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DIFFERENTIAL TRANSFORMATION METHOD FOR NONLINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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ABSTRACT

This work presents solution of nonlinear differential-difference equations such as the discretized mKdV lattice equation, the discretized nonlinear Schrödinger equation and the Toda Lattice equation by Differential Transformation Method (DTM). This method provides more realistic solutions by solving the nonlinear differential equations without any simplification and the series solutions which generally converge very rapidly in real physical models. Moreover, no linearization or perturbation is required in this method. By using this method, exact solutions may be obtained without any need of cumbersome work. This method is a useful tool for analytical and numeric solutions. The results of the present method are compared with those obtained by Adomian Decomposition Method and exact solutions. The results have shown that DTM method has better performs.
Key Words: Differential Transformation Method, Differential-Difference Equations, Nonlinear Equations.

1. INTRODUCTION

In this work, we study Nonlinear Differential-Difference Equations (NDDEs). The nonlinear differential-difference equations arise in the modeling of many phenomena in different fields, ranging from condensed matter and biophysics to mechanical engineering such as atomic chains, molecular crystals, biophysical systems, electrical lattices and optical wave guides. Nonlinear differential-difference equations are usually hard to solve analytically and exact solutions are scarce. In literature, lots of numerical techniques such as Adomian Decomposition Method [1], Laplace Method, Fourier Method, Wawelet Galerkin Method and Runge-Kutta Method exist. However, in the previous studies, there are complex integrals or methodology. Moreover, all of the previous studies require too much effort to achieve the results. But, DTM provides more realistic solutions by solving the nonlinear differential equations without any simplification and no linearization or perturbation is required. Because it is easily used various problems, DT Method is commonly used in recent published work.

Differential Transform Method (DTM) is based on Taylor series expansion. It is introduced by Zhou [2] in a study about electrical circuits. He gave exact values of the n^{th} derivative of an analytical function at a point in terms of known and unknown boundary conditions in a fast manner. Chen and Ho [3] established the basic theory of two-dimensional DT and thereby reached exact solutions to a few linear and nonlinear initial problems. Ayaz [4] considered the exact or approximate solutions to several second-order nonlinear PDEs by exploiting some properties of two-dimensional DTM. Ayaz [5] introduced three-dimensional DTM to solve some systems of PDEs.

In this paper, we extend DTM to solve NDDEs, such as the discretized mKdV Lattice Equation [6];

$$\frac{du_n}{dt} = \left(\alpha - u_n{}^2\right)\left(u_{n+1} - u_{n-1}\right)$$

the discretized nonlinear Schrödinger Equation [7];

$$i\frac{du_n}{dt} = \left(u_{n+1} + u_{n-1} - 2u_n\right) - \left|u_n\right|^2\left(u_{n+1} + u_{n-1}\right)$$

and Toda Lattice Equation [8];

$$\begin{cases} \dfrac{du_n}{dt} = u_n(v_n - v_{n-1}) \\ \dfrac{dv_n}{dt} = v_n(u_{n+1} - u_n) \end{cases}$$

2.ONE DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD

The differential transform of the function $y(x)$ is the form

$$Y(k) = \frac{1}{k!}\left[\frac{d^k}{dx^k}y(x)\right]_{x=x_0} \tag{1}$$

where $y(x)$ is original function. The differential inverse transform function of $Y(k)$ is showed as

$$y(x) = \sum_{k=0}^{\infty}\left(x - x_0\right)^k Y(k) \tag{2}$$

If Eq. (1) is substituted into the Eq. (2), Eq. (2) can be written as

$$y(x) = \sum_{k=0}^{\infty}\frac{\left(x - x_0\right)^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \tag{3}$$

If x_0 is taken as zero, differential and differential inverse transform can be shown as

$$Y(k) = \frac{1}{k!}\left[\frac{d^k}{dx^k}y(x)\right]_{x=0} \quad \text{and} \quad y(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}$$

respectively.

Because $\sum_{k=N+1}^{\infty}x^kY(k)$ is negligibly small, $y(x)$ function can be written as

$$y(x) = \sum_{k=0}^{N}x^kY(k)$$

This is truncated expression of function $y(x)$.

The fundamental operations performed by differential transform can readily be obtained and are listed in Table 1.

3. APPLICATIONS OF DIFFERENTIAL TRANSFORM METHOD

Example 1.

We consider mKdV Lattice Equation

$$\frac{du_n}{dt} = \left(\alpha - u_n{}^2\right)\left(u_{n+1} - u_{n-1}\right) \tag{4}$$

with initial condition $u_0 = \tanh(k_1)\tanh(k_1 n)$. For $\alpha = 1$, Eq. (4) has a kink-type solution [7], which reads $u_n(t) = \tanh(k_1)\tanh(k_1 n + 2\tanh(k_1)t)$.

By taking differential transform of Eq.(4), for $k \geq 0$, the following equation can be obtained

$$\begin{aligned}(k+1)U(k+1,n) &= \alpha\left(U(k,n+1) - U(k,n-1)\right)\\ &- \sum_{m=0}^{k}\sum_{l=0}^{m} U(l,n)U(m-l,n).\left[U(k-m,n+1) - U(k-m,n-1)\right]\end{aligned} \tag{5}$$

and form the differential transform of initial condition $u_n(0) = \tanh(k_1).\tanh(k_1 n)$,

$$U(0,n) = \tanh(k_1).\tanh(k_1 n)$$

can be obtained. Hence, for various value of k, the following equation system can be obtained from Eq. (5).

$$U(1,n) = \left[U(0,n+1) - U(0,n-1)\right].\left(\alpha - U^2(0,n)\right)$$

$$2U(2,n) = \alpha\left(U(1,n+1) - U(1,n-1)\right)\left[\alpha - U^2(0,n)\right] - 2U(0,n)U(1,n)\left[U(0,n+1) - U(0,n-1)\right]$$

$$\begin{aligned}3U(3,n) = \alpha\left(U(2,n+1) - U(2,n-1)\right)\left[\alpha - U^2(0,n)\right] - 2U(0,n)U(1,n)\left[U(1,n+1) - U(1,n-1)\right]\\ - \left[2U(0,n)U(2,n) + U^2(1,n)\right].\left[U(0,n+1) - U(0,n-1)\right]\end{aligned}$$

$$\begin{aligned}4U(4,n) = \alpha\left(U(3,n+1) - U(3,n-1)\right)\left[\alpha - U^2(0,n)\right] - 2U(0,n)U(1,n)\left[U(2,n+1) - U(2,n-1)\right]\\ - \left[2U(0,n)U(2,n) + U^2(1,n)\right]\left[U(1,n+1) - U(1,n-1)\right]\\ - \left[2U(0,n)U(3,n) + 2U(1,n)U(2,n)\right]\left[U(0,n+1) - U(0,n-1)\right]\end{aligned}$$

$$\begin{aligned}5U(5,n) = \alpha\left(U(4,n+1) - U(4,n-1)\right)\left[\alpha - U^2(0,n)\right] - 2U(0,n)U(1,n)\left[U(3,n+1) - U(3,n-1)\right]\\ - \left[2U(0,n)U(2,n) + U^2(1,n)\right].\left[U(2,n+1) - U(2,n-1)\right]\\ - \left[2U(0,n)U(3,n) + 2U(1,n)U(2,n)\right].\left[U(1,n+1) - U(1,n-1)\right]\\ - \left[2U(0,n)U(4,n) + 2U(1,n)U(3,n) + U^2(2,n)\right].\left[U(0,n+1) - U(0,n-1)\right]\end{aligned}$$

$$6U(6,n)=\alpha\left(U(5,n+1)-U(5,n-1)\right)\left[\alpha-U^2(0,n)\right]-2U(0,n)U(1,n)\left[U(4,n+1)-U(4,n-1)\right]$$
$$-\left[2U(0,n)U(2,n)+U^2(1,n)\right].\left[U(3,n+1)-U(3,n-1)\right]$$
$$-\left[2U(0,n)U(3,n)+2U(1,n)U(2,n)\right].\left[U(2,n+1)-U(2,n-1)\right]$$
$$-\left[2U(0,n)U(4,n)+2U(1,n)U(3,n)+U^2(2,n)\right].\left[U(1,n+1)-U(1,n-1)\right]$$
$$-\left[2U(0,n)U(5,n)+2U(1,n)U(4,n)+2U(2,n)U(3,n)\right].\left[U(0,n+1)-U(0,n-1)\right]$$

Substituting all of the $U(k,n)$, which is obtained form the solution of above, into the inverse transform function $u_n(t)$, series solution of u can be obtained as

$$u_n(t)=U(0,n)+U(1,n)t+U(2,n)t^2+U(3,n)t^3+U(4,n)t^4+U(5,n)t^5+U(6,n)t^6+\ldots$$

Exact solution, DTM solution, ADM[1] solution and absolute errors of Eq. (4) have been given in Table 2-5 and shown in Figures 1-2. From the tables and figures, it is shown that computed values and actual solutions are extremely close and the results obtained by the present method are superior from ADM[1].

Example 4.2

Next, we consider Toda Lattice Equation

$$\left.\begin{aligned}\frac{du_n}{dt}&=u_n(v_n-v_{n-1})\\ \frac{dv_n}{dt}&=v_n(u_{n+1}-u_n)\end{aligned}\right\} \qquad (6)$$

with initial conditions

$$u_n(0)=-\coth(d)c+\tanh(dn)c$$
$$v_n(0)=-\coth(d)c-\tanh(dn)c$$

which has an exact solution

$$u_n(t)=-\coth(d)c+\tanh(dn+ct)c$$
$$v_n(0)=-\coth(d)c-\tanh(dn+ct)c.$$

By taking differential transform of Eq.(6), for $k\geq 0$, the following equations can be obtained.

$$(k+1)U(k+1,n)=\sum_{l=0}^{k}U(l,n)V(k-l,n)-\sum_{l=0}^{k}U(l,n)V(k-l,n-1) \qquad (7)$$

$$(k+1)V(k+1,n)=\sum_{l=0}^{k}V(l,n)U(k-l,n+1)-\sum_{l=0}^{k}V(l,n)U(k-l,n) \qquad (8)$$

The differential transform of initial conditions

$u_n(0)=-\coth(d)c+c\tanh(dn)$ and $v_n(0)=-\coth(d)c-c\tanh(dn)$

can be written as

$$U(0,n)=-\coth(d)c+c\tanh(dn)$$

$$V(0,n) = -\coth(d)c - c\tanh(dn)$$

Substituting various values of k in to the Eq. (7) and Eq. (8), equation systems can be constructed. From the solutions of equations system, $U(k,n)$ and $V(k,n)$ differential transform functions can be obtained.

Substituting all of the $U(k,n)$ and $V(k,n)$ into the inverse transforms functions $u_n(t)$ and $v_n(t)$ respectively, series solutions of $u_n(t)$ and $v_n(t)$ can be obtained as:

$$u_n(t) = U(0,n) + U(1,n)t + U(2,n)t^2 + U(3,n)t^3 + U(4,n)t^4 + U(5,n)t^5 + U(6,n)t^6 + \ldots$$
$$v_n(t) = V(0,n) + V(1,n)t + V(2,n)t^2 + V(3,n)t^3 + V(4,n)t^4 + V(5,n)t^5 + V(6,n)t^6 + \ldots$$

Exact solution and approximate solution of the problem have been given in the Table 6, Table 7, Table 8 and Table 9 and have shown in the Fig. 3, Fig. 4, Fig. 5 and Fig. 6. From the Tables and figures, it is shown that computed values and actual solutions are extremely close.

Example 4.3

We consider nonlinear Schrödinger Equation

$$i\frac{du_n}{dt} = \left(u_{n+1} + u_{n-1} - 2u_n\right) - |u_n|^2\left(u_{n+1} + u_{n-1}\right) \tag{9}$$

with initial condition

$$u_n(0) = \tanh(k_1)\exp(ipn)\tanh(k_1 n),$$

which has an exact solution

$$u_n(t) = \tanh(k_1)\exp(i[pn + (2 - 2\cos(p)\operatorname{sec}h(k_1))t])\tanh(k_1 n + 2\sin(p)\tanh(k_1)t)$$

By taking differential transform of Eq.(9), for $k \geq 0$, the following equations can be obtained.

$$\begin{aligned} i(k+1)U(k+1,n) &= [U(k,n+1) + U(k,n-1) - 2U(k,n)] \\ &- \sum_{m=0}^{k}\sum_{l=0}^{m} U(l,n)U(m-l,n)[U(k-m,n+1) + U(k-m,n-1)] \end{aligned} \tag{10}$$

The differential transform of initial condition

$$u_n(0) = \tanh(k_1)\exp(ipn)\tanh(k_1 n)$$

can be written as

$$U(0,n) = \tanh(k_1)\exp(ipn)\tanh(k_1 n)$$

Substituting various values of k in to the Eq. (10), equation system can be constructed. From the solution of equation system, $U(k,n)$ differential transform function can be obtained.

Substituting all of the $U(k,n)$ into the inverse transform function $u_n(t)$, series solutions of $u_n(t)$ can be obtained as:

$$u_n(t) = U(0,n) + U(1,n)t + U(2,n)t^2 + U(3,n)t^3 + U(4,n)t^4 + U(5,n)t^5 + U(6,n)t^6 + \dots$$

Exact solutions and approximate solutions of the problem have been given in Fig. 7. Table 10 shows the absolute errors for various values of n and t. Absolute errors of Example 4.3 are depicted in Figure 8 $k_1 = 0.1$, $p = 0.5$ and $t = 1$. From the table and figures, it is shown that computed values and actual solutions are extremely close.

4. CONCLUSION

In this text we extend the solutions of mKdV Lattice Equation, the discretized nonlinear Schrödinger Equation and Toda Lattice equation. In the literature, a lot of methods can solve these problems such as ADM. Solving these equations by other methods were very difficult and some of them had complex integrals. But by using DTM, we turned them into easy equation system and solved them easily. From the figures, accuracy and efficiency of the method were demonstrated.

REFERENCES

[1] Wu L., Xie L.D., Zhang J.F. .Adomian Decomposition method for nonlinear differential-difference equations, ***Comm. Nonlinear. Sci. Numer. Simulat***. 14/1 12-18 (2009)
[2] Zhou J.K., Differential Transformation and Its Application for Electrical Circuits, ***Huazhong University Pres***,Wuhan, China, (1986).
[3] Chen C.K., Ho S.H., Solving partial differential equations by two-dimensional differential equations, ***Appl. Math. Comput.*** , 121, 261-270, (2001).
[4] Ayaz F., On the two-dimensional differential transform method, ***Appl. Math. Comput.***, 143, 361-374, (2003).
[5] Ayaz F., Solution of the system of differential equations by differential transform method. ***Appl. Math. Comput.,*** 147, 547-567, (2004).
[6] Ablowitz M.J., Ladic J.F., On the solution of a class of nonlinear partial difference equations, ***Stud. Apply.Math.*** 57, 1-12, (1997).
[7] Dai C.Q., Zhang J.F., Jacobian elliptic function method for nonlinear differential-difference equations. ***Chaos Solution Fract*** 27, 1042-1047, (2006).
[8] Suris Y.B., New integrable systems related to the relativistic Toda Lattice, ***J.Phys.A:Math Gen*** 30, 1745-1761, (1997).

Tables List

Table 1. Operations of differential transformation

Original function	**Transformed function**
$y(x)=\varphi(x)\pm\theta(x)$	$Y(k)=\phi(k)\pm\Theta(k)$
$y(x)=\lambda.\varphi(x)$	$Y(k)=\lambda.\phi(k)$
$y(x)=\varphi(x)\theta(x)$	$Y(k)=\sum_{l=0}^{k}\phi(l)\Theta(k-l)$
$f(x)=g_1(x)g_2(x)\ldots g_n(x)$	$F(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\ldots\sum_{k_1=0}^{k_2}G_1(k_1)G_2(k_2-k_1)\ldots G_n(k-k_{n-1})$
$y(x)=\frac{d\varphi(x)}{dx}$	$Y(k)=(k+1)\phi(k+1)$
$y(x)=\frac{d^n\varphi(x)}{dx^n}$	$Y(k)=(k+1)(k+2)\ldots(k+n)\phi(k+1)$
$y(x)=x^m$	$Y(k)=\delta(k-m)=\begin{cases}1 & ,k=m\\ 0 & ,k\neq m\end{cases}$

Table 2. For $k=3$, $t=0.5$ and $k_1=0.1$, comparing DTM and exact solutions of u_n

n	Exact Solution	DTM	Absolute error
-25	-0.0980419716685	-0.098041974540424	2.87192E-09
-15	-0.0882483737640	-0.088248370424162	3.33984E-09
-5	-0.0378970609082	-0.037897039415964	2.14922E-08
0	0.0099009464692	0.009900816293384	1.30176E-07
5	0.0535031028299	0.053503134152388	3.13225E-08
15	0.0918558740145	0.091855875966215	1.95171E-09
25	0.0985736553800	0.098573652648647	2.73135E-09

Table 3. For $k=3$, $t=1.5$ and $k_1=0.1$, comparing DTM and exact solutions of u_n

n	Exact Solution	DTM	Absolute error
-25	-0.097255114012016	-0.097255847240229	7.33228E-07
-15	-0.083118937378073	-0.083117708245280	1.22913E-06
-5	-0.019767387202623	-0.019765023755796	2.36345E-06
0	0.028943671058017	0.028913020259932	3.06508E-05
5	0.066127678611751	0.066137115576502	9.43696E-06
15	0.094355966562706	0.094356172616496	2.06054E-07
25	0.098932134016150	0.098931502773966	6.31242E-07

Table 4. For $k=5$, $t=0.5$ and $k_1=0.1$, comparing DTM, ADM and exact solutions of u_n

n	Exact Solution	ADM[1]	Absolute error (ADM)	DTM	Absolute error (DTM)
-25	-0.0980419716685	-0.09804197166	8.49999E-12	-0.0980419716679	5.99992E-13
-15	-0.0882483737640	-0.08824837298	7.84E-10	-0.0882483736885	7.55E-11
-5	-0.0378970609082	-0.03789706610	5.1918E-09	-0.0378970622547	1.3465E-09
0	0.0099009464692	0.009900946992	5.228E-10	0.0099009464588	1.04E-11
5	0.0535031028299	0.05350309813	4.6999E-09	0.0535031039771	1.1472E-09
15	0.0918558740145	0.09185587327	7.445E-10	0.0918558740072	7.30001E-12
25	0.0985736553800	0.09857365542	4E-11	0.0985736553765	3.50001E-12

Table 5. For $k=5$, $t=1.5$ and $k_1=0.1$, comparing DTM, ADM and exact solutions of u_n

n	Exact Solution	ADM[1]	Absolute error (ADM)	DTM	Absolute error (DTM)
-25	-0.097255114012016	-0.09725516662	5.2608E-08	-0.097255115097158	1.08514E-09
-15	-0.083118937378073	-0.08311834180	5.95578E-07	-0.083118839008260	9.83698E-08
-5	-0.019767387202623	-0.01977150813	4.12093E-06	-0.019768215755761	8.28553E-07
0	0.028943671058017	0.02894478018	1.10912E-06	0.028944695756441	1.0247E-06
5	0.066127678611751	0.06613063122	2.95261E-06	0.066127397190045	2.81422E-07
15	0.094355966562706	0.09435553904	4.27523E-07	0.094356034637888	6.80752E-08
25	0.098932134016150	0.09893218337	4.93538E-08	0.098932131655087	4.93538E-08

Table 6. $k=3$, $d=0.1$, $c=0.1$ and $t=1$, comparing DTM and exact solutions of u_n and v_n

n	Exact Solution u_n	DTM u_n	Abs. error(u_n)	Exact Solution v_n	DTM v_n	Abs. error (v_n)
-40	-1.103249199792	-1.103249199975	1.83E-10	-0.90341302665	-0.90341302647	1.8E-10
-20	-1.098954859038	-1.098954863828	4.79E-09	-0.90770736741	-0.90770736262	4.79E-09
-10	-1.074960900245	-1.074960852077	4.8168E-08	-0.93170132620	-0.93170137437	4.817E-08
0	-0.993364313762	-0.993364446558	1.32796E-07	-1.01329791268	-1.01329777989	1.3279E-07
10	-0.923281211049	-0.923281166651	4.4398E-08	-1.08338101540	-1.08338105979	4.439E-08
20	-0.906285919564	-0.906285924323	4.759E-09	-1.10037630688	-1.10037630212	4.76E-09
40	-0.903386028856	-0.903386029027	1.71E-10	-1.10327619759	-1.10327619742	1.7E-10

Table 7. For $k=5$, $d=0.1$, $c=0.1$ and $t=1$, comparing DTM and exact solutions of u_n and v_n

n	Exact Solution u_n	DTM u_n	Abs. error(u_n)	Exact Solution v_n	DTM v_n	Abs. error (v_n)
-40	-1.103249199792	-1.103249199792	0	-0.90341302665	-0.90341302665	0
-20	-1.098954859038	-1.098954859031	6.99996E-12	-0.90770736741	-0.90770736741	0
-10	-1.074960900245	-1.074960900276	3.10001E-11	-0.93170132620	-0.93170132617	3E-11
0	-0.993364313762	-0.993364313225	5.37E-10	-1.01329791268	-1.01329791322	5.4E-10
10	-0.923281211049	-0.923281211067	1.8E-11	-1.08338101540	-1.08338101538	2E-11
20	-0.906285919564	-0.906285919557	6.99996E-12	-1.10037630688	-1.10037630689	1E-11
40	-0.903386028856	-0.903386028856	0	-1.10327619759	-1.10327619759	0

Table 8. For $k=3$, $d=0.1$, $c=0.1$, $t=3$ comparing DTM and exact solutions of u_n and v_n

n	Exact Solution u_n	DTM u_n	Abs. error(u_n)	Exact Solution v_n	DTM v_n	Abs. error (v_n)
-40	-1.103208937353	-1.10320898500	4.7647E-08	-0.90345328909	-0.90345324144	4.765E-08

-20	-1.096872020285	-1.09687317624	1.15595E-06	-0.90979020616	-0.90978905020	1.15596E-06
-10	-1.063767890937	-1.06375536335	1.25276E-05	-0.94289433551	-0.94290686309	1.25276E-05
0	-0.97419985198	-0.97423111322	3.12612E-05	-1.03246237447	-1.03243111322	3.12613E-05
10	-0.917158797294	-0.91714893859	9.8587E-06	-1.08950342915	-1.08951328786	9.85871E-06
20	-0.905321473598	-0.90532261205	1.13845E-06	-1.10134075285	-1.10133961439	1.13846E-06
40	-0.90336792760	-0.90336796669	3.909E-08	-1.10329429884	-1.10329425975	3.909E-08

Table 9. For $k=5$, $d=0.1, c=0.1$, $t=3$ comparing DTM and exact solutions of u_n and v_n

n	Exact Solution u_n	DTM u_n	Abs. error(u_n)	Exact Solution v_n	DTM v_n	Abs. error (v_n)
-40	-1.10320893735	-1.103208937737	3.87E-10	-0.90345328909	-0.90345328871	3.8E-10
-20	-1.09687202028	-1.096872002812	1.7468E-08	-0.90979020616	-0.90979022363	1.747E-08
-10	-1.06376789093	-1.063767995498	1.04568E-07	-0.94289433551	-0.94289423095	1.0456E-07
0	-0.97419985198	-0.974198713225	1.13876E-06	-1.03246237447	-1.03246351322	1.1388E-06
10	-0.917158797294	-0.917158811862	1.4568E-08	-1.08950342915	-1.08950341458	1.457E-08
20	-0.905321473598	-0.905321461811	1.1787E-08	-1.10134075285	-1.10134076463	1.178E-08
40	-0.903367927606	-0.903367927939	3.33E-10	-1.10329429884	-1.10329429851	3.3E-10

Table 10. The absolute errors of Example 4.3 for various values of n and t

n	$t=1$	$t=1.5$	$t=2$	$t=2.5$	$t=3$
-40	3.889763e-010	9.813534e-009	9.623932e-008	5.609540e-007	2.348130e-006
-20	6.978906e-010	1.526501e-008	1.378812e-007	7.608852e-007	3.060422e-006
0	1.375211e-008	2.285186e-007	1.645200e-006	7.442950e-006	2.494993e-005
20	1.456220e-010	6.101938e-009	7.206301e-008	4.640887e-007	2.070075e-006
40	3.795945e-010	9.658237e-009	9.512766e-008	5.559651e-007	2.331590e-00

Figures List

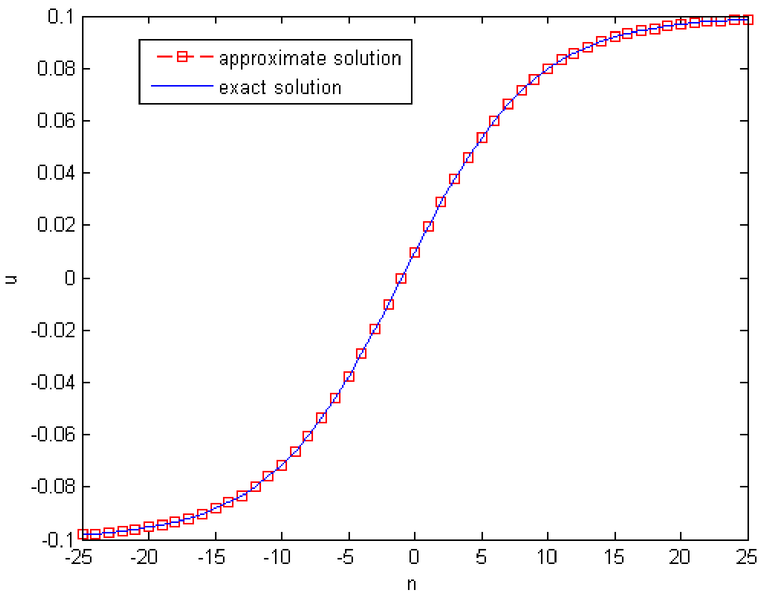

Figure 1. For $k=5$, $t=0.5$ and $k_1=0.1$, comparing DTM and exact solution of u_n

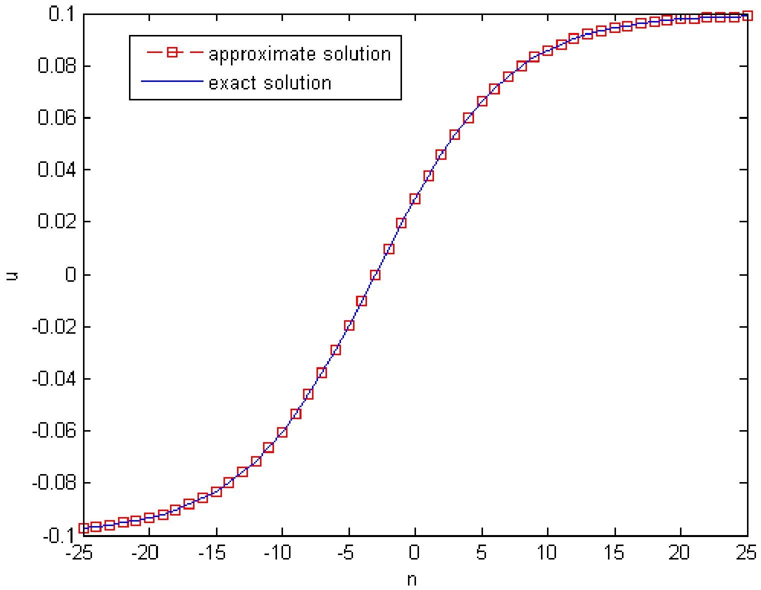

Figure 2. For $k=5$, $t=1.5$ and $k_1=0.1$, comparing DTM and exact solution of u_n

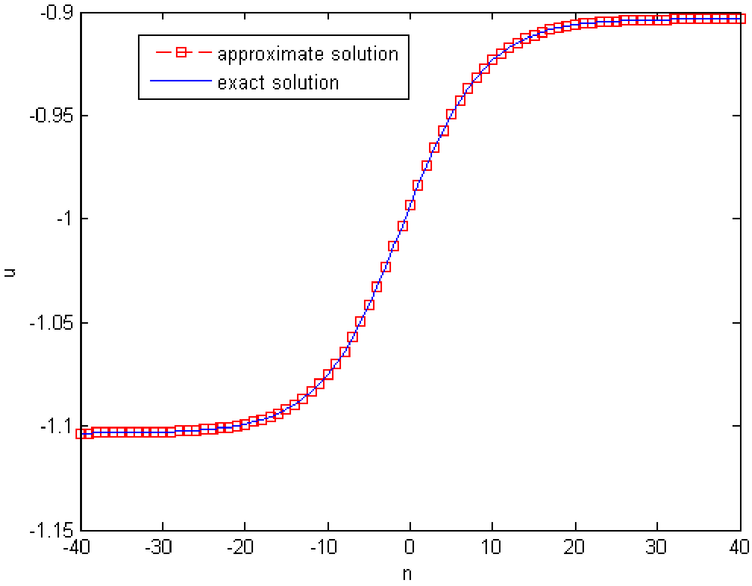

Figure 3. For $k=5$, $d=0.1$, $c=0.1$ and $t=1$, comparing DTM and exact solution of u_n

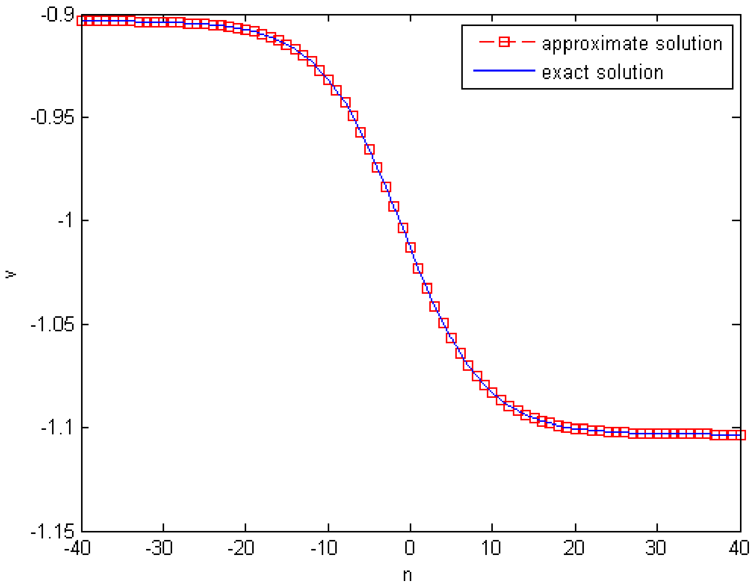

Figure 4. For $k=5$, $d=0.1$, $c=0.1$ and $t=1$, comparing DTM and exact solution of v_n

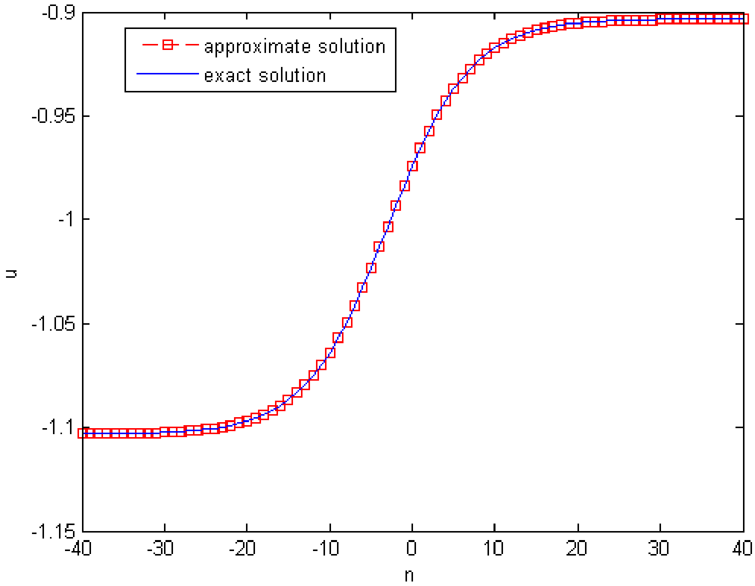

Figure 5. For $k=5$, $d=0.1$, $c=0.1$ and $t=3$, comparing DTM and exact solution of u_n

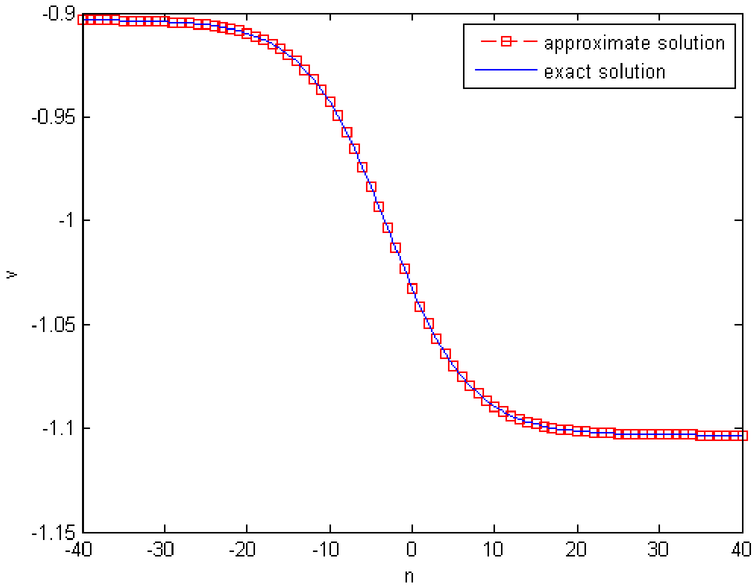

Figure 6. For $k=5$, $d=0.1$, $c=0.1$ and $t=3$, comparing DTM and exact solution of v_n

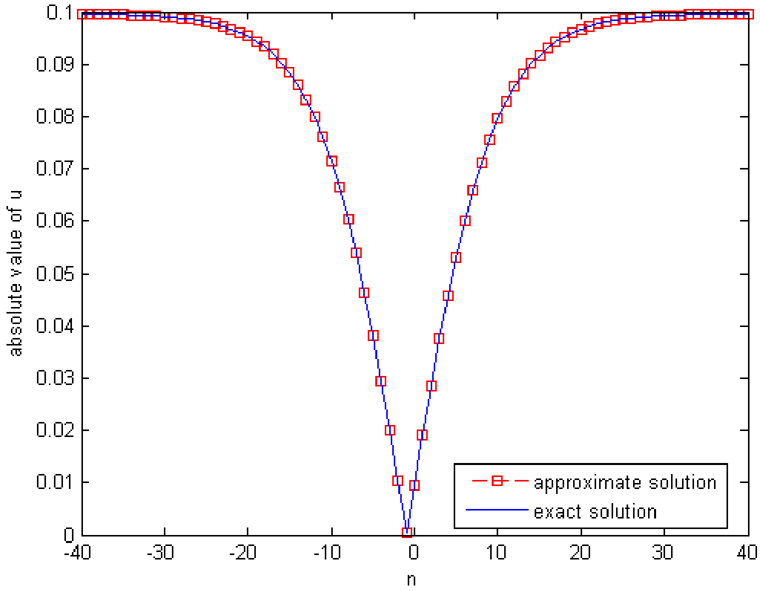

Figure 7. For $k_1 = 0.1$, $p = 0.5$ and $t = 1$, comparing DTM and exact solution of u_n

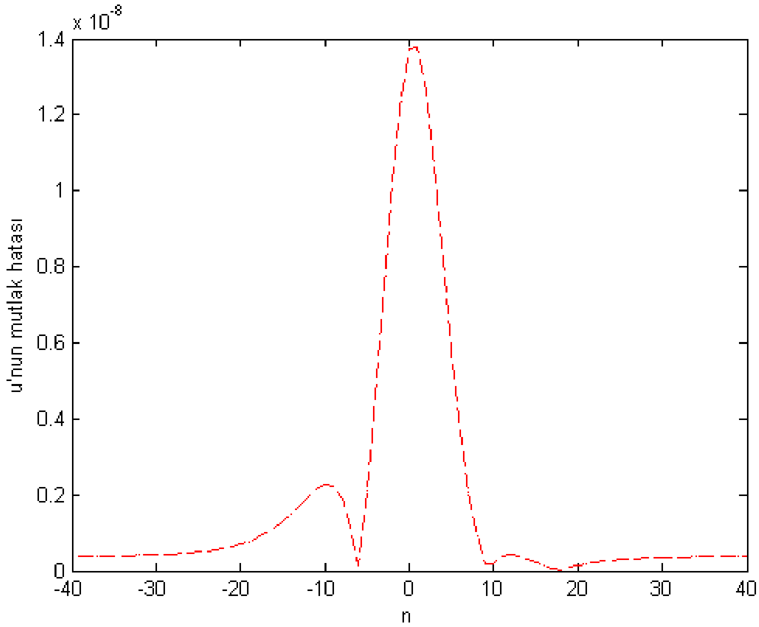

Figure 8. The absolute error of Example 4.3 for $k_1 = 0.1$, $p = 0.5$ and $t = 1$