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Research Article

Screen pseudo slant lightlike submanifolds of golden semi-Riemannian manifolds

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Abstract

In this article, we examine the term of screen pseudo-slant lightlike submanifolds of a golden semi-Riemannian manifold. Also, we obtain an example. We give some characterizations about the geometry of such submanifolds.

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1. Introduction

From ancient times, the golden proportion has played very important role in architecture, arts, music etc. J. Kepler described golden structure which was revealed by the golden proportion. The number $\sigma = \frac{1+\sqrt{5}}{2}$ which is the real positive root of the equation

$$x^2 - x - 1 = 0,$$

is the golden proportion.

In [16], golden Riemannian manifolds were introduced by M. Crasmereanu and C.E. Hretcanu (see also [17, 18]). In [2], the authors investigate the integrability of such manifolds. Also, the constancy of maps between golden Riemannian manifolds was introduced in [7]. On a golden Riemannian manifold, totally umbilical semi-invariant submanifolds were studied in [9]. Moreover, some types of lightlike submanifolds of metallic semi-Riemannian manifolds were examined in [10].

A golden structure P on a semi-Riemannian manifold $\bar{M}$ is defined as

$$P^2 - P - I = 0,$$

and if

$$\bar{g}(PU, W) = \bar{g}(U, PW),$$

then the semi-Riemannian metric is called P-compatible and $(\bar{M}, \bar{g}, P)$ is called a golden semi-Riemannian manifold [19].

In differential geometry, lightlike submanifolds of semi-Riemannian manifolds are an important research topic. This theory is developed by [12] (see also [15]). Then many authors studied lightlike submanifolds on different spaces ([4, 5, 13, 14, 21]). Moreover, on golden semi-Riemannian manifolds, lightlike submanifolds have been reported by many mathematicians (see [8, 11, 20]).

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In [6], B.Y. Chen defined slant immersions in complex geometry. In [3], A. Lotta introduced the concept of slant immersions of a Riemannian manifold into an almost contact metric manifold. Also, in [1] bi-slant submanifolds with the notion of pseudo-slant submanifolds were introduced.

The purpose of this article is to study screen pseudo-slant lightlike submanifolds of a golden semi-Riemannian manifold. The article is arranged as follows. In Section 2 there are some basic definitions about lightlike submanifolds and golden semi-Riemannian manifolds. In Section 3, we give the definition of a screen pseudo-slant lightlike submanifold and obtain a non-trivial example. In the last section, we obtain main results of our paper.

2. Preliminaries

A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called a lightlike submanifold [12], if the metric g induced from $\bar{g}$ is degenerate and the radical distribution $RadTM$ is of rank r, $1 \leq r \leq m$. Assume that $S(TM)$ is a screen distribution which is a semi-Riemannian orthogonal complementary distribution of $RadTM$, that is,

$$TM = S(TM) \perp RadTM. \tag{2.1}$$

Considering a screen transversal vector bundle $S(TM^\perp)$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^\perp$. For every local basis $\{E_i\}$ of $RadTM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^\perp)$ in $(S(TM^\perp))^\perp$ such that

$$\bar{g}(N_i, E_i) = \delta_{ij} \quad \text{and} \quad \bar{g}(N_i, N_j) = 0,$$

it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [12].

Assume that $tr(TM)$ is a complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$. Then, we get

$$tr(TM) = ltrTM \perp S(TM^\perp), \tag{2.2}$$

$$T\bar{M}|_M = TM \oplus tr(TM), \tag{2.3}$$

which gives

$$T\bar{M} = S(TM) \perp \{RadTM \oplus ltr(TM)\} \perp S(TM^\perp). \tag{2.4}$$

The Gauss and Weingarten formulae are given as

$$\bar{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.5}$$

$$\bar{\nabla}_U N = -A_N U + \nabla^t_U N, \tag{2.6}$$

for all $U, V \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$, where $\nabla_U V$, $A_N U \in \Gamma(TM)$ and $h(U, V)$, $\nabla^\perp_U N \in \Gamma(tr(TM))$. $\bar{\nabla}$, ∇, and ∇^t are linear connections on $T\bar{M}$, TM, and $tr(TM)$, respectively.

In view of (2.5) and (2.6), for all $U, V \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$, and $W \in \Gamma(S(TM^\perp))$, we get

$$\bar{\nabla}_U V = \nabla_U V + h^l(U, V) + h^s(U, V), \tag{2.7}$$

$$\bar{\nabla}_U N = -A_N U + \nabla^l_U N + D^s(U, N), \tag{2.8}$$

$$\bar{\nabla}_U W = -A_W U + \nabla^s_U W + D^l(U, W), \tag{2.9}$$

where

$$h^l(U, V) = L(h^l(U, V)), \qquad h^s(U, V) = S(h^s(U, V)),$$

$$D^l(U, W) = L(\nabla^t_U W), \qquad D^s(U, N) = S(\nabla^t_U N),$$

where L and S are the projection morphism of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$, respectively. ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^\perp)$ called the lightlike connection and screen transversal connection on M, respectively.

By the use of (2.5), (2.7)∼(2.9), and metric connection $\bar{\nabla}$, we arrive at

$$\bar{g}(h^s(U,V),W)+\bar{g}(V,D^l(U,W))=\bar{g}(A_WU,V), \tag{2.10}$$

$$\bar{g}(D^s(U,N),W)=\bar{g}(N,A_WU). \tag{2.11}$$

We will denote the projection of TM on $S(TM)$ by P. For any $U,V\in\Gamma(TM)$ and $E\in\Gamma(RadTM)$, we get

$$\nabla_U PV=\nabla^*_U PV+h^*(U,PV), \tag{2.12}$$

$$\nabla_U E=-A^*_E U+\nabla^{*t}_U E. \tag{2.13}$$

From (2.12) with (2.13), we have

$$\bar{g}(h^l(U,PV),E)=g(A^*_EU,PV), \tag{2.14}$$

$$\bar{g}(h^*(U,PV),N)=g(A_NU,PV), \tag{2.15}$$

$$\bar{g}(h^l(U,E),E)=0, \quad A^*_E E=0. \tag{2.16}$$

By using (2.7), we find

$$(\nabla_U\bar{g})(V,Z)=\bar{g}(h^l(U,V),Z)+\bar{g}(h^l(U,Z),V). \tag{2.17}$$

So, ∇ is not a metric connection.

Let $(\bar{M},\bar{g})$ be a semi-Riemannian manifold and P be golden structure on $\bar{M}$. If

$$\bar{g}(PU,V)=\bar{g}(U,PV), \tag{2.18}$$

holds, then $(\bar{M},\bar{g},P)$ called a golden semi-Riemannian manifold. Equation (2.18) is equivalent to

$$\bar{g}(PU,PV)=\bar{g}(PU,V)+\bar{g}(U,V). \tag{2.19}$$

Throughout this article, we suppose that $\bar{\nabla}P=0$.

3. Screen pseudo-slant lightlike submanifolds

Definition 3.1. Let M be a lightlike submanifold of a golden semi-Riemannian manifold $\bar{M}$. If the following conditions are satisfied then M is called a screen pseudo-slant submanifold of a golden semi-Riemannian manifold $\bar{M}$.

i) The radical distribution RadTM is an invariant distribution with respect to P, i.e.,

$$P(RadTM)=RadTM.$$

ii) There exist $\mathring{D}$ and $\hat{D}$ non-degenerate orthogonal distributions on M such that

$$S(TM)=\mathring{D}\perp\hat{D}.$$

iii) The distribution $\mathring{D}$ is anti-invariant, i.e.,

$$P(\mathring{D})\subset S(TM^\perp).$$

iv) The distribution $\hat{D}$ is slant with angle θ $(\neq\frac{\pi}{2})$, i.e., for each $x\in M$ and each non-zero vector $X\in(\hat{D})_x$, the angle θ between PX and the vector subspace $(\hat{D})_x$ is a constant $(\neq\frac{\pi}{2})$, which is independent of the choice of $x\in M$ and $X\in(\hat{D})_x$.

This constant angle θ is called the slant angle of distribution $\hat{D}$. A screen pseudo-slant lightlike submanifold is said to be proper if $\mathring{D}\neq\{0\}$, $\hat{D}\neq\{0\}$ and $\theta\neq 0$.

In view of above definition, we arrive at

$$TM=RadTM\perp\mathring{D}\perp\hat{D}. \tag{3.1}$$

Example 3.2. Let $(\mathbb{R}_5^{10}, \bar{g})$ be a semi-Riemannian manifold with signature $(-, ..., -, +, ..., +)$ and $(x_1, x_2, ..., x_{10})$ be standard coordinate system of $\mathbb{R}_5^{10}$.

Taking

$$P(x_1, ..., x_{10}) = \begin{pmatrix} \sigma x_1, (1-\sigma)x_2, (1-\sigma)x_3, (1-\sigma)x_4, (1-\sigma)x_5, \\ (1-\sigma)x_6, \sigma x_7, (1-\sigma)x_8, (1-\sigma)x_9, (1-\sigma)x_{10} \end{pmatrix}$$

then P is a golden structure on $\mathbb{R}_5^{10}$.

Assume that M is a submanifold of $\mathbb{R}_5^{10}$ given by

$$x_1 = x_7 = \omega^1, \quad x_2 = \sigma \cos\alpha\omega^5 + \cos\alpha\omega^6, \quad x_3 = \sigma\omega^2,$$

$$x_4 = \sigma\omega^3, \quad x_5 = \sigma\omega^4, \quad x_6 = \sigma \sin\alpha\omega^5 + \sin\alpha\omega^6,$$

$$x_8 = (1-\sigma)\omega^2, \quad x_9 = (1-\sigma)\omega^3, \quad x_{10} = (1-\sigma)\omega^4.$$

Then $TM = Sp\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$$Z_1 = \tfrac{\partial}{\partial x_1} + \tfrac{\partial}{\partial x_7}, \quad Z_2 = \sigma\tfrac{\partial}{\partial x_3} + (1-\sigma)\tfrac{\partial}{\partial x_8},$$

$$Z_3 = \sigma\tfrac{\partial}{\partial x_4} + (1-\sigma)\tfrac{\partial}{\partial x_9}, \quad Z_4 = \sigma\tfrac{\partial}{\partial x_5} + (1-\sigma)\tfrac{\partial}{\partial x_{10}},$$

$$Z_5 = \sigma\cos\alpha\tfrac{\partial}{\partial x_2} + \sigma\sin\alpha\tfrac{\partial}{\partial x_6}, \quad Z_6 = \cos\alpha\tfrac{\partial}{\partial x_2} + \sin\alpha\tfrac{\partial}{\partial x_6}.$$

Thus, $RadTM = Sp\{Z_1\}$ and $S(TM) = Sp\{Z_2, Z_3, Z_4, Z_5, Z_6\}$ and $ltr(TM)$ is spanned by

$$N = -\frac{1}{2}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_7}).$$

$S(TM^{\perp})$ is spanned by

$$W_1 = -\tfrac{\partial}{\partial x_3} + (1-\sigma)^2\tfrac{\partial}{\partial x_8}, \quad W_2 = -\tfrac{\partial}{\partial x_4} + (1-\sigma)^2\tfrac{\partial}{\partial x_9},$$

$$W_3 = -\tfrac{\partial}{\partial x_5} + (1-\sigma)^2\tfrac{\partial}{\partial x_{10}}.$$

It follows that $PZ_1 = \sigma Z_1$, which implies that $RadTM$ is invariant. Also, we can state that $\mathring{D} = \{Z_2, Z_3, Z_4\}$ such that $PZ_2 = W_1$, $PZ_3 = W_2$, $PZ_4 = W_3$, which gives $\mathring{D}$ is anti-invariant and $\hat{D} = \{Z_5, Z_6\}$ is a slant distribution with slant angle 2α. Therefore M is a screen pseudo-slant lightlike submanifold of $\mathbb{R}_5^{10}$.

For any vector field $U \in \Gamma(TM)$ tangent to M, we take

$$PU = RU + TU, \tag{3.2}$$

where RU and TU are the tangential and the transversal part of PU, respectively. We denote the projections on $RadTM$, $\mathring{D}$ and $\hat{D}$ in TM by R_1, R_2, and R_3 , respectively. Similarly, we show that the projections of $tr(TM)$ on $ltr(TM)$, $P(\mathring{D})$ and $D^{\prime}$ by Q_1, Q_2, and Q_3, respectively, where $D^{\prime}$ is a non-degenerate orthogonal complementary subbundle of $P(\mathring{D})$ in $S(TM^{\perp})$. So, for any $U \in \Gamma(TM)$, we have

$$U = R_1U + R_2U + R_3U. \tag{3.3}$$

Applying P to (3.3), we get

$$PU = PR_1U + PR_2U + PR_3U,$$

which yields

$$PU = PR_1U + PR_2U + wR_3U + TR_3U, \tag{3.4}$$

where wR_3U and TR_3U denotes the tangential and the transversal component of PR_3U. So, we arrive at

$$\begin{array}{ll} PR_1U \in \Gamma(RadTM), & PR_2U \in \Gamma(P(\mathring{D})) \subset S(TM^{\perp}), \\ wR_3U \in \Gamma(\hat{D}), & TR_3U \in \Gamma(D^{\prime}). \end{array}$$

Also, for any $W \in \Gamma(tr(TM))$, we get

$$W = f_1W + f_2W + f_3W. \tag{3.5}$$

Applying P to (3.5), we have

$$PW = Pf_1W + Pf_2W + Pf_3W,$$

which yields

$$PW = Pf_1W + Pf_2W + Bf_3W + Cf_3W, \tag{3.6}$$

where Bf_3W and Cf_3W denotes the tangential and the transversal component of Pf_3W. Thus we get

$$\begin{aligned} Pf_1W &\in \Gamma(ltr(TM)), \quad Pf_2W \in \Gamma(\hat{D}), \\ Bf_3W &\in \Gamma(\hat{D}), \quad\quad\quad\;\; Cf_3W \in \Gamma(\acute{D}). \end{aligned}$$

Now, using (3.4) and (3.6) with (2.7)∼(2.9), we obtain following;

$$\begin{aligned} \nabla^{*t}_U PR_1V + R_1(\nabla_U wR_3V) &= R_1(A_{TR_3V}U) + R_1(A_{PR_2V}U) \\ &\quad + PR_1\nabla_U V, \end{aligned} \tag{3.7}$$

$$\begin{aligned} R_2(A^*_{PR_1V}U) + R_2(A^*_{PR_2V}U) + R_2(A^*_{TR_3V}U) &= R_2(\nabla_U wR_3V) \\ &\quad - Pf_1h^s(U,V), \end{aligned} \tag{3.8}$$

$$\begin{aligned} R_3(A^*_{PR_1V}U) + R_3(A^*_{PR_2V}U) + R_3(A^*_{TR_3V}U) &= R_3(\nabla_U wR_3V) \\ &\quad - wR_3(\nabla_U V) - Bf_3h^s(U,V), \end{aligned} \tag{3.9}$$

$$h^l(U, PR_1V) + D^l(U, PR_2V) + h^l(U, wR_3V) + D^l(U, TR_3V) = Ph^l(U,V), \tag{3.10}$$

$$\begin{aligned} f_2\nabla^s_U PR_2V + f_2\nabla^s_U TR_3V &= PR_2\nabla_U V - f_2h^s(U, PR_1V) \\ &\quad - f_2h^s(U, wR_3V), \end{aligned} \tag{3.11}$$

$$\begin{aligned} f_3\nabla^s_U PR_2V + f_3\nabla^s_U TR_3V - TR_3\nabla_U V &= Cf_3h^s(U,V) - f_3h^s(U, wR_3V) \\ &\quad - f_3h^s(U, PR_1V). \end{aligned} \tag{3.12}$$

4. Main theorems

In this section, we give the main results of our article.

Theorem 4.1. *Let M be a screen pseudo-slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{M}$. Then the distribution $RadTM$ is integrable if and only if, for all $E_1, E_2 \in \Gamma(RadTM)$,*

i) $f_2h^s(E_2, PR_1E_1) = f_2h^s(E_1, PR_1E_2)$,
ii) $f_3h^s(E_2, PR_1E_1) = f_3h^s(E_1, PR_1E_2)$,
iii) $R_3A^*_{PR_1E_1}E_2 = R_3A^*_{PR_1E_2}E_1$.

Proof. Assume that M is a screen pseudo-slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{M}$ and $E_1, E_2 \in \Gamma(RadTM)$.

i) In view of (3.11), we get

$$PR_2\nabla_{E_1}E_2 = f_2h^s(E_1, PR_1E_2). \tag{4.1}$$

Interchanging E_1 to E_2, we obtain

$$PR_2\nabla_{E_2}E_1 = f_2h^s(E_2, PR_1E_1). \tag{4.2}$$

From (4.1) and (4.2), we obtain

$$PR_2[E_1, E_2] = f_2(h^s(E_1, PR_1E_2) - h^s(E_2, PR_1E_1)).$$

ii) Similarly, by the use of (3.12), we have

$$f_3h^s(E_1, PR_1E_2) = Cf_3h^s(E_1, E_2) + TR_3\nabla_{E_1}E_2, \tag{4.3}$$

from which we get

$$f_3h^s(E_2, PR_1E_1) = Cf_3h^s(E_2, E_1) + TR_3\nabla_{E_2}E_1. \tag{4.4}$$

From (4.3) and (4.4), we find

$$TR_3[E_1, E_2] = f_3(h^s(E_1, PR_1E_2) - h^s(E_2, PR_1E_1)).$$

iii) Moreover, from (3.9), we have

$$R_3(A^*_{PR_1E_1}E_2) = -wR_3\nabla_{E_1}E_2 - Bf_3h^s(E_1, E_2), \tag{4.5}$$

from which, we get

$$R_3A^*_{PR_1E_2}E_1 = -wR_3\nabla_{E_2}E_1 - Bf_3h^s(E_2, E_1). \tag{4.6}$$

From (4.5) and (4.6), we obtain

$$wR_3[E_1, E_2] = R_3(A^*_{PR_1E_1}E_2) - R_3(A^*_{PR_1E_2}E_1).$$

So, we arrive at the required equations. □

Theorem 4.2. *Let M be a screen pseudo-slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{M}$. Then the distribution $\mathring{D}$ is integrable if and only if, for all $U, V \in \Gamma(\mathring{D})$,*

i) $R_1(A_{PR_2U}V) = R_1(A_{PR_2V}U)$ *and* $R_3(A_{PR_2U}V) = R_3(A_{PR_2V}U)$,

ii) $f_3(\nabla^s_V PR_2U) = f_3(\nabla^s_U PR_2V)$.

Proof. Assume that M is a screen pseudo-slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{M}$ and $U, V \in \Gamma(\mathring{D})$.

i) By the use of (3.7), we get

$$R_1(A_{PR_2U}V) = -PR_1\nabla_UV. \tag{4.7}$$

Interchanging U and V, we have

$$R_1(A_{PR_2V}U) = -PR_1\nabla_VU. \tag{4.8}$$

From (4.1) and (4.2), we obtain

$$R_1(A_{PR_2U}V) - R_1(A_{PR_2V}U) = PR_1[U, V].$$

In view of (3.9), we get

$$R_3(A_{PR_2U}V) + Bf_3h^s(U, V) = -wR_3(\nabla_UV),$$

from which we have

$$R_3(A_{PR_2U}V) - R_3(A_{PR_2V}U) = wR_3[U, V].$$

ii) Moreover, using (3.12), we have

$$f_3\nabla^s_U PR_2V + Cf_3h^s(U, V) = TR_3\nabla_UV,$$

from which, we arrive at

$$f_3\nabla^s_U PR_2V - f_3\nabla^s_V PR_2U = TR_3[U, V].$$

Thus, we obtain the desired results. □

Theorem 4.3. *Let M be a screen pseudo-slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{M}$. Then the distribution $\hat{D}$ is integrable if and only if, for all $U, V \in \Gamma(\hat{D})$,*

i) $R_1(\nabla_UwR_3V - \nabla_VwR_3U) = R_1(A_{TR_3V}U - A_{TR_3U}V)$,

ii) $f_2(\nabla^s_U TR_3V - \nabla^s_V TR_3U) = f_2(h^s(V, wR_3U) - h^s(U, wR_3V))$.

Proof. Assume that M is a screen pseudo-slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{M}$ and $U, V \in \Gamma(\hat{D})$.

i) If we consider (3.7), we have

$$R_1(\nabla_U wR_3V) = R_1(A_{TR_3V}U) + PR_1\nabla_U V. \tag{4.9}$$

Interchanging U and V, we have

$$R_1(\nabla_V wR_3U) = R_1(A_{TR_3U}V) + PR_1\nabla_V U. \tag{4.10}$$

In view of (4.9) with (4.10), we find

$$R_1(\nabla_U wR_3V - \nabla_V wR_3U) - R_1(A_{TR_3V}U - A_{TR_3U}V) = PR_1[U, V].$$

ii) Also, using (3.11), we get

$$f_2\nabla^s_U TR_3V + f_2h^s(U, wR_3V) = PR_2\nabla_U V,$$

with

$$f_2\nabla^s_V TR_3U + f_2h^s(V, wR_3U) = PR_2\nabla_V U.$$

From last two equations, we arrive at

$$f_2(\nabla^s_U TR_3V - \nabla^s_V TR_3U) + f_2(h^s(U, wR_3V) - h^s(V, wR_3U)) = PR_2[U, V].$$

So, we obtain the required results. □

Now, we find some conditions for foliations determined by distributions to be totally geodesic.

Theorem 4.4. *Let M be a screen pseudo-slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{M}$. Then the distribution $RadTM$ defines a totally geodesic foliation if and only if*

$$\bar{g}(D^l(E_1, R_2Z) + D^l(E_1, TR_3Z), PE_2) = -\bar{g}(h^l(E_1, wR_3Z), PE_2),$$

for all $E_1, E_2 \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$.

Proof. Let M be a screen pseudo-slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{M}$. We know that $RadTM$ defines a totally geodesic foliation if and only if

$$\nabla_{E_1}E_2 \in \Gamma(RadTM),$$

for all $E_1, E_2 \in \Gamma(RadTM)$.

Because of $\bar{\nabla}$ is a metric connection, by the use of (2.7), (2.19) with (3.4), we have

$$\bar{g}(\nabla_{E_1}E_2, Z) = \bar{g}(P\bar{\nabla}_{E_1}E_2, PZ) - \bar{g}(P\bar{\nabla}_{E_1}E_2, Z).$$

So, we get

$$\begin{aligned}
\bar{g}(\nabla_{E_1}E_2, Z) &= \bar{g}(\bar{\nabla}_{E_1}PE_2, PZ) - \bar{g}(\bar{\nabla}_{E_1}PE_2, Z) \\
&= \bar{g}(\bar{\nabla}_{E_1}PR_2Z, PE_2) + \bar{g}(\bar{\nabla}_{E_1}wR_3Z, PE_2) \\
&\quad +\bar{g}(\bar{\nabla}_{E_1}TR_3Z, PE_2) - \bar{g}(\bar{\nabla}_{E_1}PR_2Z, E_2) \\
&\quad -\bar{g}(\bar{\nabla}_{E_1}wR_3Z, PE_2) - \bar{g}(\bar{\nabla}_{E_1}TR_3Z, E_2) \\
&= \bar{g}(D^l(E_1, PR_2Z), PE_2) + \bar{g}(h^l(E_1, wR_3Z), PE_2) \\
&\quad +\bar{g}(D^l(E_1, TR_3Z), PE_2) - \bar{g}(D^l(E_1, PR_2Z), E_2) \\
&\quad -\bar{g}(h^l(E_1, wR_3Z), E_2) - \bar{g}(D^l(E_1, TR_3Z), E_2).
\end{aligned}$$

In view of last equation, we get the proof of theorem. □

Theorem 4.5. *Let M be a screen pseudo-slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{M}$. Then the distribution $\mathring{D}$ defines a totally geodesic foliation if and only if*

i) $\bar{g}(h^s(U, wZ), PV) = -\bar{g}(\nabla^s_U TZ, PV)$, *and* $h^s(U, Z) \in \Gamma(\acute{D})$, *ii)* $D^s(U, PN)$ *has no component in* $P(\mathring{D})$ *and* $D^s(U, N) \in \Gamma(\acute{D})$,

for all $U, V \in \Gamma(\mathring{D})$ *and* $Z \in \Gamma(\hat{D})$ *and* $N \in \Gamma(ltr(TM))$.

Proof. Let M be a screen pseudo-slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{M}$. We know that $\mathring{D}$ defines a totally geodesic foliation if and only if

$$\nabla_U V \in \Gamma(\mathring{D}),$$

for all $U, V \in \Gamma(\mathring{D})$.

By the use of (2.7) with (2.19), we have

$$\begin{aligned} \bar{g}(\nabla_U V, Z) &= -\bar{g}(P\nabla_U Z, PV) + \bar{g}(\nabla_U Z, PV) \\ &= -\bar{g}(\bar{\nabla}_U PZ, PV) + \bar{g}(\bar{\nabla}_U Z, PV) \\ &= \bar{g}(h^s(U, wZ), PV) + \bar{g}(\nabla^s_U TZ, PV) \\ &\quad +\bar{g}(h^s(U, Z), PV), \end{aligned}$$

which gives (i).

Also, from (2.7) with (2.19), we get

$$\begin{aligned} \bar{g}(\nabla_U V, Z) &= -\bar{g}(\bar{\nabla}_U PN, PV) + \bar{g}(\bar{\nabla}_U N, PV) \\ &= -\bar{g}(D^s(U, PN), PV) + \bar{g}(D^s(U, N), PV), \end{aligned}$$

which implies (ii). □

Theorem 4.6. *Let* M *be a screen pseudo-slant lightlike submanifold of a golden semi-Riemannian manifold* $\bar{M}$. *Then the distribution* $\hat{D}$ *defines a totally geodesic foliation if and only if*

i) $A_{PZ}U$ *has no component in* $\hat{D}$ *and* $\nabla^s_U PZ \in \Gamma(\mathring{D})$,

ii) $A_{PN}U$ *has no component in* $\hat{D}$ *and* $D^s(U, PN) \in \Gamma(\mathring{D})$,

for all $U, V \in \Gamma(\hat{D})$ *and* $Z \in \Gamma(\mathring{D})$ *and* $N \in \Gamma(ltr(TM))$.

Proof. Let M be a screen pseudo-slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{M}$. We know that $\hat{D}$ defines a totally geodesic foliation if and only if

$$\nabla_U V \in \Gamma(\hat{D}),$$

for all $U, V \in \Gamma(\hat{D})$.

In view of (2.7) with (2.19), we have

$$\begin{aligned} \bar{g}(\nabla_U V, Z) &= -\bar{g}(\bar{\nabla}_U PZ, PV) + \bar{g}(\bar{\nabla}_U PZ, V) \\ &= \bar{g}(A_{PZ}U, wV) + \bar{g}(\nabla^s_U PZ, TV) \\ &\quad +\bar{g}(A_{PZ}U, V), \end{aligned}$$

which yields (i).

On the other hand, using (2.7) with (2.19), we get

$$\begin{aligned} \bar{g}(\nabla_U V, Z) &= -\bar{g}(\bar{\nabla}_U PN, PV) + \bar{g}(\bar{\nabla}_U N, PV) \\ &= -\bar{g}(A_{PN}U, wV) + \bar{g}(D^s(U, PN), TV) \\ &\quad -\bar{g}(A_{PN}U, V), \end{aligned}$$

which implies (ii). □

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