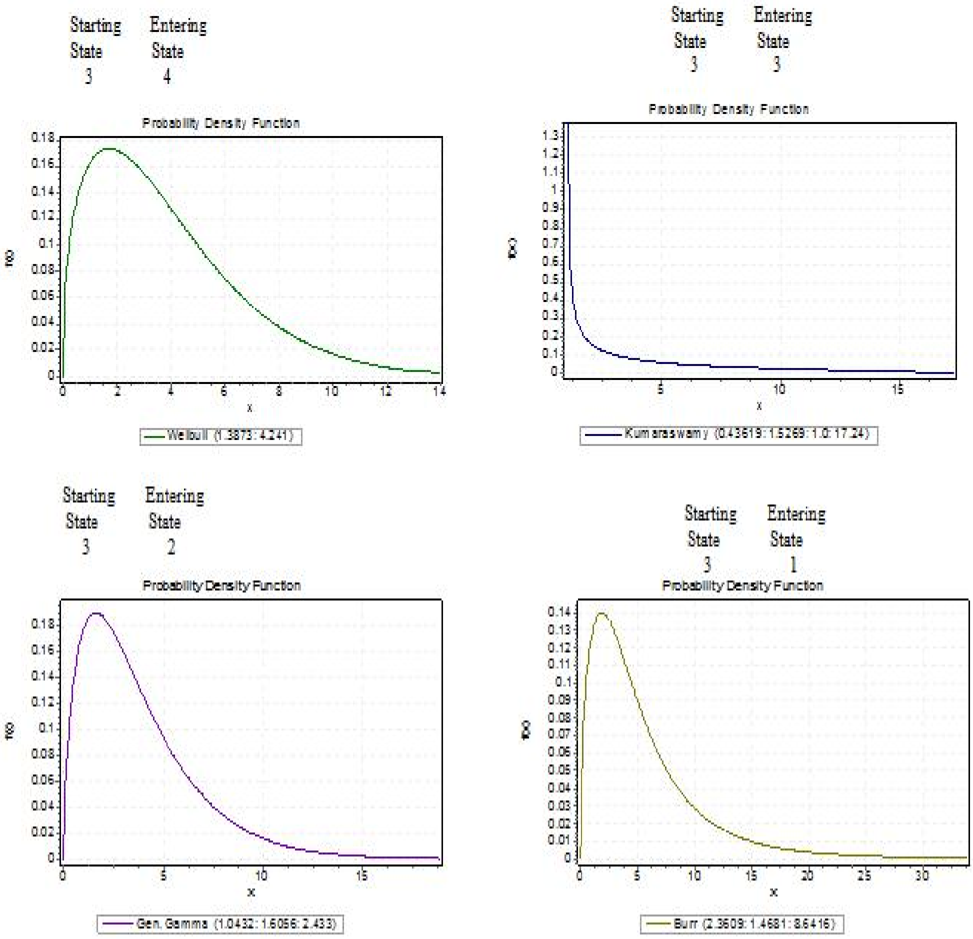

Figure 4. Graphs of fitting distribution for first passage time starting state 3

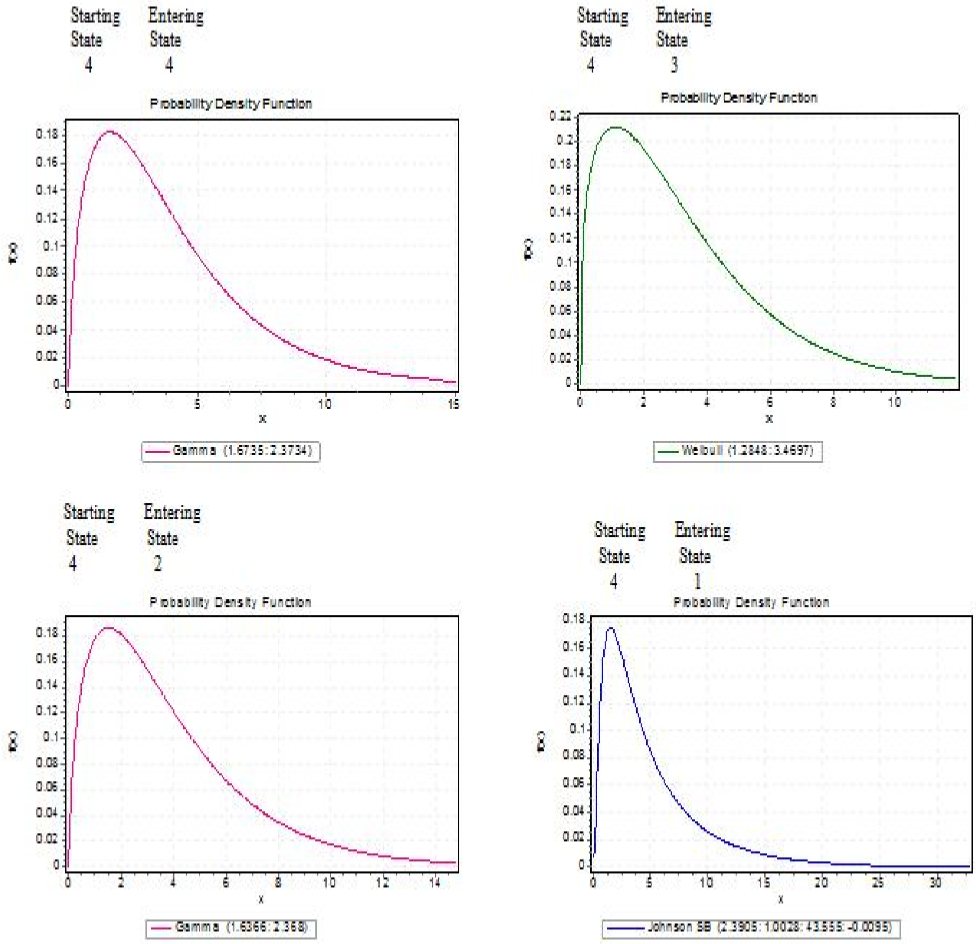

Figure 5. Graphs of fitting distribution for first passage time starting state 4

3. CONCLUSIONS AND SUGGESTIONS

In this study, the concept of first passage was examined. It was also emphasized the importance of the first passage time in Markov Chains and Stochastic Processes. Later by using data from the ISE for an application, transition matrix of Markov chain was estimated. Under the assumption that the frequency distribution of the first passage time fits with the geometric distribution, the fittings of the first passage time obtained from empirical information and the first passage time obtained from asymptotic information to the geometric distribution was compared with a chi-square analysis. With respect to the results of this comparision, for some states of the first passage time the values obtained from the asymptotic information comply better with the geometric distribution. Looking at the results in the Table 3.4, approximately 40 percent of cases the asymptotic information is superior to the empirical information. Researchers who wish to use the first passage times might need to do their analysis by taking asymptotic information into account. Using Easy Fit software, fitness of the first passage time distributions according to Kolmogorov-Smirnov Goodness of Fit test were examined and thus, the first passage time distributions corresponding to cases were found. Graphics of continuous distribution which comply with frequency of first passage time were also provided. From these graphs first passage time distribution was seen to be a positively skewed or reverse j-shaped distribution.

Since it is also possible to obtain geometric distributions from some continuous distributions [21]. We will attempt to obtain the first passage time distributions in our subsequent papers.

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