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On the structure of finite groups with the given numbers of involutions

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Abstract

Let G be a finite non-solvable group. In this paper, we show that if $1/8$ of elements of G have order two, then G is either a simple group isomorphic to $PSL_2(q)$, where $q \in \{7, 8, 9\}$ or $G \cong GL_2(4).\mathbb{Z}_2$. In fact in this paper, we answer Problem 132 in [1].

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1. Introduction

An involution in a group G is an element of order two. For a finite group G, let $T(G)$ denote the set of involutions of a group G, $t(G) = |T(G)|$ and let $t_0(G) = t(G)/|G|$. It is an elementary fact in group theory that a group G in which all of its non-identity elements have order two, is abelian. We can also see that every finite group which at least $3/4$ of its elements have order two is abelian. But this result can not be extended for the case $t_0(G) < 3/4$. So finding the structure of the finite groups according to the number of their involutions can be an interesting question. This problem has received some attention in existing literature. For instance, Wall [8] classified all finite groups in which more than half of the elements are involutions.After that, Berkovich in [1] described the structure of all finite non-solvable groups which at least $1/4$ of its elements have order two. Also, in addition to classifying all finite groups G with $t_0(G) = 1/4$, he put forward he following problem (Problem 132 in [1]):
Problem. What is the structure of finite groups G with $t_0(G) = 1/8$?
In this paper, we show that:
Main Theorem. If G is a finite non-solvable group with $t(G) = |G|/8$, then either G is a simple group isomorphic to $PSL_2(q)$, where $q \in \{7, 8, 9\}$ or $G \cong GL_2(4).\mathbb{Z}_2$.

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Actually, in [4], Davoudi Monfared has shown that the non-abelian finite simple group G with $t_0(G) = 1/8$ is isomorphic to $PSL_2(q)$, where $q \in \{7, 8, 9\}$. In [4], the author has completed the proof using the simplicity of the considered groups. In this paper, we prove the main theorem using the classification of groups with given 2-Sylow subgroups of order 8 and the relation between the set of involutions of the given group and its normal subgroups.

2. Notation and preliminary results

Throughout this paper, we use the following notation: for a finite group G, by $O(G)$ and $O'(G)$ we denote the maximal normal subgroup of odd order in G and the maximal normal subgroup of odd index in G, respectively. Also, the set of all p-Sylow subgroups of G is denoted by $\mathrm{Syl}_p(G)$, and the set of all involutions of G by $T(G)$. The p-part of G, denoted by $|G|_p$, is the order of any $P \in \mathrm{Syl}_p(G)$. We indicate by $n_p(G)$ the number of p-Sylow subgroups of G. Let $t(G) = |T(G)|$ and let $t_0(G) = t(G)/|G|$. It is evident that for every finite group G, $t_0(G) < 1$. If $H \leq G$ and $x, g \in G$, then for simplicity of notation, we write H^g and x^g instead of $g^{-1}Hg$ and $g^{-1}xg$, respectively. All further unexplained notation is standard and can be found in [3].

We start with some known facts about the structure of the finite group G with given 2-Sylow subgroups and some facts about the Frobenius groups:

2.1. Lemma. *Let G be a finite group and $S \in \mathrm{Syl}_2(G)$. Then:*

- (i) [7] *If S is cyclic, then G is 2-nilpotent (i.e. G has a normal complement to a 2-Sylow subgroup). In particular, G is solvable.*
- (ii) [9] *If S is abelian, then $O'(G/O(G))$ is a direct product of a 2-group and simple groups of one of the following types:*
  - (a) *$PSL_2(2^n)$, where $n > 1$;*
  - (b) *$PSL_2(q)$, where $q \equiv 3$ or $5 \pmod 8$ and $q > 3$;*
  - (c) *a simple group S such that for each involution J of S, $C_S(J) = \langle J\rangle \times R$, where R is isomorphic to $PSL_2(q)$, where $q \equiv 3$ or $5 \pmod 8$.*
- (iii) [7, P. 462] (The dihedral theorem) *If S is a dihedral group, then $G/O(G)$ is isomorphic to one of the following groups:*
  - (a) *a 2-Sylow subgroup of G;*
  - (b) *the alternating group A_7;*
  - (c) *a subgroup of $\mathrm{Aut}(PSL_2(q))$ containing $PSL_2(q)$, where q is odd.*

  In particular if G is simple, then G is isomorphic to either A_7 or $PSL_2(q)$, where $q > 3$ is odd.
- (iv) [2] *If G is a non-abelian simple group, S is an elementary belian group of order $q = 2^n$ and for every $x \in S - \{1\}$, $C_G(x) = S$, then $G \cong PSL_2(q)$.*

2.2. Remark. [7, P. 480] In Lemma 2.1(ii), for the simple groups of third type, Janko and Thompson showed that q is either 5 or 3^{2n+1}. In the case where $q = 5$, R has been shown to be a particular group of order $2^3.3.5.7.11.19$, by Janko, which is named the Janko group J_1. If $q = 3^{2n+1}$, then there is one other infinite family of simple groups, ${}^2G_2(3^{2n+1})$, for $n > 0$, discovered by Ree. It has been proved that these groups are the only examples.

2.3. Lemma. [7, Theorem 3.1, P. 339] *Let G be a Frobenius group with the kernel K and the complement H. Then K is nilpotent. In particular, if $|H|$ is even, then K is abelian.*

In 1993, Berkovich proved the following lemmas:

2.4. Lemma. [1] *If G is a non-solvable group, then $t_0(G) \geq 1/4$ if and only if $G = PSL(2,5) \times E$, where $\exp(E) \leq 2$;*

In the following two lemmas, we collect some facts about finite groups and the number of their involutions which are known and we just give hints of their proofs.

2.5. Lemma. *Let G_1 and G_2 be two finite groups and let H be a normal subgroup of G_1. Then the following hold:*

(i) $t(G_1 \times G_2) = t(G_1)t(G_2) + t(G_1) + t(G_2)$;

(ii) *if $[G_1 : H]$ is an odd number, then $t(G_1) = t(H)$;*

(iii) *if $|G_1|$ is even, then $t(G_1)$ is an odd number. In particular if $t_0(G_1) = 1/8$, then $|G_1|_2 = 8$;*

(vi) *let $x \in G_1$ and $cl_H(x) = \{h^{-1}xh : h \in H\}$. Then $|cl_H(x)| = [H : C_H(x)]$.*

Proof. Since $T(G_1 \times G_2) = \{(x, y) : x \in T(G_1),\ y \in T(G_2)\} \cup \{(x, 1) : x \in T(G_1)\} \cup \{(1, y) : y \in T(G_2)\}$, the proof of (i) is straightforward. For the proof of (ii), it is easy to see that if $[G_1 : H]$ is an odd number, then $\mathrm{Syl}_2(G_1) = \mathrm{Syl}_2(H)$ and hence, (ii) follows. For the proof of (iii), since for every $x \in G_1 - T(G_1)$, x and x^{-1} are distinct elements in $G_1 - T(G_1)$, and $1 \in G_1 - T(G_1)$, we deduce that $|G_1 - T(G_1)|$ is an odd number and hence, $t(G_1) = |G_1| - |G_1 - T(G_1)|$ is an odd number, too. Also if $t_0(G_1) = 1/8$, then $|G_1| = 8t(G_1)$ and hence, $|G_1|_2 = 8$, so (iii) follows. It remains to prove (iv). We can see that the function $\varphi : \{hC_H(x) : h \in H\} \longrightarrow cl_H(x)$ such that for every $hC_H(x)$, $\varphi(hC_H(x)) = hxh^{-1}$ is a bijection and hence, $|cl_H(x)| = [H : C_H(x)]$, as claimed in (iv). □

2.6. Lemma. *Let G be a finite group with a normal subgroup H of odd order and $\bar{G} = G/H$. If by $\bar{x}$, we denote the image of an element x of G in $\bar{G}$, then we have the following:*

(i) *For every involution $\bar{x} \in T(\bar{G})$, there exists an involution $y \in G$ such that $\bar{x} = \bar{y}$;*

(ii) *if $H \neq 1$ and $P \in \mathrm{Syl}_2(G)$ is an abelian non-cyclic group or a dihedral group of order 8, then $t(\bar{G})|H| \gneqq t(G)$;*

(iii) *if $cl_{\bar{G}}(\bar{x}_1), ..., cl_{\bar{G}}(\bar{x}_n)$ are distinct conjugacy classes of involutions in $\bar{G}$, then we can assume that $x_1, ..., x_n$ are distinct involutions in G and there exist natural number $t_1, ..., t_n$ such that $t_1, ..., t_n \leq |H|$ and $\sum_{i=1}^{n} |cl_{\bar{G}}(\bar{x}_i)|t_i = t(G)$.*

Proof. For the proof of (i), it suffices to establish that for the case where $\bar{x}$ is an arbitrary element in $T(\bar{G})$ and x is not an involution. Since $\bar{x} \in T(\bar{G})$ and $|H|$ is odd, we can see that $x^2 \in H$ and $O(x) = 2m$, where m is an odd number greater than 1. Thus there exist the integer numbers r and $s = 2k+1$ (for some integer k) such that $2r+sm = 1$ and hence, $x = (x^2)^r(x^2)^{mk}x^m$, which implies that $\bar{x} = \bar{x}^m$ and hence, (i) follows, because $y = x^m$ is an involution in G. For the proof of (ii), suppose, contrary to our claim, that $t(G) \geq t(\bar{G})|H|$. A routine argument shows that $T(G) = \{xh : \bar{x} \in T(\bar{G}),\ h \in H\}$. According to the structure of P, we can choose distinct involutions x and y in P such that $O(xy) = 2$. Since $x,\ y,\ xy \in T(G)$, for every $h \in H$, we have $O(xh) = O(yh) = O(xyh) = 2$. This implies that $h^{-1} = xyhyx = xyhxhyh = xyxyh = h$. But $|H|$ is odd, so $H = 1$. Now, we are going to prove (iii). By (i), we can assume that $x_1, ..., x_n$ are distinct involutions in G such that $T(\bar{G}) = \bigcup_{i=1}^{n} cl_{\bar{G}}(\bar{x}_i)$ and for every $1 \leq i, j \leq n$ such that $i \neq j$, we have $cl_{\bar{G}}(\bar{x}_i) \cap cl_{\bar{G}}(\bar{x}_j) = \emptyset$. For $x \in T(G)$, put $\mathrm{Inv}(x, H) := \{h \in H : O(xh) = 2\}$ and for every $1 \leq i \leq n$, put $t_i := |\mathrm{Inv}(x_i, H)|$. Obviously, for every $x \in T(G)$, $1 \in \mathrm{Inv}(x, H) \subseteq H$. Thus $t_1, ..., t_n \leq |H|$ are natural numbers. Also, it is evident that for every $g \in G$ and $x \in T(G)$, $|\mathrm{Inv}(x, H)| = |\mathrm{Inv}(x^g, H)|$ and for every $y \in T(G)$, there exists $1 \leq i \leq n$ such that $\bar{y} \in cl_{\bar{G}}(\bar{x}_i)$, so there exists $\bar{x}_i^{g} \in cl_{\bar{G}}(\bar{x}_i)$ such that $(x_i^g)^{-1}y \in \mathrm{Inv}(x_i^g, H)$ and hence,

$T(G) = \{x_i^g h : h \in \mathrm{Inv}(x_i^g, H)\}$. Now, we can see that $t(G) = \sum_{i=1}^{n} |cl_{\bar{G}}(\bar{x}_i)|t_i$, which is the desired conclusion. $\square$

2.7. Lemma. *Let q be an odd number and $q > 3$. Then*

(i) $t(PSL_2(q)) = \begin{cases} q(q-1)/2, & \textit{if } q \equiv 3 \pmod 4 \\ q(q+1)/2, & \textit{if } q \equiv 1 \pmod 4 \end{cases}$;

(ii) *there exist involutions $x_1, x_2 \in T(PGL_2(q))$ such that $t(PGL_2(q)) = |cl_{PGL_2(q)}(x_1)| + |cl_{PGL_2(q)}(x_2)|$ and, $|cl_{PGL_2(q)}(x_1)| = q(q-1)/2$ and $|cl_{PGL_2(q)}(x_2)| = q(q+1)/2$.*

Proof. The Dickson's result about the maximal subgroups of $PSL_2(q)$ (see [5]) shows that the dihedral group D_{q+1} of order $(q+1)$, where $q \neq 7,\ 9$ and the dihedral group D_{q-1} of order $(q-1)$, where $q \geq 13$ are maximal subgroups of $PSL_2(q)$. First let $q \geq 13$. Obviously, if $q \equiv 3 \pmod 4$, then $(q+1)/2$ is even and hence, the center of D_{q+1} contains an involution x^+ and if $q \equiv 1 \pmod 4$, then $(q-1)/2$ is even and hence, the center of D_{q-1} contains an involution x^-. This shows that if $q \equiv 3 \pmod 4$, then $C_{PSL_2(q)}(x^+) = D_{q+1}$ and if $q \equiv 1 \pmod 4$, then $C_{PSL_2(q)}(x^-) = D_{q-1}$. But $\gcd(2, q) = 1$ and hence, every involution of $PSL_2(q)$ is a semi-simple element of $PSL_2(q)$. Thus for every involution x of $PSL_2(q)$, there exists a maximal torus containing x. It is known that the maximal torus of $PSL_2(q)$ are cyclic groups of order $(q \pm 1)/\gcd(2, q-1)$. Thus in the case where $q \equiv 3 \pmod 4$, the involutions are contained in the maximal torus of order $(q+1)/2$ and in the case where $q \equiv 1 \pmod 4$, the involutions are contained in the maximal torus of order $(q-1)/2$. On the other hand, it is known that the maximal torus of $PSL_2(q)$ of the same order are conjugate and hence, all involutions in $PSL_2(q)$ are conjugate in $PSL_2(q)$. Therefore, if $q \equiv 3 \pmod 4$, then $t(PSL_2(q)) = |cl_{PSL_2(q)}(x^+)| = q(q-1)/2$ and if $q \equiv 1 \pmod 4$, then $t(PSL_2(q)) = |cl_{PSL_2(q)}(x^-)| = q(q+1)/2$, as claimed in (i). If $q \leq 11$, then ATLAS [3], completes the proof of (i). For the proof of (ii), since the dihedral group $D_{2(q+1)}$ of order $2(q+1)$ and the dihedral group $D_{2(q-1)}$ of order $2(q-1)$, where $q > 5$ are the maximal subgroups of $PGL_2(q)$, and the maximal torus of $PGL_2(q)$ are cyclic groups of order $(q \pm 1)$, the same argument as in the proof of (i) shows that $T(PGL_2(q)) = cl_{PGL_2(q)}(x^+) \cup cl_{PGL_2(q)}(x^-)$, where x^+ and x^- are central involutions of $D_{2(q+1)}$ and $D_{2(q-1)}$, and also $C_{PGL_2(q)}(x^+) = D_{2(q+1)}$ and $C_{PGL_2(q)}(x^-) = D_{2(q-1)}$. Thus $t(PGL_2(q)) = |cl_{PGL_2(q)}(x^+)| + |cl_{PGL_2(q)}(x^-)| = q(q-1)/2 + q(q+1)/2$. This completes the proof of (ii). Also, if $q = 5$, then according to ATLAS [3], the result is obvious. $\square$

3. Proof of the main theorem

Let $P \in \mathrm{Syl}_2(G)$. Under the assumption of the main theorem, $t_0(G) = 1/8$ and hence, Lemma 2.5(iii) shows that $|P| = 8$. Thus, the proof falls naturally into three cases: P is an abelian group, the quaternion group or the dihedral group of order 8:

Case 1. If P is abelian, then since G is non-solvable, we deduce from Lemma 2.1(i) that P is not cyclic. Also, Lemma 2.1(ii) implies that $O'(G/O(G))$ is a direct product of a 2-group and simple groups of one of the types mentioned in 2.1(ii)(a-c). For abbreviation, put $H := O(G)$, $K/H := O'(G/O(G))$ and let $\bar{x}$ be the image of an element x of G in G/H. Obviously $|H|$ and $[G : K]$ are odd numbers. So Lemma 2.5(i) guarantees that $t(G) = t(K)$. On the other hand, $t(G) = |G|/8$ and hence, $t(K) = |G|/8$. This implies that $t_0(K) = [G : K]/8$. If $[G : K] \neq 1$, then since $[G : K]$ is an odd number and $t_0(K) < 1$, we deduce that $[G : K] \in \{3, 5, 7\}$. This forces $t_0(K)$ to be greater than $1/4$. Now, we apply Lemma 2.4 to conclude that $K \cong PSL_2(5) \times E$, where $\exp(E) \leq 2$. Since $|G|_2 = 8$ and $[G : K]$ is odd, we have $|K|_2 = 8$ as well and hence, $|E| = 2$ and $|K| = 120$. Thus Lemmas 2.5(i) and 2.7(i) allow us to conclude that $t(K) = 31$. But as was obtained above, $t(K) = t_0(K)|K| = [G : K]|K|/8 \in \{3|K|/8, 5|K|/8, 7|K|/8\} = \{45, 75, 105\}$, which is a contradiction. This shows that $[G : K] = 1$. Thus $G/H = O'(G/H)$. Now according to Lemma 2.1(ii) and Remark 2.2, we have the following possibilities for G/H:

(i) $G/H \cong F \times PSL_2(2^n)$, where $n > 1$ and F is a 2-group. Then since $|G|_2 = 8$, we obtain that either $(n, |F|) = (2, 2)$ or $(n, |F|) = (3, 1)$. If $(n, |F|) = (2, 2)$, then by [3], all involutions in $PSL_2(4)$ are conjugate and hence, Lemma 2.6(iii) leads us to find involutions x_1, x_2, x_3 in $T(G)$ and natural numbers r, s, t such that $1 \leq r, s, t \leq |H|$ and $t(G) = r|cl_{G/H}(\bar{x}_1)| + s|cl_{G/H}(\bar{x}_2)| + t|cl_{G/H}(\bar{x}_3)|$, so $r/15 + s + t = |H|$. We claim that either $s > |H|/3$ or $t > |H|/3$. Suppose, contrary to our claim, that $s \leq |H|/3$ and $t \leq |H|/3$. Thus, we have $r/15 \geq |H|/3$, which implies that $r \geq 5|H|$, a contradiction. Therefore, without loss of generality, we can assume that $s > |H|/3$. According to the proof of Lemma 2.6(iii), $s = |H_2|$, where $H_2 = \{h \in H : O(x_2h) = 2\}$. Put $G_2 := \langle x_2, H\rangle$. Then since for every $h \in H_2$, $x_2hx_2h = 1$, we have $h^{-1}x_2h = h^{-2}x_2$. So $s = |H_2| \leq |cl_H(x_2)| = [H : C_H(x_2)]$ and hence, $|C_H(x_2)| < 3$, which forces $C_H(x_2) = 1$. Thus G_2 is a Frobenius group with the kernel H. It follows from Lemma 2.3, H is abelian. Thus it is easy to see that H_2 is a subgroup of H and hence, $|H_2|$ divides $|H|$. Moreover, since $s > |H|/3$, we have $|H| = |H_2| = s$. Thus $r/15 + t + s = |H|$ forces $r = t = 0$, which is a contradiction. It remains to consider the case where $(n, |F|) = (3, 1)$. Since ATLAS [3] shows that $t(G/H) = |PSL_2(8)|/8$, $|H|t(G/H) = t(G)$. Thus by Lemma 2.6(ii), $H = 1$ and hence, in this case, $G \cong PSL_2(8)$.

(ii) $G/H \cong F \times PSL_2(q)$, where F is a 2-group and, $q \equiv 3$ or $5 \pmod 8$ and $q > 3$. Then since $|G/H|_2 = |G|_2 = 8$, we obtain that $|F| = 2$. Since $PSL_2(5) \cong PSL_2(4)$, as mentioned in (i), we can see that $G/H \ncong PSL_2(5) \times F$. This allows us to assume that $q > 5$. Thus by Lemmas 2.5(i) and 2.7(i), $t(G/H) = 2(|cl_{PSL_2(q)}(x)|) + 1 = 2q(q \pm 1)/2 + 1$, where $x \in T(PSL_2(q))$. Thus Lemma 2.6(ii) gives that either $H \neq 1$ and $2(|cl_{PSL_2(q)}(x)|)+1 > 2|PSL_2(q)|/8$ or $H = 1$ and $2(|cl_{PSL_2(q)}(x)|) + 1 = 2|PSL_2(q)|/8$. Obviously, $2(|cl_{PSL_2(q)}(x)|) + 1 \neq 2|PSL_2(q)|/8$. Therefore, $H \neq 1$ and hence, $4 \mid |C_{PSL_2(q)}(x)| \leq 8$. This ensures that $|C_{PSL_2(q)}(x)| = 4$. Now applying Lemma 2.1(iv) to $PSL_2(q)$ shows that $q = 4$, which is a contradiction.

(iii) G/H is isomorphic to the Janko group J_1. Then by ATLAS [3], $t(G/H) = 7.11.19$ and hence, $t(G/H) < \frac{|J_1|}{8}$, which is a contradiction with Lemma 2.6(ii).

(iv) G/H is isomorphic to the Ree group ${}^2G_2(q)$, where $q = 3^{2n+1}$, for $n > 0$. Then applying Lemma 2.1(ii)(c) and Remark 2.2 show that for each involution J of ${}^2G_2(q)$, $C_{{}^2G_2(q)}(J) = \langle J\rangle \times R$, where R is isomorphic to $PSL_2(q)$ and the 2-Sylow subgroups of ${}^2G_2(q)$ are 2-elementary abelian. Thus the 2-Sylow subgroup PH/H of G/H contains 7 elements of order 2. On the other hand, all 2-Sylow subgroups of G/H are conjugate in G/H and hence, every involution of G/H is conjugate with one of the involutions in PH/H. Thus $t(G/H) \leq |\bigcup_{\bar{x} \in PH/H - \{\bar{1}\}} cl_{G/H}(\bar{x})| \leq \frac{7|G/H|}{2|PSL_2(q)|} = \frac{7|G/H|}{q(q^2-1)}$, which is less than $\frac{|G/H|}{8}$, a contradiction with Lemma 2.6(ii).

Consequently, the above results show that if P is abelian, then G is a simple group isomorphic to $PSL_2(8)$.

Case 2. Let P be a quaternion group of order 8. Then since $P \leq N_G(P)$, $8 \mid |N_G(P)|$. Thus $n_2(G) \leq |G|/8$. If there exist $P_1, P_2 \in \mathrm{Syl}_2(G)$ such that $P_1 \cap P_2 \neq 1$, then $P_1 \cap P_2$ contains an element of order 2. On the other hand, the quaternion group has exactly one element of order 2. This implies that $t(G) < n_2(G) \leq |G|/8$, which is a contradiction. Thus the intersection of every two 2-Sylow subgroups of G is trivial and hence, $t(G) = n_2(G)$. So by our assumption, $n_2(G) = |G|/8$, which forces $|N_G(P)| = 8$ and hence, $N_G(P) = P$. This implies that for every $x \in G - P$, $x^{-1}Px \cap P = 1$ and hence, G is a Frobenius group and P is its complement. It follows immediately from the fact which the kernel of G and P are nilpotent that G is solvable, which is a contradiction with our assumption.

Case 3. Let P be a dihedral group of order 8. Put $H := O(G)$. Then by Lemma 2.1(iii), we have the following possibilities for G/H:

(i) G/H is a 2-group. Thus G/H is solvable. But $H = O(G)$ is a solvable group, because by our assumption $|H|$ is odd. We thus get that G is solvable, which is a contradiction with our assumption.

(ii) $G/H \cong A_7$. Then it is easy to check that $t(G/H) = t(A_7) = \frac{7.6.5.4}{8} < \frac{|A_7|}{8}$, which is a contradiction with Lemma 2.6(ii).

(iii) G/H is isomorphic to a subgroup of $\mathrm{Aut}(PSL_2(q))$ containing $PSL_2(q)$, where q is odd. It is known that $\mathrm{Aut}(PSL_2(q)) \cong PGL_2(q).\mathbb{Z}_n$, where $q = p^n$, for a prime p. In the following, we will consider the cases $q \equiv \pm 1 \pmod 8$ and $q \equiv \pm 3 \pmod 8$, separately:

(a) Let $q \equiv \pm 1 \pmod 8$. Then since $|PSL_2(q)|_2 = |G|_2 = 8$, we deduce that $G/H = K/H.\mathbb{Z}_m$, where $K/H \cong PSL_2(q)$ and m is an odd divisor of n and hence, the same conclusion as that of in the proof of Case 1 can be drawn to conclude that $[G : K] = 1$. Thus $G/H \cong PSL_2(q)$. So from Lemma 2.7(i), we obtain that

$$t(G/H) = \begin{cases} q(q-1)/2, & \text{if } q \equiv -1 \pmod 8 \\ q(q+1)/2, & \text{if } q \equiv 1 \pmod 8 \end{cases}.$$

If $H \neq 1$, then Lemma 2.6(ii) yields $t(G/H) > |G/H|/8$ and hence,

$$\begin{cases} q+1 < 8, & \text{if } q \equiv -1 \pmod 8 \\ q-1 < 8, & \text{if } q \equiv 1 \pmod 8 \end{cases},$$

which is impossible. This forces H to be a trivial group and hence, $G \cong PSL_2(q)$. Thus Lemma 2.7(i) leads to

$$|PSL_2(q)|/8 = t(PSL_2(q)) = \begin{cases} q(q-1)/2, & \text{if } q \equiv -1 \pmod 8 \\ q(q+1)/2, & \text{if } q \equiv 1 \pmod 8 \end{cases},$$

so

$$\begin{cases} q+1 = 8, & \text{if } q \equiv -1 \pmod 8 \\ q-1 = 8, & \text{if } q \equiv 1 \pmod 8 \end{cases}.$$

Consequently, $q \in \{7, 9\}$ and hence, $G \cong PSL_2(7)$ or $PSL_2(9)$.

(b) Let $q \equiv \pm 3 \pmod 8$. Since $q = p^n \equiv \pm 3 \pmod 8$, an easy computation shows that n is odd and hence, according to $|PSL_2(q)|_2 = 4$, we deduce that $G/H = K_0/H.\mathbb{Z}_m$, where $K_0/H \cong PGL_2(q)$ and m is an odd divisor of n. Now, as in the proof of Case 1, we see that $m = 1$ and hence, $G/H = K_0/H = K/H.\mathbb{Z}_2$, where $K/H \cong PSL_2(q)$. Now we conclude from Lemmas 2.6(ii) and 2.7(ii) that $t(PGL_2(q)) = q(q-1)/2 + q(q+1)/2 = q^2 \geq \frac{|G|}{8|H|} = \frac{|PGL_2(q)|}{8}$. This shows that $q^2 \geq q(q^2-1)/8$, so $q \in \{3, 5\}$. If $q = 3$, then $PGL_2(3) \cong S_4$ and hence, G/H is solvable. Also the fact that $|H|$ is odd, forces H to be solvable and hence, G is solvable, which is a contradiction with our assumption.

It remains to consider the case where $q = 5$. Let $q = 5$. For abbreviation assume that $G/H = PGL_2(5)$ and $K/H = PSL_2(5)$. According to Lemma 2.7(ii), $t(PGL_2(5)) \neq |PGL_2(5)|/8$ and hence, $H \neq 1$. Let $\bar{x}$ be the image of an element x of G in G/H. Note that $PSL_2(q)$ is a normal subgroup of $PGL_2(q)$ and hence, every 2-Sylow subgroup $\bar{P}$ of $PGL_2(q)$ contains an involution $\bar{x} \in \bar{P} \cap PSL_2(q)$. Thus, we conclude from Lemmas 2.6(iii) and 2.7(ii) that there exist involutions $\bar{x}_1 \in T(PSL_2(q))$ and $\bar{x}_2 \in T(PGL_2(q))$, and natural numbers s, t such that $1 \leq s, t \leq |H|$, $|cl_{PGL_2(q)}(\bar{x}_1)| = \frac{|PGL_2(5)|}{2(5-1)}$ and $|cl_{PGL_2(q)}(\bar{x}_2)| = \frac{|PGL_2(5)|}{2(5+1)}$, and $\frac{s|PGL_2(5)|}{2(5-1)} + \frac{t|PGL_2(5)|}{2(5+1)} = s|cl_{PGL_2(q)}(\bar{x}_1)| + t|cl_{PGL_2(q)}(\bar{x}_2)| = t(G) = |G|/8$. This gives

$$3s + 2t = 3|H|. \tag{3.1}$$

By Lemma 2.6(i), we have $x_1, x_2 \in T(G)$. For $i \in \{1, 2\}$, put $H_i := \{h \in H : O(x_i h) = 2\}$. Then as mentioned in the proof of Lemma 2.6(iii), $|H_1| = s$ and $|H_2| = t$. Put $G_1 := \langle x_1, H \rangle$. Since $t \leq |H|$, we see $|H| \leq 3|H| - 2t = 3s$ and hence, $s \geq |H|/3$. Thus one of the following possibilities holds:

(I) $s > |H|/3$. Obviously if $h \in H_1$, then $x_1 h x_1^{-1} = h^{-1}$, so $h x_1 h^{-1} = x_1 h^{-2}$ and hence, $|H|/3 < s = |H_1| \leq |cl_{G_1}(x_1)|$. Thus $[G_1 : C_{G_1}(x_1)] > |H|/3$. This shows that $|C_{G_1}(x_1)| < 6$. On the other hand, $2 \mid |C_{G_1}(x_1)|$, $|G_1| = 2|H|$ and $|H|$ is an odd number. This gives $|C_{G_1}(x_1)| = 2$ and hence, $C_{G_1}(x_1) = \langle x_1 \rangle$. It follows that $C_H(x_1) = 1$. This forces G_1 to be a Frobenius group with the kernel H. Thus H is abelian, by Lemma 2.3 and hence, it follows easily that H_1 is a subgroup of H, so $|H_1|$ divides $|H|$. But $|H|$ is an odd number and $|H_1| = s > |H|/3$. Thus $|H_1| = |H|$ and hence, (3.1) shows that $t = 0$, which is impossible.

(II) $s = |H|/3$. Then (3.1) shows that $|H| + 2t = 3|H|$ and hence, $t = |H|$. This gives that $H_2 = H$. Thus for every $h \in H$, $O(x_2 h) = 2$, so $h^{x_2} = h^{-1}$. Let $h_1, h_2 \in H$, then $h_2^{-1} h_1^{-1} = (h_1 h_2)^{x_2} = h_1^{x_2} h_2^{x_2}$ and hence, $h_2^{-1} h_1^{-1} = h_1^{-1} h_2^{-1}$. This shows that H is abelian. Thus for every $k_1, k_2 \in H_1$, $(k_1 k_2)^{x_1} = k_1^{x_1} k_2^{x_1} = k_1^{-1} k_2^{-1} = (k_1 k_2)^{-1}$, so $O(x_1 k_1 k_2) = 2$ and hence, $k_1 k_2 \in H_1$. Obviously, $k_1^{-1} \in H_1$ and hence, $H_1 \leq H$. As above, we can assume that there exist $g, y \in K$ and $h \in H$ such that $x_1 x_1^g = x_1^y h$, because there exists $\bar{g} \in PSL_2(q)$ such that $\bar{x}_1 \bar{x}_1^{\bar{g}}$ is an involution in $PSL_2(q)$ and hence, since $T(PSL_2(q)) = cl_{PSL_2(q)}(\bar{x}_1)$, we deduce that there exists $y \in K$ such that $x_1 x_1^g \in x_1^y H$. First let $H_1 H_1^g \lneqq H$, then we see that $|H|/3$ divides $3|H_1 \cap H_1^g|$ and $|H_1 \cap H_1^g|$ divides $|H_1| = |H|/3$. Thus either $|H_1 \cap H_1^g| = |H|/9$ or $|H_1 \cap H_1^g| = |H|/3$. If $|H_1 \cap H_1^g| = |H|/9$, then $|H_1 H_1^g| = |H|$ and hence, $H_1 H_1^g = H$, which is a contradiction. Thus $|H_1 \cap H_1^g| = |H|/3$. This forces $[H_1 : H_1 \cap H_1^g] = 1$, which shows that $H_1 = H_1 \cap H_1^g$. So $H_1 = H_1^g$. It follows immediately that for every $h \in H_1$, there exists $h_1 \in H_1$ such that $h = h_1^g$ and hence, $(x_1 x_1^g)^{-1} h (x_1 x_1^g) = (x_1^g)^{-1} h^{-1} x_1^g = (x_1^{-1} h_1^{-1} x_1)^g = h_1^g = h$, because if $h \in H_1$, then $O(x_1 h) = 2$, which shows that $x_1^{-1} h x_1 = h^{-1}$. This implies that $H_1 \leq C_H(x_1 x_1^g)$ and hence, $|cl_H(x_1 x_1^g)| \leq 3$. But as mentioned above, $x_1 x_1^g = x_1^y h$. Thus, since H is an abelian group of an odd order, we deduce that for every $u \in H_1^y$, $u^{-1} x_1 x_1^g u = u^{-1} x_1^y h u = u^{-1} x_1^y u h = u^{-2} x_1^y h$, so $|H|/3 = |H_1^y| \leq |cl_H(x_1 x_1^g)|$. From this, we conclude that $|H|/3 \leq 3$ and hence, $|H| \in \{3, 9\}$, because $3 \mid |H|$ and $2 \nmid |H|$. If $|H| = 9$, then $|cl_H(x_1 x_1^g)| = 3$. But H is an abelian normal subgroup of G and $x_1 x_1^g = x_1^y h$.

Thus $C_H(x_1) = C_H(x_1^y)^{y^{-1}} = C_H(x_1^y h)^{y^{-1}}$ and hence, $|C_H(x_1)| = 3$. Thus

$$x_1 \notin C_G(H). \tag{3.2}$$

Also, $H \le C_G(H)$ and hence, $\frac{C_G(H)}{H} \trianglelefteq \frac{G}{H} = PGL_2(5)$. Thus $\frac{C_G(H)}{H} = PGL_2(5)$, $\frac{C_G(H)}{H} = PSL_2(5)$ or $\frac{C_G(H)}{H} = 1$. If $\frac{C_G(H)}{H} = PSL_2(5) = K/H$ or $\frac{C_G(H)}{H} = PGL_2(5) = G/H$, then $K \le C_G(H)$ and hence, $x_1 \in C_G(H)$, contrary to (3.2). Thus $\frac{C_G(H)}{H} = 1$, which means that $C_G(H) = H$. But $|H| = 9$ and hence, $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $H \cong \mathbb{Z}_9$. So by $N - C$-theorem, we obtain $PGL_2(5) = G/H = N_G(H)/C_G(H) \lesssim \mathrm{Aut}(H) \cong GL_2(3)$ or $\mathbb{Z}_8$. This forces 5 to divide $|GL_2(3)|$ or $|\mathbb{Z}_8|$, a contradiction. If $|H| = 3$, then applying the same reasoning as above shows that $\frac{C_G(H)}{H} = PSL_2(5) \cong SL_2(4)$, which leads us to see that $G \cong (\mathbb{Z}_3 \times SL_2(4)).\mathbb{Z}_2 \cong GL_2(4).\mathbb{Z}_2$. Now, a trivial verification in GAP [6] shows that $t(GL_2(4).\mathbb{Z}_2) = 2|GL_2(4)|/8$. Thus G can be isomorphic to $GL_2(4).\mathbb{Z}_2$. Now consider the case where $H_1 H_1^g = H$, then a slight change in the above statements shows that $|H| \in \{9, 27\}$ and $|H_1 \cap H_1^g| = |H|/9$. If $|H| = 9$, then applying the previous argument leads us to get a contradiction. Thus $|H| = 27$ and hence, our assumption forces H to be isomorphic to $\mathbb{Z}_9 \times \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Now, the same argument as above gives that $H = C_G(H)$. Thus $PGL_2(5) = \frac{G}{H} = \frac{N_G(H)}{C_G(H)} \lesssim \mathrm{Aut}(H)$. But $|\mathrm{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)| = 11232$ and $|\mathrm{Aut}(\mathbb{Z}_9 \times \mathbb{Z}_3)| = |GL_3(3)| = 108$, so $|PGL_2(5)| \nmid |\mathrm{Aut}(H)|$, a contradiction.

Consequently, the above cases show that either G is a simple group isomorphic to $PSL_2(q)$, where $q \in \{7, 8, 9\}$ or $G \cong GL_2(4).\mathbb{Z}_2$, as desired.

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