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Research Article

Embedding the weighted space $Hv_0(G,E)$ of holomorphic functions into the sequence space $c_0(E)$

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Abstract

We embed almost isometrically the generalized weighted space $Hv_0(G,E)$ of holomorphic functions on an open subset G of $\mathbb{C}^N$ with values in a Banach space E, into $c_0(E)$, the space of all null sequences in E, where v is an operator-valued continuous function on G vanishing nowhere. This extends and generalizes some known results in the literature. We then deduce the non 1-Hyers-Rassias stability of the isometry functional equation in the framework of Banach spaces.

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1. Introduction

A interesting issue when studying Banach spaces is whether such a space can be embedded isometrically into a simpler Banach space. Such a problem has been considered by several authors, especially in weighted spaces of holomorphic functions on an open subset of $\mathbb{C}$ [2, 3, 9–11].

The first author who dealt with embedding weighted spaces of holomorphic functions on an open subset of $\mathbb{C}$ into sequence spaces seems to be W. Lusky [9]. There, the author showed that, whenever G is the unit open disc D of $\mathbb{C}$ and v is a radial (i.e. $v(z) = v(|z|)$, $z \in D$) strictly positive continuous function on D, the Banach space $Hv_0(D)$ of all holomorphic functions f on D such that $v|f|$ vanishes at infinity, endowed with the weighted sup-norm $\|\cdot\|_v$, is always isomorphic to a subspace of c_0. He then showed in [10] that there are weights v such that $Hv_0(D)$ is not isomorphic to the whole c_0. Actually, as Lusky showed in [11], there are exactly two situations in such a case: either $Hv_0(D)$ is isomorphic to ℓ_∞ or it is isomorphic to the Hardy space $H_\infty \subset c_0$. He even gave instances where each situation occurs.

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Concerning the case of several variables, J. Bonet and E. Wolf extended in [2] the result of Lusky to the case where G is an arbitrary open subset of $\mathbb{C}^N$, N being a positive integer, without any further condition on the weight v. They showed that if v is any strictly positive continuous function on a nonempty open set $G \subset \mathbb{C}^N$, then $Hv_0(G)$ is almost isometrically isomorphic to a subspace of c_0. This means that, for every $\varepsilon \in]0,1[$, there is an isomorphism T from $Hv_0(G)$ into c_0 such that:

$$(1-\varepsilon)\|f\|_v \leq \|T(f)\|_{c_0} \leq \|f\|_v, \quad (\forall f \in Hv_0(G)).$$

This seems to be the maximum one can obtain in general since, in [3], C. Boyd and P. Rueda showed that, whenever $G \subset \mathbb{C}^N$ is balanced and v is radial, the isomorphism of $Hv_0(G)$ into c_0 cannot be an isometry.

Recently, C. Shekhar and B. S. Komal [14] and subsequently M. Klilou and L. Oubbi [7] introduced systems V of weights with values in the set of positive operators on a Hilbert space H. They then studied some questions concerning multiplication operators in the corresponding weighted spaces of continuous functions $CV(G,H)$. This study has been enlarged to weights with values in continuous operators on a normed space [8].

In this note, we deal with the question whether for a nonempty open subset G of $\mathbb{C}^N$, a Banach space E, and a continuous mapping v from G into the algebra $B(E)$ of bounded operators on E, the weighted space $Hv_0(G,E)$ can be embedded into the space $c_0(E)$ of all null sequences of E. We mainly show that, if v is continuous with respect to the norm topology on $B(E)$ and takes values in the bounded below operators on E, then the Banach space $Hv_0(G,E)$, endowed with the weighted sup-norm

$$\|f\|_v := \sup\{\|v(z)(f(z))\|, z \in G\},$$

is almost isometrically isomorphic to a closed subspace of the space $c_0(E)$. This extends and generalizes the result, alluded to above, of J. Bonet and E. Wolf [2].

We obtain as an application, the non 1-Hyers-Rassias stability of the isometry functional equation $\|f(x)\| = \|x\|$ between Banach spaces.

2. Preliminaries

Let N be a positive integer, G a nonempty open subset of $\mathbb{C}^N$, and $(E, \|\ \|)$ a Banach space. We write z for $z = (z_1, \ldots, z_N) \in G$ and $z_j = x_j + iy_j$ for $j = 1, \ldots, N$. We will denote by $\mathbb{N}$ the set of all non-negative integers, by $\mathbb{N}^*$ the set $\mathbb{N} \setminus \{0\}$, and by $c_0(E)$ the linear space of all null sequences of E. The space $c_0(E)$ will be endowed with its natural sup-norm.

We first recall some facts related to holomorphic functions. We refer to [6] and [13] for ample details.

Definition 2.1 ([13])**.** A function $f : G \to \mathbb{C}$ is said to be holomorphic in G provided
1. f is continuous (i.e., $f \in C(G)$), and
2. f is holomorphic in each variable separately.

If f is continuously differentiable in the variables x_j and y_j, $j = 1, \ldots, N$, it is said to be holomorphic in G in the Cauchy-Riemann sense (see [6, Definition 2.1.1]) if

$$\frac{\partial f}{\partial \overline{z}_j} = 0, \quad (1 \leq j \leq N),$$

in G, where

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

The following theorem is due to Hartogs [6, Theorem 2.2.8].

Theorem 2.2. *Let f be a function from G to $\mathbb{C}$. The following properties are equivalent:*
1. f is holomorphic in G.
2. f is holomorphic in the Cauchy-Riemann sense.

Theorem 2.3 ([12, p. 400, Theorem 8])**.** *Let f be a function from G into E. The following properties are equivalent:*
1. The $\mathbb{C}$-valued function $\varphi \circ f$ is holomorphic in G for each φ in the topological dual E' of E.
2. For every $w \in G$, there exists a neighborhood U of w and elements $x_\alpha \in E$ with $\alpha \in \mathbb{N}^N$ such that $f(z) = \sum_{\alpha \in \mathbb{N}^N} x_\alpha (z-w)^\alpha$.
3. f is holomorphic in each variable separately in the sense described in 1.

We will denote by $H(G,E)$ the linear space of all E-valued functions on G satisfying one of (and then all) the assertions in Theorem 2.3, while $C(G,E)$ will denote the space of all continuous functions from G into E.

For $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, denote $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \ldots \alpha_N!$, and $z^\alpha = z_1^{\alpha_1} \ldots z_N^{\alpha_N}$. For $w \in \mathbb{C}^N$, we will let $D(w,r)$ denote the polydisc $D(w,r) := \{z \in \mathbb{C}^N : |z_k - w_k| \leq r, k = 1, \ldots, N\}$. We then have the Cauchy integral formula [12, p. 400]. Let $f \in H(G,E)$, $w \in G$, and $r > 0$ such that $D(w,r) \subset G$. Then

$$f(w) = \frac{1}{(2\pi i)^N} \int_{\partial D(w,r)} \frac{f(z)}{(z_1 - w_1) \ldots (z_N - w_N)} dz_1 \ldots dz_N. \tag{2.1}$$

Therefore, for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ with $|\alpha| = 1$,

$$D^\alpha f(w) = \frac{1}{(2\pi i)^N} \int_{\partial D(w,r)} \frac{f(z)}{(z-w)^{\alpha+1}} dz_1 \ldots dz_N.$$

We will denote by $B(E)$ the Banach algebra of all bounded linear operators from E into itself. The strong operator (resp. the norm) topology on $B(E)$ will be denoted by β (resp. σ).

Recall that a linear mapping T from the Banach space E into another one F is said to be bounded below if there exists $r > 0$ such that $r\|x\|_E \leq \|T(x)\|_F$, for every $x \in E$. We will denote by $\mathcal{L}_{bb}(E)$ the subset of $B(E)$ consisting of all continuous and bounded below operators.

A mapping $\mu : G \to E$ is said to vanish at infinity if for every $\varepsilon > 0$, there is a compact subset $K_\varepsilon \subset G$ such that $\|\mu(z)\| < \varepsilon$ for all $z \notin K_\varepsilon$.

Here we consider generalized Nachbin families consisting of a single weight. Unlike [14] and [7], we no more consider Hilbert spaces but arbitrary Banach spaces.

Definition 2.4. A generalized weight on G is any β-continuous mapping $v : G \to B(E)$ such that $v(z)$ is injective for every z in some dense subset G_0 of G. The weight v is said to be equibounded below on a subset A of G if there is $r = r_A > 0$ such that $r\|x\| \leq \|v(z)x\|$ for every $z \in A$ and every $x \in E$.

With a generalized weight v on G are associated the following weighted spaces :

$$Cv_0(G,E) := \{f \in C(G,E), vf : z \longmapsto v(z)(f(z)) \text{ vanishes at infinity on } G\}$$
$$Hv_0(G,E) := \{f \in H(G,E), vf : z \longmapsto v(z)(f(z)) \text{ vanishes at infinity on } G\}.$$

From now on, we will denote the mapping $z \mapsto v(z)(f(z))$ by vf, $f \in Hv_0(G,E)$. Since vf is bounded on G for every $f \in Hv_0(G,E)$, the quantity

$$\|f\|_v = \sup_{z \in G} \|v(z)(f(z))\|$$

defines a semi-norm on $Hv_0(G,E)$. Actually $\|f\|_v$ turns out to be a norm on $Hv_0(G,E)$, because for every nonzero f in $Hv_0(G,E)$, there is $z_0 \in G$, such that $f(z_0) \neq 0$. By the continuity of f, there exists a neighborhood Ω of z_0 and $\varepsilon > 0$ such that $\|f(z)\| > \epsilon$ for every $z \in \Omega$. But the density of G_0 in G yields some $z_1 \in G_0$ so that $\|f(z_1)\| > \epsilon$. Now, $v(z_1)$ is injective, then $v(z_1)(f(z_1)) \neq 0$. Hence $\|f\|_v \neq 0$ and $(Hv_0(G,E), \| \ \|_v)$ is a normed space. From now on, $Hv_0(G,E)$ will be endowed with this norm.

Whenever u is a strictly positive continuous function on G, if we consider on E the operator $T_z : x \mapsto u(z)x$, then the mapping $v : z \mapsto T_z$ is a generalized weight on G and the generalized weighted space $Hv_0(G,E)$ is nothing but the usual weighted space $Hu_0(G,E)$ algebraically and topologically. Obviously, if E is the complex field, $Hv_0(G,E)$ is nothing but $Hv_0(G)$. More generally, we have the following.

Example 2.5. Let $u : G \longrightarrow (0,\infty)$ be a continuous mapping vanishing nowhere on G and $T \in B(E)$. If T is injective, then the mapping $v : z \longrightarrow u(z)T$ is a generalized weight on G. Moreover, if T is bounded below, then v is equibounded below on every compact subset of G. Indeed, let K be such a compact set. Since T is bounded below, there is $r > 0$ such that $\|T(x)\| \geq r\|x\|$ for all $x \in E$. Therefore, for each $z \in K$ and $x \in E$, we have

$$\|v(z)x\| = u(z)\|T(x)\| \geq u(z)r\|x\| \geq [\inf_{z\in K} u(z)]r\|x\|,$$

whence the result.

Example 2.6. Let v be a generalized weight on G and ϱ be the real function assigning to any $z \in G$ the minimum modulus $\mu(v(z))$ of $v(z)$ [5], where

$$\mu(v(z)) := \inf\{\|v(z)x\|, \|x\| = 1\}.$$

If ϱ is lower semi-continuous and does not vanish on G, then v is equibounded below on each compact subset K of G. Indeed, if $r_K = \inf\{\varrho(z), z \in K\}$, then $\|v(z)\frac{x}{\|x\|}\| \geq r_K$ for all $z \in K$ and all $x \in E$, hence $\|v(z)x\| \geq r_K\|x\|$ for all $x \in E$.

Proposition 2.7. *Let $v : G \to B(E)$ be a generalized weight on G. If v is equibounded below on the compact subsets of G, then the space $Hv_0(G,E)$, endowed with the norm $\| \ \|_v$, is a Banach space.*

Proof. The space $Cv_0(G,E)$ is a Banach space by Theorem 3.3 of [7]. Then it is sufficient to prove that $Hv_0(G,E)$ is a closed subspace of $Cv_0(G,E)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $Hv_0(G,E)$ converging to some function f in $Cv_0(G,E)$. Fix a compact $K \subset G$ and $\varepsilon > 0$. Since v is equibounded below on K, there exists $r_K > 0$, such that $\|x\| < r_K\|v(z)(x)\|$, $z \in K$, $x \in E$. But also, there exists $N \in \mathbb{N}$, such that, for all $n \geq N$, $\|f_n - f\|_v < r_K^{-1}\varepsilon$. Therefore, for every $z \in G$, we have

$$\|f_n(z) - f(z)\| < r_K\|v(z)(f_n(z) - f(z))\| \leq r_K\|f_n - f\|_v < \varepsilon.$$

Hence $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on every K. Then f is holomorphic. Since $f \in Cv_0(G,E)$, $f \in Hv_0(G,E)$. □

3. Embedding $Hv_0(G,E)$ into $c_0(E)$

Our main result gives instances where $Hv_0(G,E)$ is almost isometrically isomorphic to a subspace of $c_0(E)$, extending and generalizing a result of [2]. From now on, let us denote by d the sup-norm metric on $\mathbb{C}^N$. This is $d(z,w) := \max_{i=1,\ldots,N} |z_i - w_i|$, $z, w \in \mathbb{C}^N$.

Theorem 3.1. *Let E be a Banach space, G a nonempty open subset of $\mathbb{C}^N$, and $v : G \to B(E)$ a generalized weight. If v is σ-continuous and maps G into $\mathcal{L}_{bb}(E)$, then the space $Hv_0(G,E)$ is almost isometrically isomorphic to a closed subspace of $c_0(E)$.*

Proof. Fix $\varepsilon \in]0,1[$ and consider an exhaustion of G by an increasing sequence $(K_k)_{k\in\mathbb{N}}$ of compact subsets of G. Since v is continuous, $M_k := \sup\{\|v(z)\|_{B(E)},\ z \in K_k\} < +\infty$. Set

$$a_k := \min\left(1, \frac{1}{2}d(K_k, \mathbb{C}^N \setminus K_{k+1})\right).$$

As $v(G) \subset \mathcal{L}_{bb}(E)$, for all $z \in K_k$, there exists $r_z > 0$ such that $r_z\|x\| \le \|v(z)x\|$, for all $x \in E$. By Theorem 2.1 of [1], $\mathcal{L}_{bb}(E)$ is open in $(B(E),\sigma)$. Then, by the σ-continuity of v, there exists a neighborhood U_z of z in G, such that $v(U_z) \subset B(v(z), \frac{r_z}{2}) \cap \mathcal{L}_{bb}(E)$, $B(v(z), \frac{r_z}{2})$ being the ball in $(B(E),\sigma)$ centered at $v(z)$ with radius $\frac{r_z}{2}$. Then for all $w \in U_z$, we have

$$\|v(z)x\| - \|v(w)x - v(z)x\| \le \|v(w)x\|, \quad \forall x \in E.$$

Hence

$$\|v(z)x\| - \|v(z) - v(w)\|\|x\| \le \|v(w)x\|, \quad \forall x \in E.$$

Therefore

$$r_z\|x\| - \frac{r_z}{2}\|x\| = \frac{r_z}{2}\|x\| \le \|v(w)x\|, \quad \forall x \in E.$$

Since K_k is compact, we can find a finite set $\{z_1, \dots, z_n\} \subset K_k$ such that $K_k \subset \cup_{i=1}^n U_{z_i}$. Now, for $r_k = \inf_{i=1,\dots,n} \frac{r_{z_i}}{2}$, we have

$$r_k\|x\| \le \|v(z)x\|, \quad \forall z \in K_k, \quad \forall x \in E.$$

For arbitrary $f \in Hv_0(G,E)$ and $k \in \mathbb{N}$, with $\|f\|_v = 1$, we have

$$1 = \sup_{z\in G}\|v(z)f(z)\| \ge r_k \sup_{z\in K_k}\|f(z)\|.$$

Hence, for each $z \in K_k$, the following inequality holds:

$$\|f(\zeta)\| \le \frac{1}{r_{k+1}}, \qquad \forall \zeta \in D(z, a_k). \tag{3.1}$$

For $\alpha \in \mathbb{N}^N$ with $|\alpha| = 1$, we have

$$\|D^\alpha f(z)\| = \left\| \frac{1}{(2\pi i)^N} \int_{\partial D(z,a_k)} \frac{f(\zeta)}{(\zeta - z)^{\alpha+1}} d\zeta_1 \dots d\zeta_N \right\| \le \frac{1}{r_{k+1}a_k}. \tag{3.2}$$

Whereby

$$\|D^\alpha f(z)\| \le \frac{1}{r_{k+1}a_k}, \qquad \forall z \in K_k. \tag{3.3}$$

If $A_k := K_k \setminus \overset{\circ}{K}_{k-1}$ and $\delta_k > 0$ satisfy

$$\delta_k < \min\left(a_k, \varepsilon\left(\frac{1}{r_k} + \frac{M_{k+1}N}{a_{k+1}r_{k+2}},\right)^{-1}\right), \tag{3.4}$$

then

$$A_k \subset \bigcup_{z\in A_k} \{z' \in G, d(z', z) < \delta_k \text{ and } \|v(z') - v(z)\| < \delta_k\}.$$

By the compactness of A_k, there is a finite subset F_k of A_k such that

$$A_k \subset \bigcup_{z\in F_k} \{z' \in G, d(z', z) < \delta_k \text{ and } \|v(z') - v(z)\| < \delta_k\}.$$

Consequently, for each $z \in A_k$, there is $w \in F_k$ with $d(w,z) < \delta_k$ and $\|v(w) - v(z)\| < \delta_k$. On the other hand, $D(z,\delta_k) \subset D(z,a_k) \subset K_{k+1}$. This implies

$$\begin{aligned} \|v(z)f(z)\| &\le \|v(z)f(z) - v(w)f(z)\| + \|v(w)f(z)\| \\ &\le \|v(z) - v(w)\|\|f(z)\| + \|v(w)(f(z) - f(w))\| + \|v(w)f(w)\|. \end{aligned} \tag{3.5}$$

We then have, denoting $\alpha_i = (0, \dots, 1, 0, \dots)$, where 1 is in the i^{th} place :

$$\begin{aligned}\|f(z) - f(w)\| &= \|f(z_1, \dots, z_N) - f(w_1, \dots, w_N)\| \\ &\leq \|f(z_1, \dots, z_N) - f(w_1, z_2, \dots, z_N) + f(w_1, z_2, \dots, z_N) \\ &\quad - f(w_1, w_2, z_3, \dots, z_N) + \dots + f(w_1, w_2, \dots, w_{N-1}, z_N) - f(w_1, \dots, w_N)\| \\ &\leq \sup_{\zeta \in D(z,\delta_k)} \|D^{\alpha_1} f(\zeta)\| |z_1 - w_1| + \dots + \sup_{\zeta \in D(z,\delta_k)} \|D^{\alpha_N} f(\zeta)\| |z_N - w_N|.\end{aligned}$$

Taking (3.3) into consideration, it follows that

$$\|f(z) - f(w)\| \leq \frac{N\delta_k}{a_{k+1} r_{k+2}}. \tag{3.6}$$

Now, since w belongs to K_{k+1}, it follows from (3.1), (3.4), (3.5), and (3.6) that

$$\sup_{z \in A_k} \|v(z)f(z)\| \leq \varepsilon + \max_{w \in F_k} \|v(w)f(w)\|.$$

Setting $F := \bigcup\{F_k, k \in \mathbb{N}\}$, we conclude

$$1 \leq \varepsilon + \sup_{w \in F} \|v(w)f(w)\|.$$

Denote the elements of F as a sequence $(z_n)_{n\in\mathbb{N}} \subset G$. Then z_n tends to the boundary ∂G of G, i.e., for each $k \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that $z_n \notin K_k$ for every $n > n_0$. Since F does not depend on the function f, the correspondence $g \longmapsto (v(z_n)g(z_n))_{n\in\mathbb{N}}$ defines an operator T from $Hv_0(G, E)$ into $c_0(E)$. Now, if $g \in Hv_0(G, E)$ with $g \neq 0$, we have

$$\left\| \frac{g}{\|g\|_v} \right\|_v = 1 \leq \varepsilon + \left\| T\left(\frac{g}{\|g\|_v}\right) \right\|_{c_0(E)}.$$

Thus

$$(1 - \varepsilon)\|g\|_v \leq \|T(g)\|_{c_0(E)}.$$

Since $\|T(f)\|_{c_0(E)} = \sup\{\|v(w)f(w)\|, w \in F\} \leq \|f\|_v$, we obtain

$$(1 - \varepsilon)\|f\|_v \leq \|T(f)\|_{c_0(E)} \leq \|f\|_v, \quad \forall f \in Hv_0(G, E)$$

showing that T is an almost isometry. □

Remark 3.2. It comes out from the proof of Theorem 3.1 that v is equibounded below on compact subsets of G if and only if its range lies in $\mathcal{L}_{bb}(E)$.

If $u : G \to (0, +\infty)$ is continuous, T is the identity of E, and $v := uT$, as in Example 2.5, we get, as a corollary, the vector-valued version of J. Bonet and E. Wolf's theorem.

Corollary 3.3. *Let E be a Banach space, G a nonempty open subset of $\mathbb{C}^N$, and u a strictly positive and continuous weight on G. Then the space $Hu_0(G, E)$ is isomorphic to a closed subspace of $c_0(E)$. Actually, $Hu_0(G, E)$ embeds almost isometrically into $c_0(E)$.*

In case $E = \mathbb{C}$, Corollary 3.3 is nothing but the result of J. Bonet and E. Wolf [2].

Corollary 3.4. *Let G be an open subset of $\mathbb{C}^{\mathbb{N}}$ and v be a strictly positive and continuous weight on G. Then the space $Hv_0(G)$ embeds almost isometrically into c_0.*

Notice that, for every nonzero $x \in E$ and every generalized weight v on G, the mapping $v_x : z \mapsto \|v(z)\|_{\{x\}} := \|v(z)(x)\|$ is continuous. Therefore the normed weighted space $Hv_{x_0}(G)$ $(:= H(v_x)_0(G))$ is complete provided $v(z)$ is injective for every $z \in G$. Since the correspondence $f \mapsto x \otimes f$ is an isometry from $Hv_{x_0}(G)$ into $Hv_0(G, E)$, the space $Hv_{x_0}(G)$, identified with $x \otimes Hv_{x_0}(G) := \{x \otimes f, f \in Hv_{x_0}(G)\}$, is a closed subspace of $Hv_0(G, E)$, where $(x \otimes f)(z) := f(z)x$ for every $z \in G$ and every $f \in Hv_{x_0}(G)$.

Now, recall the following result.

Lemma 3.5 ([2, Corollary 2])**.** *Let G be an open subset of $\mathbb{C}^N$, and let v be a strictly positive and continuous weight on G. If the space $Hv_0(G)$ is infinite dimensional, then $Hv_0(G)$ is not reflexive.*

We then obtain the following theorem extending the lemma above.

Theorem 3.6. *Let G be an open subset of $\mathbb{C}^N$, and let v be a generalized weight on G such that $v(z)$ is injective for every $z \in G$. If the space $Hv_{x_0}(G)$ is infinite dimensional for some $x \in E$, then $Hv_0(G,E)$ is not reflexive.*

Proof. Since $x \otimes Hv_{x_0}(G)$ is a closed subspace of $Hv_0(G,E)$ and, by Lemma 3.5, $x \otimes Hv_{x_0}(G)$ is not reflexive, then $Hv_0(G,E)$ is not reflexive as well. □

As an application, we will show that the isometry equation $\|f(x)\| = \|x\|$ is not 1-Hyers-Rassias-stable. It is known that the Cauchy equation $f(x+y) = f(x) + f(y)$ is p-Hyers-Rassias stable for every real number $p \neq 1$, see [4]. This means that, for every such p and every real $\theta > 0$, if a function $f : X \to Y$ between Banach spaces X and Y satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p), \quad x, y \in X,$$

then there exists a unique additive function $g : X \to Y$ such that

$$\|f(x) - g(x)\| \leq \frac{2\theta\epsilon_p}{2^p - 2}\|x\|^p, \quad x \in X,$$

where $\epsilon_p = \text{sign}(p-1)$ is the sign of $p-1$. The same equation fails to be stable for $p = 1$, as shown in [4].

Here, we will show that the isometry functional equation $\|T(f)\| = \|f\|$, where T is a (even linear) mapping from the Banach space $Hv_0(G)$ into c_0 is not 1-Hyers-Rassias stable as well. Indeed, let $\theta > 0$ be arbitrary, using Corollary 3.3, there exists a linear mapping $T_\theta : Hv_0(G) \to c_0$ such that

$$(1-\theta)\|f\|_v \leq \|T_\theta(f)\|_{c_0} \leq \|f\|_v, \quad f \in Hv_0(G).$$

It follows from this that T_θ is an approximate isometry, this is $|\|T_\theta(f)\| - \|f\|_v| \leq \theta\|f\|_v$, $f \in Hv_0(G)$. However, for $G = \Delta$, the unit disc of $\mathbb{C}$, and a positive continuous and radial weight v (i.e., $v(z) = v(\lambda z)$ for every $\lambda \in \mathbb{T}$), there exists no isometry, at all, from $Hv_0(G)$ into c_0 as shown in [3], Corollary 17. Hence there exists no isometry approximating T_θ. Therefore the isometry equation fails to be 1-Hyers-Rassias stable.

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