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GENERALIZED IDENTITIES OF BIVARIATE FIBONACCI AND BIVARIATE LUCAS POLYNOMIALS

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ABSTRACT

In this paper, we present generalized identities of bivariate Fibonacci polynomials and bivariate Lucas polynomials and related identities consisting even and odd terms. Binet's formula will employ to obtain the identities. Also we describe and derive sums and connection formula.

Keywords: Bivariate Fibonacci polynomials; Bivariate Lucas polynomials; Binet's formula.

ÖZET

Bu çalışmada, iki değişkenli Fibonacci polinomlarının ve iki değişkenli Lucas polinomlarının genelleştirilmiş tanımlarını ve çift ve tek terimleri içeren ilgili tanımları sunmaktayız. Tanımları elde etmek için Binet'in formülü kullanılmıştır. Ayrıca toplamlar ve bağlantı formülleri açıklanmış ve türetilmiştir.

Anahtar Kelimeler: İki değişkenli Fibonacci polinomları; iki değişkenli Lucas polinomları; Binet formülü.

1. INTRODUCTION

Catalani, 2014, define generalized bivariate polynomials, from which specifying initial conditions the bivariate Fibonacci and Lucas polynomials are obtained and derived many interesting identities. Also derive a collection of identities for bivariate Fibonacci and Lucas polynomials using essentially a matrix approach as well as properties of such polynomials when the variables x and y are replaced by polynomials. A wealth of combinatorial identities can be obtained for selected values of the variables. Tuglu et al., 2011, study the bivariate Fibonacci and Lucas p–polynomials ($p \geqslant 0$ is integer) from which, specifying x, y and p, bivariate Fibonacci and Lucas polynomials and obtain some properties of the bivariate Fibonacci and Lucas p–polynomials. Tasci et al., 2012; Jacob et al. 2006; Belbachir & Bencherif, 2007; Inoue & Aki, 2011; Kaygisiz & Sahin, 2011, theories of the bivariate Fibonacci and Lucas polynomials are developed. Panwar & Singh, 2014, present generalized bivariate Fibonacci-Like polynomials sequence and its properties like Catalan’s identity, Cassini’s identity or Simpson’s identity and d’ocagnes’s identity for generalized bivariate Fibonacci-Like polynomials through Binet’s formulas. Taşköprü & Altıntaş, 2015, define the HOMFLY polynomial of (2, n)-torus link and show that the HOMFLY polynomial of (2, n)-torus link can be obtained from its Alexander-Conway polynomial and give the matrix representations and prove important identities. Alves & Catarino, 2016, present a set of mathematical definitions related to a generalized sequence model that allow an understanding of the evolutionary epistemological and mathematical process of a second order recurrent sequence, with the introduction of one or two variables, we begin to discuss the family of the Bivariate Lucas Polynomials (BLP) and the Bivariate Fibonacci Polynomials (BFP). Finally, throughout the work they bring several figures that represent some examples of commands and algebraic operations with the CAS Maple that allow to compare properties of the Lucas polynomials, taking as a reference the classic of Fibonacci´s model that still serves as inspiration for several current studies in Mathematics. Altıntaş & Taşköprü, 2019, study the two-variable Kauffman polynomials L and F, and the one-variable BLM/Ho polynomial Q of (2,n)-torus link as the Fibonacci-type polynomials and to express the Kauffman polynomials in terms of the BLM/Ho polynomial and prove that each of the examined polynomials of (2,n)-torus link can be determined by a third-order recurrence relation and give the recursive properties of them. Also correlate these polynomials with the Fibonacci-type polynomials. By using the relations between the BLM/Ho polynomials and Fibonacci-type polynomials, and express the Kauffman polynomials in terms of the BLM/Ho polynomials. Cakmak & Karaduman, 2018, present the new algebraic properties related to bivariate Fibonacci polynomials have been given and the partial derivatives of these polynomials in the form of convolution of bivariate Fibonacci polynomials. Also, they define a new recurrence relation for the r-th partial derivative sequence of bivariate Fibonacci polynomials. In this paper, we present generalized identities of bivariate Fibonacci polynomials and bivariate Lucas polynomials and related identities consisting even and odd terms.

2. RESULTS AND DISCUSSION

For $n \geq 2$, the bivariate Fibonacci polynomials sequence is defined by

$$F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y) \tag{1}$$

So, the first bivariate Fibonacci polynomials are

$$\{F_n(x,y)\} = \{0, 1, x, x^2+y, x^3+2xy, x^4+3x^2y+y^2, ...\}$$

Binet's formula for the bivariate Fibonacci polynomials:

$$F_n(x,y) = \frac{\Re_1^n - \Re_2^n}{\Re_1 - \Re_2} \tag{2}$$

Generating function for the bivariate Fibonacci polynomials:

$$F_n(x,y) = \frac{t}{(1-xt-yt^2)} \tag{3}$$

For $n \geq 2$, the bivariate Lucas polynomials sequence is defined by

$$L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y) \tag{4}$$

So, the first bivariate Lucas polynomials are

$$\{L_n(x,y)\} = \{2, x, x^2+2y, x^3+3xy, x^4+4x^2y+2y^2, ...\}$$

Binet's formula for the bivariate Lucas polynomials:

$$L_n(x,y) = \Re_1^n + \Re_2^n \tag{5}$$

Generating function for the bivariate Lucas polynomials:

$$L_n(x,y) = \frac{2-xt}{(1-xt-yt^2)} \tag{6}$$

The characteristic equation of recurrence relation (1) and (4) is: $t^2 - xt - y = 0$ (7)

Where $x \neq 0,\ y \neq 0,\ x^2+4y \neq 0$.

This equation has two real roots: $\Re_1 = \dfrac{x+\sqrt{x^2+4y}}{2}$ and $\Re_2 = \dfrac{x-\sqrt{x^2+4y}}{2}$

Note that $\Re_1 + \Re_2 = x, \Re_1\Re_2 = -y, \Re_1 - \Re_2 = \sqrt{x^2+4y}$.

Also $F_{-n}(x,y)=\dfrac{-1}{(-y)^n}F_n(x,y)$ and $L_{-n}(x,y)=\dfrac{1}{(-y)^n}L_n(x,y)$.

Proposition 2.1: For any integer $n\geq 0$,
$$\begin{aligned}\Re_1^{n+2}&=x\Re_1^{n+1}+y\Re_1^{n}\\ \Re_2^{n+2}&=x\Re_2^{n+1}+y\Re_2^{n}\end{aligned}\tag{8}$$

Proof: Since $\Re_1$ and $\Re_2$ are the roots of the characteristic equation (7), then

$$\Re_1^2=x\Re_1+y \text{ and } \Re_2^2=x\Re_2+y$$

now, multiplying both sides of these equations by $\Re_1$ and $\Re_2$ respectively, we obtain the desired result.

3. SUMS OF BIVARIATE FIBONACCI POLYNOMIALS AND BIVARIATE LUCAS POLYNOMIALS

In this section, we study the sums of bivariate Fibonacci polynomials and bivariate Lucas polynomials. This enables us to give in a straightforward way several formulas for the sums of such polynomials.

3.1. Sums of Bivariate Fibonacci Polynomials

Lemma 3.1.1: For fixed integers $p,\ q$ *with* $0\leq q\leq p-1$, the following equality holds

$$F_{p(n+2)+q}(x,y)=L_p(x,y)F_{p(n+1)+q}(x,y)-(-y)^pF_{pn+q}(x,y)\tag{9}$$

Proof: From the Binet's formula of bivariate Fibonacci and bivariate Lucas polynomials,

$$L_p(x,y)F_{p(n+1)+q}(x,y)=\left(\Re_1^p+\Re_2^p\right)\left(\frac{\Re_1^{p(n+1)+q}-\Re_2^{p(n+1)+q}}{\Re_1-\Re_2}\right)$$

$$=\frac{1}{\Re_1-\Re_2}\left[\Re_1^{p(n+2)+q}+(-y)^p\Re_1^{pn+q}-(-y)^p\Re_2^{pn+q}-\Re_2^{p(n+2)+q}\right]$$

$$=\frac{1}{\Re_1-\Re_2}\left[\left\{\Re_1^{p(n+2)+q}-\Re_2^{p(n+2)+q}\right\}+(-y)^p\left(\Re_1^{pn+q}-\Re_2^{pn+q}\right)\right]$$

$$=F_{p(n+2)+q}(x,y)+(-y)^pF_{pn+q}(x,y)$$

then, the equality becomes,

$$F_{p(n+2)+q}(x,y)=L_p(x,y)F_{p(n+1)+q}(x,y)-(-y)^pF_{pn+q}(x,y)$$

Proposition 3.1.2: For fixed integers p, q *with* $0 \le q \le p-1$, the following equality holds

$$\sum_{i=0}^{n} F_{pi+q}(x,y) = \frac{F_{p(n+1)+q}(x,y) + (-y)^q L_{p-q}(x,y) - F_q(x,y) - (-y)^p F_{pn+q}(x,y)}{L_p(x,y) - (-y)^p - 1} \qquad (10)$$

Proof: From the Binet's formula of bivariate Fibonacci polynomials,

$$\sum_{i=0}^{n} F_{pi+q}(x,y) = \sum_{i=0}^{n} \frac{\Re_1^{pi+q} - \Re_2^{pi+q}}{\Re_1 - \Re_2}$$

$$= \frac{1}{\Re_1 - \Re_2}\left[\sum_{i=0}^{n} \Re_1^{pi+q} - \sum_{i=0}^{n} \Re_2^{pi+q}\right]$$

$$= \frac{1}{\Re_1 - \Re_2}\left[\frac{\Re_1^{pn+q+p} - \Re_1^q}{\Re_1^p - 1} - \frac{\Re_2^{pn+q+p} - \Re_2^q}{\Re_2^p - 1}\right]$$

$$= \frac{1}{(-y)^p - L_p(x,y) + 1}\left[(-y)^p F_{pn+q}(x,y) - F_{p(n+1)+q}(x,y) + F_q(x,y) - (-y)^q L_{p-q}(x,y)\right]$$

$$= \frac{F_{p(n+1)+q}(x,y) + (-y)^q L_{p-q}(x,y) - F_q(x,y) - (-y)^p F_{pn+q}(x,y)}{L_p(x,y) - (-y)^p - 1}$$

This completes the proof.

Corollary 3.1.3: Sum of odd bivariate Fibonacci polynomials, If $p = 2m+1$, then Equation (10) is

$$\sum_{i=0}^{n} F_{(2m+1)i+q}(x,y) = \frac{F_{(2m+1)(n+1)+q}(x,y) + (-y)^q L_{2m+1-q}(x,y) - F_q(x,y) - (-y)^{(2m+1)} F_{(2m+1)n+q}(x,y)}{L_{(2m+1)}(x,y) - (-y)^{(2m+1)} - 1} \qquad (11)$$

Corollary 3.1.4: Sum of even bivariate Fibonacci polynomials, If $p = 2m$, then Equation (10) is

$$\sum_{i=0}^{n} F_{2mi+q}(x,y) = \frac{F_{2m(n+1)+q}(x,y) + (-y)^q L_{2m-q}(x,y) - F_q(x,y) - (-y)^{2m} F_{2mn+q}(x,y)}{L_{2m}(x,y) - (-y)^{2m} - 1} \qquad (12)$$

Proposition 3.1.5: For fixed integers p, q *with* $0 \le q \le p-1$, the following equality holds

$$\sum_{i=0}^{n} (-1)^i F_{pi+q}(x,y) = \frac{(-1)^n F_{p(n+1)+q}(x,y) + (-1)^n (-y)^p F_{pn+q}(x,y) - (-y)^q F_{p-q}(x,y) + F_q(x,y)}{L_p(x,y) + (-y)^p + 1} \qquad (13)$$

Proof: From the Binet's formula of bivariate Fibonacci and bivariate Lucas polynomials, the proof is clear.

3.2. Sums of Bivariate Lucas Polynomials

Lemma 3.2.1: For fixed integers $p,\ q\ with\ 0 \le q \le p-1$, the following equality holds

$$L_{p(n+1)+q}(x,y) = L_p(x,y)L_{pn+q}(x,y) - (-y)^p L_{p(n-1)+q}(x,y) \tag{14}$$

Proposition 3.2.2: For fixed integers $p,\ q\ with\ 0 \le q \le p-1$, the following equality holds

$$\sum_{i=0}^{n} L_{pi+q}(x,y) = \frac{L_{p(n+1)+q}(x,y) + (-y)^q L_{p-q}(x,y) - L_q(x,y) - (-y)^p L_{pn+q}(x,y)}{L_p(x,y) - (-y)^p - 1} \tag{15}$$

Corollary 3.2.3: Sum of odd bivariate Lucas polynomials, If $p = 2m+1$ then Equation (15) is

$$\sum_{i=0}^{n} L_{(2m+1)i+q}(x,y) = \frac{L_{(2m+1)(n+1)+q}(x,y) + (-y)^q L_{2m+1-q}(x,y) - L_q(x,y) - (-y)^{(2m+1)} L_{(2m+1)n+q}(x,y)}{L_{(2m+1)}(x,y) - (-y)^{(2m+1)} - 1} \tag{16}$$

Corollary 3.2.4: Sum of even bivariate Fibonacci polynomials, If $p = 2m$ then Equation (15) is

$$\sum_{i=0}^{n} L_{2mi+q}(x,y) = \frac{L_{2m(n+1)+q}(x,y) + (-y)^q L_{2m-q}(x,y) - L_q(x,y) - (-y)^{2m} L_{2mn+q}(x,y)}{L_{2m}(x,y) - (-y)^{2m} - 1} \tag{17}$$

Proposition 3.2.5: For fixed integers $p,\ q\ with\ 0 \le q \le p-1$, the following equality holds

$$\sum_{i=0}^{n} (-1)^i L_{pi+q}(x,y) = \frac{(-1)^n L_{p(n+1)+q}(x,y) + (-1)^n (-y)^p L_{pn+q}(x,y) + (-y)^q L_{p-q}(x,y) + L_q(x,y)}{L_p(x,y) + (-y)^p + 1} \tag{18}$$

3.3. Sum and difference of squares of Bivariate Fibonacci Polynomials and Bivariate Lucas Polynomials

In this section, sum and difference of bivariate Fibonacci and Lucas polynomials are treated in the following propositions.

Proposition 3.3.1:

$$(x^2+4y)\left\{F_{n+1}^2(x,y) + F_{n-1}^2(x,y)\right\} = L_{2n+2}(x,y) + L_{2n-2}(x,y) - 2(-y)^{n-1}(1+y^2) \tag{19}$$

Proposition 3.3.2:

$$(x^2+4y)\left\{F_{n+1}^2(x,y) - F_{n-1}^2(x,y)\right\} = L_{2n+2}(x,y) - L_{2n-2}(x,y) - 2(-y)^{n-1}(y^2-1) \tag{20}$$

Proof: From the Binet's formula of bivariate Fibonacci and bivariate Lucas polynomials, the proof is clear.

4. GENERALIZED IDENTITIES ON THE PRODUCT OF BIVARIATE FIBONACCI POLYNOMIALS AND BIVARIATE LUCAS POLYNOMIALS

In this section, we present identities involving product of bivariate Fibonacci and bivariate Lucas polynomials and related identities consisting even and odd terms.

Theorem 4.1: If $F_n(x,y)$ and $L_n(x,y)$ are the bivariate Fibonacci polynomials and bivariate Lucas polynomials, then

$$F_{2n+p}(x,y)L_{2n+1}(x,y)=F_{4n+p+1}(x,y)+(-y)^{2n+1}F_{p-1}(x,y) \tag{21}$$

where $n\geq 0$ and $p\geq 0$.

Proof: By the Binet's formula of bivariate Fibonacci and bivariate Lucas polynomials,

$$\begin{aligned}F_{2n+p}(x,y)L_{2n+1}(x,y)&=\left(\frac{\Re_1^{2n+p}-\Re_2^{2n+p}}{\Re_1-\Re_2}\right)\left(\Re_1^{2n+1}+\Re_2^{2n+1}\right)\\&=\left(\frac{\Re_1^{4n+p+1}-\Re_2^{4n+p+1}}{\Re_1-\Re_2}\right)+\frac{(\Re_1\Re_2)^{2n}}{(\Re_1-\Re_2)}\left(\Re_1^p\Re_2-\Re_2^p\Re_1\right)\\&=\left(\frac{\Re_1^{4n+p+1}-\Re_2^{4n+p+1}}{\Re_1-\Re_2}\right)+(\Re_1\Re_2)^{2n}(-y)\left(\frac{\Re_1^{p-1}-\Re_2^{p-1}}{\Re_1-\Re_2}\right)\\&=F_{4n+p+1}(x,y)+(-y)^{2n+1}F_{p-1}(x,y)\end{aligned}$$

This completes the proof.

Corollary 4.2: For different values of p, (21) can be expressed for even and odd numbers:

(i) *If* $p=0$, *then*: $F_{2n}(x,y)L_{2n+1}(x,y)=F_{4n+1}(x,y)-y^{2n}$

(ii) *If* $p=1$, *then*: $F_{2n+1}(x,y)L_{2n+1}(x,y)=F_{4n+2}(x,y)$

(iii) *If* $p=2$, *then*: $F_{2n+2}(x,y)L_{2n+1}(x,y)=F_{4n+3}(x,y)+(-y)^{2n+1}$

Following theorem can be solved by Binet's formula of Bivariate Fibonacci and Bivariate Lucas polynomials.

Theorem 4.3: If $F_n(x,y)$ and $L_n(x,y)$ are the bivariate Fibonacci polynomials and bivariate Lucas polynomials, then

(i) $$F_{2n+p}(x,y)L_{2n+2}(x,y)=F_{4n+p+2}(x,y)+y^{2n+2}F_{p-2}(x,y) \quad (22)$$

(ii) $$F_{2n+p}(x,y)L_{2n}(x,y)=F_{4n+p}(x,y)+y^{2n}F_{p}(x,y) \quad (23)$$

(iii) $$F_{2n-p}(x,y)L_{2n+1}(x,y)=F_{4n-p+1}(x,y)+(-y)^{2n+1}F_{-p-1}(x,y) \quad (24)$$

(iv) $$F_{2n-p}(x,y)L_{2n}(x,y)=F_{4n-p}(x,y)+y^{2n}F_{-p}(x,y) \quad (25)$$

(v) $$F_{2n}(x,y)L_{2n+p}(x,y)=F_{4n+p}(x,y)-y^{2n}F_{p}(x,y) \quad (26)$$

(vi) $$(x^2+4y)F_{2n}(x,y)F_{2n+p}(x,y)=L_{4n+p}(x,y)-y^{2n}L_{p}(x,y) \quad (27)$$

(vii) $$L_{2n}(x,y)L_{2n+p}(x,y)=L_{4n+p}(x,y)+y^{2n}L_{p}(x,y) \quad (28)$$

5. CONCLUSIONS

In this paper we have derived many identities of bivariate Fibonacci and Lucas polynomials through Binet's formulas. We describe sums of bivariate Fibonacci and Lucas polynomials. This enables us to give in a straightforward way several formulas for the sums of such generalized polynomials. These identities can be used to develop new identities of numbers and polynomials. We describe some generalized identities involving product of new generalization of Fibonacci and Lucas numbers. Also we present identities related to their sum and difference of squares involving them.

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