Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 33 – 39

# Inequalities with conjugate exponents in grand Lebesgue spaces

René Erlín Castillo* and Humberto Rafeiro†

**Abstract**

In this paper we want to show the validity of the generalized double-parametric Hilbert inequality with conjugate exponents in the framework of grand Lebesgue spaces as a particular case of a more general result involving an homogeneous kernel. We also study the boundedness of an integral operator with the aforementioned kernel.

*2000 AMS Classification:* Primary 46E30; Secondary 26D15

**Keywords:** Lebesgue spaces, grand Lebesgue spaces, Hilbert inequality

Received 02/11/2013 : Accepted 13/12 /2013 Doi : 10.15672/HJMS.2015449087

## 1. Introduction

The classical Hilbert inequality on double series is the following

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_n b_m}{n+m} < \pi\csc(\pi/p)\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}b_m^{p'}\right)^{\frac{1}{p'}} \tag{1.1}$$

where $p$ and $p'$ are conjugate exponents and $a_n, b_m \geqslant 0$. In accordance to [10], this inequality was included by Hilbert for $p = p' = 2$ in his lectures, and it was published by H. Weyl [25]. The estimate was later on improved by Schur [24] obtaining the sharp constant $\pi$. It is possible to show that the constant $\pi\csc(\pi/p)$ is sharp for the case $p > 1$. The integral analogue for (1.1) is the so-called *double-parametric Hilbert inequality*

$$\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+}\frac{f(y)g(x)}{x+y}\,\mathrm{d}x\,\mathrm{d}y < \pi\csc(\pi/p)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^{p'}(\mathbb{R}_+)}, \tag{1.2}$$

which was shown in [10].

*Universidad Nacional de Colombia, Departamento de Matemáticas, Cra. 45 No. 26–85, Bogotá, Colombia.
Email: recastillo@unal.edu.co
†Pontificia Universidad Javeriana, Departamento de Matemáticas, Cra. 7 No. 43–82, Bogotá, Colombia.
Email:silva-h@javeriana.edu.co

We can generalize (1.2) substituting $1/(x+y)$ by a suitable kernel $k$, in this case the constant will depend on $k$ and $p$. Another generalization is to drop the requirement that the exponents are conjugate, this leads to the so-called *inequalities with non-conjugate exponents*.

In 1992 T. Iwaniec and C. Sbordone [12], in their studies related with the integrability properties of the Jacobian in a bounded open set $\Omega$, introduced a new type of function spaces $L^{p)}(\Omega)$, called *grand Lebesgue spaces*. A generalized version of them, $L^{p),\theta}(\Omega)$ appeared in L. Greco, T. Iwaniec and C. Sbordone [9]. Harmonic analysis related to these spaces and their associate spaces (called *small Lebesgue spaces*), was intensively studied during last years due to various applications, we mention e.g. [1, 3, 4, 5, 6, 7, 14, 17, 22] and continue to attract attention of various researchers.

For example, in the theory of PDE's, it turned out that these are the right spaces in which some nonlinear equations have to be considered (see [8, 9]). Also noteworthy to mention the extension of the ideas regarding grand Lebesgue spaces into the framework of the so-called grand Morrey spaces, e.g. [16, 15, 18, 20, 21].

In this paper we show the validity of the generalized double-parametric Hilbert inequality with conjugate exponents in the framework of grand and small Lebesgue spaces as a particular case of a more general result involving an homogeneous kernel. We also study the boundedness in grand Lebesgue spaces of an integral operator with the aforementioned kernel. Finally, we state an open problem related with the so-called *inequalities with non-conjugate exponents*.

## 2. Grand and Small Lebesgue Spaces

In this section we will introduce grand and small Lebesgue spaces, for more properties, see e.g. [2, 4, 14].

**2.1. Definition.** By $\Phi$ we denote the class of continuous positive functions on $(0, p-1)$ such that $\lim_{x\to 0+}\varphi(x) = 0$.

**2.2. Definition.** Let $(\mathsf{X}, \mathscr{A}, \mu)$ be a finite measure space and $\varphi \in \Phi$. The *grand Lebesgue space*, denoted by $L^{p),\varphi}(\mathsf{X})$, is the set of all real-valued measurable functions for which

$$\|f\|_{L^{p),\varphi}(\mathsf{X})} := \sup_{0<\varepsilon<p-1}\left(\varphi(\varepsilon)\fint_{\mathsf{X}} |f(x)|^{p-\varepsilon}\,\mathrm{d}\mu(x)\right)^{\frac{1}{p-\varepsilon}} < \infty,$$

where $\fint_\omega f\,\mathrm{d}\mu(x) := \frac{1}{\mu(\omega)}\int_\omega f\,\mathrm{d}\mu(x)$ stands for the integral average of the function $f$ in $\omega$.

Taking $\varphi : (0, p-1) \to (0, +\infty)$ with $x \mapsto x$ and the induced Lebesgue measure in a bounded subset of the Euclidean space, we recover the space introduced by T. Iwaniec and C. Sbordone in [12] and we get the space introduce in [9] when $x \mapsto x^\theta$.

**2.3. Definition.** Let $(\mathsf{X}, \mathscr{A}, \mu)$ be a finite measure space and $1 < p < \infty$. If $\varphi \in \Phi$, we define the *small Lebesgue space* $L^{p)',\varphi}(\mathsf{X})$ as

$$L^{p)',\varphi}(\mathsf{X}) = \left\{g \in \mathcal{M}_0 \mid \|g\|_{L^{p)',\varphi}}(\mathsf{X}) < +\infty\right\}$$

where $\mathcal{M}_0$ is the set of all measurable functions, whose values lie in $[-\infty, \infty]$ finite a.e. in $\mathsf{X}$, and

$$\|g\|_{L^{p)',\varphi}}(\mathsf{X}) = \sup_{0\leqslant\psi\leqslant|g|;\psi\in L^{(p'}(\mathsf{X})} \|\psi\|_{L^{(p',\varphi}(\mathsf{X})}$$

with

$$\|g\|_{L^{(p',\varphi}(\mathsf{X})} = \inf_{g=\sum_{k=1}^{\infty} g_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \frac{1}{\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}}} \left( \fint_{\mathsf{X}} |g_k|^{(p-\varepsilon)'} \, \mathrm{d}\mu(x) \right)^{\frac{1}{(p-\varepsilon)'}} \right\}.$$

We want to mention that there is another characterization of grand and small Lebesgue spaces, using rearrangement functions, see e.g. [6].

## 3. Hilbert integral inequality

In this section we show the validity of the generalized double-parametric Hilbert integral inequality.

We need some definitions, namely:

**3.1. Definition.** Let $g$ be a function with domain in $[a, b] \subset \mathbb{R}$ and range $[\alpha, \beta] \subset \mathbb{R}$. If $\mu(E) = 0$ implies that $\mu(g(E)) = 0$ for all $E \subset [a, b]$, then $g$ is said to be an $N$-function or to satisfy the condition $N$.

**3.2. Definition.** We will call to a measure $\mu$, a *$N$-dilation measure*, if $\mu(E) = 0$ implies that $\mu(tE) = 0$, $\mathbb{R} \ni t > 0$, where $tE$ is the dilation of the set $E$.

We will also need the following proposition, see e.g. [11, Corollaries 20.4 and 20.5].

**3.3. Proposition.** *Let $[a, b]$ be an interval in $\mathbb{R}$ and let $\varphi$ be a monotone continuous $N$-function with domain $[a, b]$ and range $[\alpha, \beta]$ $(\alpha < \beta)$. Thus for $f \in L^1([\alpha, \beta], \mathscr{A}, \mu)$, we have $(f \circ \varphi)|\varphi'| \in L^1([a, b], \mathscr{A}, \mu)$, and*

$$\int_{\alpha}^{\beta} f(y) \, \mathrm{d}\mu(y) = \int_{a}^{b} f \circ \varphi(x)|\varphi'(x)| \, \mathrm{d}\mu(x).$$

We now prove the validity of the generalized double-parametric Hilbert inequality.

**3.4. Theorem.** *Let $\mu$ be a $N$-dilation finite measure on $[0, \infty)$, $\varphi \in \Phi$ and $k : [0, \infty) \times [0, \infty) \to [0, \infty)$ be homogeneous of degree -1. If $f \in L^{p),\varphi}(\mathbb{R}_+, \mu)$ and $g \in L^{p)',\varphi}(\mathbb{R}_+, \mu)$ then*

$$\left| \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} k(x, y) f(y) g(x) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \right| \leqslant c \, \|f\|_{L^{p),\varphi}(\mathbb{R}_+,\mu)} \|g\|_{L^{p)',\varphi}(\mathbb{R}_+,\mu)}. \tag{3.1}$$

*with*

$$c = \inf_{0<\varepsilon<p-1} \int_{\mathbb{R}_+} w^{-\frac{1}{p-\varepsilon}} k(1, w) \, \mathrm{d}\mu(w). \tag{3.2}$$

*Proof.* We first prove the theorem for $g \in L^{(p',\varphi}(\mathbb{R}_+, \mu)$ and then bootstrap the result to $g \in L^{p)',\varphi}(\mathbb{R}_+, \mu)$.

Let $g \in L^{p)',\varphi}(\mathbb{R}_+,\mu)$ and $|g| = \sum_{k\in\mathbb{N}} g_k$ be any decomposition with $g_k \geq 0$ for all $k \in \mathbb{N}$. Taking

$$I_{M,N} = \left| \int_0^M \int_0^N f(y) g_k(x) k(x,y)\, \mathrm{d}\mu(y)\, \mathrm{d}\mu(x) \right|,$$

then we have

$$\begin{aligned} I_{M,N} &\leqslant \int_0^M \mathrm{d}\mu(x) \int_0^N \left| f(y) g_k(x) k(x,y) \right| \mathrm{d}\mu(y) \\ &\leq \int_{\mathbb{R}_+} k(1,w)\, \mathrm{d}\mu(w) \int_0^M \left| f(wx) g_k(x) \right| \mathrm{d}\mu(x) \end{aligned}$$

where we used an appropriate change of variables, the fact that the kernel $k$ is of degree -1 and also the Proposition 3.3 together with the assumption that $\mu$ is a $N$-dilation measure.

For each $0 < \varepsilon < p-1$, we have

$$\begin{aligned} I_{M,N} \leqslant \mu(\mathbb{R}_+) \int_{\mathbb{R}_+} k(1,w) \left( \frac{1}{\mu(\mathbb{R}_+)} \int_0^M |f(wx)|^{p-\varepsilon}\, \mathrm{d}\mu(x) \right)^{\frac{1}{p-\varepsilon}} \mathrm{d}\mu(w) \\ \left( \frac{1}{\mu(\mathbb{R}_+)} \int_0^M |g_k(x)|^{(p-\varepsilon)'}\, \mathrm{d}\mu(x) \right)^{\frac{1}{(p-\varepsilon)'}} \end{aligned}$$

which gives

$$\begin{aligned} I_{\infty,\infty} \lesssim \int_{\mathbb{R}_+} \frac{k(1,w)}{w^{\frac{1}{p-\varepsilon}}}\, \mathrm{d}\mu(w) \left( \varphi(\varepsilon) \fint_{\mathbb{R}_+} |f(x)|^{p-\varepsilon}\, \mathrm{d}\mu(x) \right)^{\frac{1}{p-\varepsilon}} \\ \frac{1}{\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}}} \left( \fint_{\mathbb{R}_+} |g_k(x)|^{(p-\varepsilon)'}\, \mathrm{d}\mu(x) \right)^{\frac{1}{(p-\varepsilon)'}}. \end{aligned} \tag{3.3}$$

Since

$$\left| \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f(y) g(x) k(x,y)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \right| \leqslant \int_{\mathbb{R}_+} \sum_{k\in\mathbb{N}} \int_{\mathbb{R}_+} |f(y) g_k(x) k(x,y)|\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \tag{3.4}$$

we get from (3.3), the definition of grand and small Lebesgue spaces that

$$\begin{aligned} \left| \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f(y) g(x) k(x,y)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \right| \leqslant \\ \int_{\mathbb{R}_+} \frac{k(1,w)}{w^{\frac{1}{p-\varepsilon}}}\, \mathrm{d}\mu(w) \|f\|_{L^{p),\varphi}(\mathbb{R}_+,\mu)} \|g\|_{L^{(p',\varphi}(\mathbb{R}_+,\mu)}. \end{aligned} \tag{3.5}$$

Taking now infimum in (3.5), we obtain (3.1) for the case of $g \in L^{(p',\varphi}(\mathbb{R}_+,\mu)$.

Let now $g \in L^{p'),\varphi}(\mathbb{R}_+,\mu)$ and let us define the truncated function

$$\psi_n(x) = \begin{cases} |g(x)| & \text{if } |g(x)| \leqslant n, \\ n & \text{if } |g(x)| > n, \end{cases}$$

which is in $L^{(p'}(\mathsf{X})$ due to the embedding $L^\infty(\mathsf{X}) \subset L^{(p'}(\mathsf{X})$.

Moreover, for all $0 \leqslant \psi \leqslant |g|$ we have $\|\psi\|_{L^{(p'}}(\mathsf{X}) \leqslant \|g\|_{L^{p)'}}(\mathsf{X})$ which yields

$$\begin{aligned} \left| \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} k(x,y)f(y)\psi(x)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \right| &\leqslant c\|f\|_{L^{p),\varphi}(\mathbb{R}_+,\mu)}\|\psi\|_{L^{(p',\varphi}(\mathbb{R}_+,\mu)} \\ &\leqslant c\|f\|_{L^{p),\varphi}(\mathbb{R}_+,\mu)}\|g\|_{L^{p)',\varphi}(\mathbb{R}_+,\mu)}. \end{aligned}$$

Since $g(x) = \lim_{n\to\infty}\psi_n(x)$, by the Beppo-Levi theorem we get

$$\begin{aligned} \left| \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} k(x,y)f(y)g(x)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \right| &\leqslant \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |k(x,y)f(y)g(x)|\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\ &= \lim_{n\to\infty}\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |k(x,y)f(y)\psi_n(x)|\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\ &\leqslant c\|f\|_{L^{p),\varphi}(\mathbb{R}_+,\mu)}\|g\|_{L^{p)',\varphi}(\mathbb{R}_+,\mu)}. \end{aligned}$$

□

**3.5. Remark.** As an example, taking $\mathrm{d}\mu(x) = \mathrm{e}^{-x}\,\mathrm{d}x$ with the usual Lebesgue measure, and $k(x,y) = 1/(x+y)$ we get

$$\int\limits_{\mathbb{R}_+} \frac{x^{\frac{1}{p-\varepsilon}}}{1+x}\mathrm{e}^{-x}\,\mathrm{d}x \leqslant \int\limits_{\mathbb{R}_+} \frac{x^{\frac{1}{p-\varepsilon}}}{1+x}\,\mathrm{d}x = \pi\csc(\pi/(p-\varepsilon)).$$

thus satisfying the conditions in the Theorem 3.4.

We now prove a theorem regarding the boundedness of an integral operator based on the kernel $k$.

**3.6. Theorem.** *Let $\mu$ be a $N$-dilation finite measure on $[0,\infty)$, $\varphi \in \Phi$ and $k : [0,\infty) \times [0,\infty) \to [0,\infty)$ be homogeneous of degree -1. If $f \in L^{p),\varphi}(\mathbb{R}_+,\mu)$ and define the integral operator $K$ has*

$$Kf(y) = \int\limits_{\mathbb{R}_+} k(x,y)f(x)\,\mathrm{d}\mu(x).$$

*Then*

$$\|Kf\|_{L^{p,\varphi)}(\mathbb{R}_+,\mu)} \leqslant c\,\|f\|_{L^{p),\varphi}(\mathbb{R}_+,\mu)} \tag{3.6}$$

*with*

$$c = \sup_{0<\varepsilon<p-1}\int\limits_{\mathbb{R}_+} w^{-\frac{1}{p-\varepsilon}}k(1,w)\,\mathrm{d}\mu(w). \tag{3.7}$$

*Proof.* For fixed $0 < \varepsilon < p-1$, we have

$$\begin{aligned} \|Kf\|_{L^{p-\varepsilon}(\mathbb{R}_+,\mu)} &= \left( \int\limits_{\mathbb{R}_+} \left| \int\limits_{\mathbb{R}_+} k(z,1) f(zy)\, \mathrm{d}\mu(z) \right|^{p-\varepsilon} \mathrm{d}\mu(y) \right)^{\frac{1}{p-\varepsilon}} \\ &\leqslant \int\limits_{\mathbb{R}_+} k(z,1) \left( \int\limits_{\mathbb{R}_+} |f(zy)|^{p-\varepsilon}\, \mathrm{d}\mu(y) \right)^{\frac{1}{p-\varepsilon}} \mathrm{d}\mu(z) \\ &\leqslant \int\limits_{\mathbb{R}_+} k(z,1) z^{-\frac{1}{p-\varepsilon}}\, \mathrm{d}\mu(z) \|f\|_{L^{p-\varepsilon}(\mathbb{R}_+,\mu)} \end{aligned} \tag{3.8}$$

where we have used the Minkowski integral inequality, a proper change of variable and the homogeneity of the kernel $k$. From (3.8) we get

$$\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|Kf\|_{L^{p-\varepsilon}(\mathbb{R}_+,\mu)} \leqslant \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(\mathbb{R}_+,\mu)} \int\limits_{\mathbb{R}_+} k(z,1) z^{-\frac{1}{p-\varepsilon}}\, \mathrm{d}\mu(z)$$

and now (3.6) follows taking (3.7) into account applying supremum over $0 < \varepsilon < p-1$ in both sides. □

Open Problem: The case of non-conjugate exponents is an open problem. For example, what is the analogue in grand Lebesgue spaces for the following result.

Let $p > 1$, $r > 1$ and $\lambda = \frac{1}{p} + \frac{1}{r}$, then

$$\int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} k^{\lambda}(x,y) f(x) g(y)\, \mathrm{d}x\, \mathrm{d}y \leqslant \left( \int\limits_0^{\infty} u^{\frac{1}{\lambda q}} k(1,u)\, \mathrm{d}u \right)^{\lambda} \|f\|_{L^p(\mathbb{R}_+)} \|g\|_{L^r(\mathbb{R}_+)}$$

where $k(x,y)$ is a non-negative homogeneous kernel of degree -1.


## Acknowledgements

The authors would like to thank the anonymous referee for careful reading the paper. The second named author was partially supported by research project *Inequalities in grand Lebesgue spaces*, `ID PRY: 6054`, of the Faculty of Sciences of the *Pontificia Universidad Javeriana*, Bogotá, Colombia.


## References


[1] C. Capone, A. Fiorenza, *On small Lebesgue spaces*, J. Funct. Spaces Appl., **3** (2005), 73–89.

[2] C. Capone, M.R. Formica and R. Giova, *Grand Lebesgue spaces with respect to measurable functions*, Nonlinear Anal. **85** (2013), 125–131.

[3] G. Di Fratta, A. Fiorenza, *A direct approach to the duality of grand and small Lebesgue spaces.* Nonlinear Anal. **70**(7) (2009), 2582–2592.

[4] A. Fiorenza, *Duality and reflexivity in grand Lebesgue spaces.* Collect. Math. **51**(2) (2000), 131–148.

[5] A. Fiorenza, B. Gupta, P. Jain, *The maximal theorem in weighted grand Lebesgue spaces.* Studia Math. **188**(2) (2008), 123–133.

[6] A. Fiorenza, G. E. Karadzhov, *Grand and small Lebesgue spaces and their analogs.* Z. Anal. Anwend. **23**(4) (2004), 657–681.

[7] A. Fiorenza, J. M. Rakotoson, *Petits espaces de Lebesgue et leurs applications.* C. R., Math., Acad. Sci. Paris **333**(1) (2002), 23–26.

[8] A. Fiorenza, C. Sbordone, *Existence and uniqueness results for solutions of nonlinear equations with right hand side in $L^1$*, Studia Math. **127**(3) (1998), 223–231.

[9] L. Greco, T. Iwaniec, C. Sbordone, *Inverting the p-harmonic operator.* Manuscripta Math. **92** (1997), 249–258.

[10] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge Univ. Press, 1934.

[11] E. Hewitt, K. Stromberg, *Real and abstract analysis. A modern treatment of the theory of functions of a real variable.* Springer-Verlag, New York-Heidelberg, 1975.

[12] T. Iwaniec, C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses.* Arch. Rational Mech. Anal. **119** (1992), 129–143.

[13] V. Kokilashvili, Weighted estimates for classical integral operators, *Nonlinear analysis, function spaces and application, IV*, Teubner-Texte Math., Teubner-Leipzig (1990), 86–113.

[14] V. Kokilashvili, *Weighted problems for operators of harmonic analysis in some Banach function spaces.* Lecture course of Summer School and Workshop "Harmonic Analysis and Related Topics" (HART2010), Lisbon, June 21-25, 2010.

[15] V. Kokilashvili, A. Meskhi, H. Rafeiro, *Boundedness of commutators of singular and potential operators in generalized grand Morrey spaces and some applications.* Studia Math. **217**(2) (2013), 159-178.

[16] V. Kokilashvili, A. Meskhi, H. Rafeiro, *Estimates for nondivergence elliptic equations with VMO coefficients in generalized grand Morrey spaces.* Complex Var. Elliptic Equ. **59**(8) (2014), 1169-1184.

[17] V. Kokilashvili, A. Meskhi, H. Rafeiro, *Grand Bochner-Lebesgue space and its associate space.* J. Funct. Anal. **266**(4) (2014), 2125-2136.

[18] V. Kokilashvili, A. Meskhi, H. Rafeiro, *Riesz type potential operators in generalized grand Morrey spaces.* Georgian Math. J. **20**(1) (2013), 43-64.

[19] E. Liflyand, E. Ostrovsky, L. Sirota, *Structural properties of Bilateral Grand Lebesque Spaces.* Turk. J. Math. **34** (2010), 207–219.

[20] A. Meskhi, *Maximal functions, potentials and singular integrals in grand Morrey spaces.* Complex Var. Elliptic Equ. **56**(10-11) (2011), 1003–1019.

[21] H. Rafeiro, A note on boundedness of operators in Grand Grand Morrey spaces, *Advances in Harmonic Analysis and Operator Theory, The Stefan Samko Anniversary Volume*, Operator Theory: Advances and Applications, Vol. 229 (2013), 343–350, Birkhäuser.

[22] H. Rafeiro, A. Vargas, *On the compactness in grand spaces.* Georgian Math. J. **22**(1) (2015) 141-152.

[23] S.G. Samko, S.M. Umarkhadzhiev, *On Iwaniec-Sbordone spaces on sets which may have infinite measure.* Azerb. J. Math. **1**(1) (2010) 67–84.

[24] I. Schur, *Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen.* J. Reine Angew. Math. **140** (1911) 1–28.

[25] H. Weyl, *Singuläre Integralgleichungen mit besonderer Berücksichtigung des Fourierschen Integraltheorems* , Göttingen (1908) (Thesis)