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On Characterization of Being a Matrix $Q^{(k)}_{g(a_2,b_2)}$ of Linear Combinations of a Matrix $Q^{(n)}_{g(a_1,b_1)}$ and a Matrix Q^m

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Abstract

It is given a characterization of all solution of the matrix equation $c_1Q^{(n)}_{g(a_1,b_1)} + c_2Q^m = Q^{(k)}_{g(a_2,b_2)}$ with unknowns $c_1, c_2 \in \mathbb{C}^*$. Here the matrix $Q^{(l)}_{g(a,b)}$, called an l-generalized Fibonacci Q-matrix, is defined by means of the Fibonacci Q-matrix, where l is an integer, and $a,b \in \mathbb{R}^*$.

Keywords: *Fibonacci Numbers, Fibonacci Q-matrix, Generalized Fibonacci Numbers, Linear Combination, Matrix Equations*
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1. Introduction and Preliminaries

It is a well known fact that the Fibonacci and generalized Fibonacci numbers have a very common usage in mathematics and applied sciences (see, for example, [17], [18], and [20]). Also, Fibonacci sequences have amazing applications in coding, encryption, and decryption (see, for example, [12], [16]). Besides these, Fibonacci numbers arise in the solution of many combinatorial problems, and they are extensively used in many research areas such as architecture, nature, art, physics and engineering (see, for example, [8] and [17]. In addition, many authors have been intensively studying these topics [2], [4], [6], [10–12], [14–16], [21–24]. In a word, there are so many works related to these topics in the literature, for example, [1–24].

In this work, a special problem related to the Fibonacci and generalized Fibonacci numbers is considered. For this reason, it will be sufficient to remind some concepts and some results without proof to be used in the work.

The Fibonacci sequence is defined by the initial conditions $F_0 = 0$ and $F_1 = 1$, and the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ (see, for example, [3], [20]).
There is the relation

$$F_{-n} = (-1)^{n+1}F_n \tag{1.1}$$

for all integers $n \geq 1$ between the Fibonacci numbers and the Fibonacci numbers with negative subscripts [18]. In addition, the identity

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n \tag{1.2}$$

holds for all $n,m \in \mathbb{Z}$ [20].
On the other hand, the identity

$$F_aF_b - F_cF_d = (-1)^r(F_{a-r}F_{b-r} - F_{c-r}F_{d-r}) \tag{1.3}$$

is well known, where a,b,c and d are integers with $a+b = c+d$ [9].
The sequences, called as generalized Fibonacci sequences, were defined in several ways by different authors. For example, the sequence defined as

$$H_n = H_{n-1} + H_{n-2} \quad \text{for} \quad n \geq 3 \quad H_1 = p, \quad H_2 = p+q$$

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is said to be generalized Fibonacci number sequence, where p, q are any integers [1]. Similarly, Gupta et al. described the generalized Fibonacci sequence as

$$F_k = pF_{k-1} + qF_{k-2}$$

for integers $k \geq 2$ together with the initial conditions $F_0 = a, F_1 = b$, where p, q, a, and b are positive integers [21].
By examining a number sequence that provides the Fibonacci recurrence relation but whose initial conditions are any two numbers, it can be seen that this number sequence is directly related to the Fibonacci numbers: Let $\{G_n\}$ be a sequence such that

$$G_n = G_{n-1} + G_{n-2} \quad \text{for} \quad n \geq 3$$

with $G_1 = a$ and $G_2 = b$. The elements of this number sequence, clearly, are

$$a, b, a+b, a+2b, 2a+3b, 3a+5b, \ldots \quad .$$

This number sequence is called the generalized Fibonacci sequence. Looking at the coefficients, it is seen that they have an interesting pattern. The coefficients of a and b are Fibonacci numbers. The elements of this sequence are determined by the formula $G_n = aF_{n-2} + bF_{n-1}$ for integers $n \geq 3$ [17].
From now on, the elements of generalized Fibonacci sequence defined as based on a and b, that is, the numbers $G_n = aF_{n-2} + bF_{n-1}$, for the sake of simplicity, will be denoted by $G_{(a,b)}^{(n)}$.
For all integers n, it was established the relation

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \tag{1.4}$$

between the matrix $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ known as the Fibonacci Q-matrix in the literature and classical Fibonacci sequence $\{F_n\}$ in [3]. For detailed information about the Fibonacci Q-matrix, see, for example, [5].
According to this property, it is clear that, for all integers n,

$$aQ^{n-2} + bQ^{n-1} = \begin{bmatrix} aF_{n-1} + bF_n & aF_{n-2} + bF_{n-1} \\ aF_{n-2} + bF_{n-1} & aF_{n-3} + bF_{n-2} \end{bmatrix}$$

with $a, b \in \mathbb{R}^*$. For the sake of simplicity, throughout the work, we will call this matrix as n-generalized Fibonacci Q-matrix, and denote it by $Q_{g(a,b)}^{(n)}$.

2. Being a Matrix $Q_{g(a_2,b_2)}^{(k)}$ of Linear Combination of a Matrix $Q_{g(a_1,b_1)}^{(n)}$ and a Matrix Q^m

Let's consider the problem of characterizing the linear combination of the matrices $Q_{g(a_1,b_1)}^{(n)}$ and Q^m as a matrix $Q_{g(a_2,b_2)}^{(k)}$, where n, m, and k are integers, c_1, c_2 are unknowns, and $a_i, b_i \in \mathbb{R}^*, i = 1, 2$:

$$c_1 Q_{g(a_1,b_1)}^{(n)} + c_2 Q^m = Q_{g(a_2,b_2)}^{(k)} \tag{2.1}$$

It is clear that this matrix equation leads to the linear equations system

$$\begin{array}{l} c_1(a_1F_{n-2} + b_1F_{n-1}) + c_2F_m = a_2F_{k-2} + b_2F_{k-1} \\ c_1(a_1F_{n-3} + b_1F_{n-2}) + c_2F_{m-1} = a_2F_{k-3} + b_2F_{k-2} \end{array} \tag{2.2}$$

in the variables c_1 and c_2. The determinant of the coefficients matrix of the system (2.2) is

$$\begin{vmatrix} a_1F_{n-2} + b_1F_{n-1} & F_m \\ a_1F_{n-3} + b_1F_{n-2} & F_{m-1} \end{vmatrix} = (-1)^n(-a_1F_{m-n+2} + b_1F_{m-n+1}). \tag{2.3}$$

The determinant (2.3) is zero if $a_1F_{m-n+2} = b_1F_{m-n+1}$. Otherwise, it is not zero. If the determinant is not zero, then it is obvious that the matrix equation (2.1) has the unique solution
$c_1 = \frac{(-1)^{k-n}(-a_2F_{m-k+2} + b_2F_{m-k+1})}{-a_1F_{m-n+2} + b_1F_{m-n+1}}$ and $c_2 = \frac{-a_1G_{(a_2,b_2)}^{(k-n+2)} + b_1G_{(a_2,b_2)}^{(k-n+1)}}{-a_1F_{m-n+2} + b_1F_{m-n+1}}$.

Now, let $a_1F_{m-n+2} = b_1F_{m-n+1}$. In this case, it must be $m - n \neq -2$. Otherwise, we get the contradiction $b_1 = 0$. Hence $a_1 = \frac{b_1F_{m-n+1}}{F_{m-n+2}}$ is obtained. Thus, the augmented matrix of the linear equations system (2.2) is obtained as

$$\left[\begin{array}{cc|c} \frac{b_1(F_{m-n+1}F_{n-2} + F_{n-1}F_{m-n+2})}{F_{m-n+2}} & F_m & a_2F_{k-2} + b_2F_{k-1} \\ \frac{b_1(F_{m-n+1}F_{n-3} + F_{n-2}F_{m-n+2})}{F_{m-n+2}} & F_{m-1} & a_2F_{k-3} + b_2F_{k-2} \end{array}\right].$$

To rearrange this matrix considering the equality (1.2) leads to the augmented matrix

$$\left[\begin{array}{cc|c} \frac{b_1F_m}{F_{m-n+2}} & F_m & a_2F_{k-2} + b_2F_{k-1} \\ \frac{b_1F_{m-1}}{F_{m-n+2}} & F_{m-1} & a_2F_{k-3} + b_2F_{k-2} \end{array}\right]. \tag{2.4}$$

Now, let us first assume that $m \neq 0$. Then it is clear that the matrix in (2.4) is row equivalent to the matrix

$$\left[\begin{array}{cc|c} \frac{b_1F_m}{F_{m-n+2}} & F_m & a_2F_{k-2}+b_2F_{k-1} \\ 0 & 0 & \frac{(-1)^{m-1}G^{(k-m)}_{(a_2,b_2)}}{F_m} \end{array}\right]$$

in view of the equality (1.3). Hence, if $G^{(k-m)}_{(a_2,b_2)} \neq 0$, then there is no solution of the system (2.2). Thus, for the equation system to have a solution, it must be $G^{(k-m)}_{(a_2,b_2)} = a_2F_{k-m-2}+b_2F_{k-m-1} = 0$, or equivalently, $k-m \neq 2$. Otherwise, we get the contradiction $b_2 = 0$. Consequently, we obtain the general solution as

$$(c_1,c_2) = \left(t, -\frac{b_2(-1)^{k-m}}{F_{k-m-2}} - \frac{tb_1}{F_{m-n+2}}\right), t \in \mathbb{R}^*$$

taking (1.3) into account.
Next, if $m = 0$, then the augmented matrix (2.4) turns into the matrix

$$\left[\begin{array}{cc|c} 0 & 0 & a_2F_{k-2}+b_2F_{k-1} \\ \frac{b_1}{F_{2-n}} & 1 & a_2F_{k-3}+b_2F_{k-2} \end{array}\right].$$

According to this, in case $a_2F_{k-2}+b_2F_{k-1} \neq 0$, there is no solution of the matrix equation (2.1). So, let us consider the case $a_2F_{k-2} + b_2F_{k-1} = 0$. In this case, it is obvious that $k \neq 2$. Otherwise, the contradiction $b_2 = 0$ is obtained. Thus, we get the general solution of the matrix equation (2.1) as

$$(c_1,c_2) = \left(t, -\frac{b_2(-1)^k}{F_{k-2}} - \frac{tb_1}{F_{2-n}}\right), \quad t \in \mathbb{R}^*$$

taking (1.3) into account. So, we have proved the following theorem.

Theorem 2.1. *Consider the matrix equation*

$$c_1Q^{(n)}_{g(a_1,b_1)} + c_2Q^m = Q^{(k)}_{g(a_2,b_2)}, \tag{2.5}$$

where n, m, and k are integers, $c_1,c_2 \in \mathbb{C}^*$ are unknowns, and $a_i,b_i \in \mathbb{R}^*, i = 1,2$. Then, the following statements are true.

(i) If $a_1F_{m-n+2} \neq b_1F_{m-n+1}$, then the matrix equation (2.5) has a unique solution such that $c_1 = \frac{(-1)^{k-n}(-a_2F_{m-k+2}+b_2F_{m-k+1})}{-a_1F_{m-n+2}+b_1F_{m-n+1}}$ and $c_2 = \frac{-a_1G^{(k-n+2)}_{(a_2,b_2)}+b_1G^{(k-n+1)}_{(a_2,b_2)}}{-a_1F_{m-n+2}+b_1F_{m-n+1}}$.

(ii) If $a_1F_{m-n+2} = b_1F_{m-n+1}$, then the matrix equation (2.5) has no solution for $G^{(k-m)}_{(a_2,b_2)} \neq 0$, and has finitely many solution such that $(c_1,c_2) = \left(t, -\frac{b_2(-1)^{k-m}}{F_{k-m-2}} - \frac{tb_1}{F_{m-n+2}}\right)$, $t \in \mathbb{R}^*$, provided however that $G^{(k-m)}_{(a_2,b_2)} = 0$.

Example 2.2. *The following three cases illustrate Theorem 2.1.*

1. *Let $a_1 = 5, b_1 = 7, a_2 = 3, b_2 = 4, m = 7, n = 5, k = 4$. Since $5F_4 \neq 7F_3$, according to (i) of Theorem 2.1, it is obtained $c_1 = -3$ and $c_2 = 8$. In fact, it is obvious that c_1 and c_2 satisfy the equality*

$$c_1Q^{(5)}_{g(5,7)} + c_2Q^7 = Q^{(4)}_{g(3,4)}$$

that is, the equality

$$c_1\begin{bmatrix} 50 & 31 \\ 31 & 19 \end{bmatrix} + c_2\begin{bmatrix} 21 & 13 \\ 13 & 8 \end{bmatrix} = \begin{bmatrix} 18 & 11 \\ 11 & 7 \end{bmatrix}.$$

2. *Let $a_1 = 6, b_1 = 9, a_2 = 3, b_2 = 4, m = 7, n = 5, k = 4$. Since $6F_4 = 9F_3$ and $G^{(-3)}_{(3,4)} = 3F_{-5}+4F_{-4} \neq 0$ (in view of (1.1)), there is no solution of the system (2.1) according to (ii) of Theorem 2.1. In fact, it is obvious that there is no solution of the matrix equation*

$$c_1Q^{(5)}_{g(6,9)} + c_2Q^7 = Q^{(4)}_{g(3,4)},$$

or equivalently, the matrix equation

$$c_1\begin{bmatrix} 63 & 39 \\ 39 & 24 \end{bmatrix} + c_2\begin{bmatrix} 21 & 13 \\ 13 & 8 \end{bmatrix} = \begin{bmatrix} 18 & 11 \\ 11 & 7 \end{bmatrix}.$$

3. *Let $a_1 = 6, b_1 = 9, a_2 = 1, b_2 = 2, m = 7, n = 5, k = 6$. Since $6F_4 = 9F_3$ and $G^{(-1)}_{(1,2)} = F_{-3}+2F_{-2} = 0$ (in view of (1.1)), according to (ii) of Theorem 2.1, the general solution of the system (2.1) is obtained as $(c_1,c_2) = (t,1-3t)$, $t \in \mathbb{R}^*$. In fact, it is clearly seen that the matrix equation*

$$c_1Q^{(5)}_{g(6,9)} + c_2Q^7 = Q^{(6)}_{g(1,2)},$$

or equivalently the matrix equation

$$c_1\begin{bmatrix} 63 & 39 \\ 39 & 24 \end{bmatrix} + c_2\begin{bmatrix} 21 & 13 \\ 13 & 8 \end{bmatrix} = \begin{bmatrix} 21 & 13 \\ 13 & 8 \end{bmatrix}$$

has the general solution such that $(c_1,c_2) = (t,1-3t)$, $t \in \mathbb{R}^*$.

NOTE. In case $a_1 = b_1$, $a_2 = b_2$, the matrix equation (2.1) turns into the matrix equation $d_1Q^n + d_2Q^m = Q^k$. So, it this case, the problem considered in this work is reduced to the problems handled in [6] and [15]. Therefore, the problem discussed here can be considered as, in a sense, a generalization of the problems dealt with in [6] and [15]. On the other hand, in case $a_2 = b_2$, the matrix equation (2.1) turns into the matrix equation $e_1Q^{(n)}_{g(a_1,b_1)} + e_2Q^m = Q^k$. Notice that by using Theorem 2.1, we can have an idea about the solutions of the latter matrix equation.

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References

[1] A. Horadam, A Generalized Fibonacci Sequence, *The American Mathematical Monthly*, 68(5) (1961), pp. 455-459.
[2] B. Demirtürk, Fibonacci and Lucas Sums with Matrix Method, *International Mathematical Forum*, 5(3) (2010), pp. 99-107.
[3] C.H. King, *Some Properties of the Fibonacci Numbers*, Master's Thesis, San Jose State College, June, (1960).
[4] G. Cerda-Morales, On Generalized Fibonacci and Lucas Numbers by Matrix Methods, *Hacettepe Journal of Mathematics and Statistics*, 42(2) (2013), pp. 173-179.
[5] H.W. Gould, A History of the Fibonacci *Q*-Matrix and a Higher-Dimensional Problem, *The Fibonacci Quarterly*, 19(3) (1981), pp. 250-257.
[6] H. Özdemir, S. Karakaya, and T. Petik, On Characterization of Some Linear Combinations Involving the Matrices *Q* and *R*, *Honam Mathematical Journal*, 42(2) (2020), pp. 235-249.
[7] J.R. Silvester, Fibonacci properties by matrix methods, *Mathematical Gazette*, 63(425) (1979), pp. 188-191.
[8] A.M. Meinke, *Fibonacci Numbers and Associated Matrices*, Master's Thesis, Kent States University, (2011).
[9] R. Euler and J. Sadek, Elementary Problems and Solutions, *Fibonacci Quarterly*, 41(2) (2003), pp. 181-186.
[10] R. P. Grimaldi, *Fibonacci and Catalan Numbers an Introduction*, John Wiley and Sons, (2012).
[11] R. Keskin, B. Demirtürk, Some new Fibonacci and Lucas identities by matrix methods, *Internat. J. Math. Ed. Sci. Tech.* 41(3) (2010), pp. 379-387.
[12] S. Sinha, The Fibonacci Numbers and Its Amazing Applications, *International Journal of Engineering Science Invention*, 6(9) (2017), pp. 07-14.
[13] S.L. Basin, V.E. Hoggatt JR., A Primer on the Fibonacci Sequence Part I, *The Fibonacci Quarterly*, 1(1) (1963), pp. 65-72.
[14] S. Karakaya, H. Özdemir, and T. Petik, 3×3 Dimensional Special Matrices associated with Fibonacci and Lucas numbers, *Sakarya University Journal of Science* 22(6) (2018), pp. 1917-1922.
[15] S. Karakaya, On the Linear Combinations of Matrices associated with Fibonacci and Lucas numbers, Master's Thesis, (2018).
[16] S. Uçar, N. Taş, and N. Yılmaz Özgür, A New Application to Coding Theory via Fibonacci and Lucas Numbers, *Mathematical Sciences and Applications E-Notes*, 7(1) (2019), pp. 62-70.
[17] T. Koshy, *Fibonacci and Lucas Numbers With Applications*, John Wiley and Sons, (2001).
[18] T. Koshy, *Fibonacci and Lucas Identities, Fibonacci and Lucas Numbers With Applications*, John Wiley & Sons, (2018).
[19] V.E. Hoggatt JR., I.D. Ruggles, A Primer for the Fibonacci numbers-Part IV, *Fibonacci Quarterly*, 1(4) (1963), pp. 65-71.
[20] V.E. Hoggatt, *Fibonacci and Lucas Numbers*, The Fibonacci Association, University of Santa Clara, (1969).
[21] V. Gupta, Y.K. Panwar, and O. Sikhwal, Generalized Fibonacci Sequences, *Theoretical Mathematics & Applications*, 2(2), (2012), pp. 115-124.
[22] Y.K. Panwar, B. Singh, V.K. Gupta, Generalized Fibonacci Sequences and its Properties, *Palestine Journal of Mathematics*, 3(1) (2014), pp. 141-147.
[23] Z. Akyüz, S. Halıcı, On Some Combinatorial Identities Involving the terms of Generalized Fibonacci and Lucas Sequences, *Hacettepe Journal of Mathematics and Statistics*, 42(4) (2013), pp. 431-435.
[24] Z. Şiar, R. Keskin, Some New Identities Concerning Generalized Fibonacci and Lucas Numbers, *Hacettepe Journal of Mathematics and Statistics*, 42(3) (2013), pp. 211-222.