*Araştırma Makalesi / Research Article*

# On Torsion Units in Integral Group Ring of A Dicyclic Group

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**Abstract**
Let $G$ become an any group. We recall that any two elements of integral group ring $\mathbb{Z}G$ are rational conjugate provided that they are conjugate in terms of units in $\mathbb{Q}G$. Zassenhaus introduced as a conjecture that any unit of finite order in $\mathbb{Z}G$ is rational conjugate to an element of the group $G$. This is known as the first conjecture of Zassenhaus [4]. We denote this conjecture by ZC1 throughout the article. ZC1 has been satisfied for some types of solvable groups and metacyclic groups. Besides one can see that there exist some counterexamples in metabelian groups. In this paper, the main aim is to characterize the structure of torsion units in integral group ring $\mathbb{Z}T_3$ of dicyclic group $T_3 = \langle a, b{:}\, a^6 = 1, a^3 = b^2, bab^{-1} = a^{-1}\rangle$ via utilizing a complex 2nd degree faithful and irreducible representation of $\mathbb{Z}T_3$ which is lifted from a representation of the group $T_3$. We show by ZC1 that non-trivial torsion units in $\mathbb{Z}T_3$ are of order 3, 4 or 6 and each of them can be stated by 3 free parameters.

**Keywords:** Torsion units, Integral group rings, Dicyclic group, Zassenhaus conjectures.

# İki-devirli Bir Grubun İntegral Grup Halkasındaki Burulmalı Birimsel Elemanlar Üzerine

**Öz**
$G$ bir grup olsun. $\mathbb{Z}G$ integral grup halkasındaki herhangi iki birimsel elemanın, $\mathbb{Q}G$ grup cebrindeki birimseller bakımından eşlenik olması durumunda rasyonel eşlenik olarak ifade edildiklerini anımsayalım. Zassenhaus, bir konjektür olarak sunmuştur ki $\mathbb{Z}G$'deki herhangi bir sonlu mertebeli birimsel eleman, $G$ grubunun bir elemanı ile rasyonel eşleniktir. Bu, Zassenhaus'un ilk konjektürü olarak bilinir [4]. Biz bu konjektürü makale boyunca ZC1 ile göstereceğiz. ZC1, çözülebilir ve meta-devirli grupların bazı sınıfları için çözülmüştür. Bunun yanı sıra, biri görebilir ki metabelyen gruplarda bazı aksine örnekler vardır. Bu makalede temel amaç, $T_3 = \langle a, b{:}\, a^6 = 1, a^3 = b^2, bab^{-1} = a^{-1}\rangle$ iki-devirli grubunun $\mathbb{Z}T_3$ integral grup halkasındaki burulmalı birimsel elemanların yapısını, ikinci dereceden bir kompleks indirgenemez güvenilir temsil kullanarak karakterize etmektir. Birinci Zassenhaus konjektürü (ZC1) ile göstereceğiz ki $\mathbb{Z}T_3$ integral grup halkasındaki aşikar olmayan burulmalı birimsel elemanlar 3, 4 veya 6 mertebeli ve bunların her biri üç serbest parametre cinsinden ifade edilebilir.

**Anahtar kelimeler**: Burulmalı birimsel elemanlar, İntegral grup halkaları, İki-devirli grup, Zassenhaus konjektürleri.

## 1. Introduction

Throughout the paper, $\boldsymbol{\mathcal{U}_1}(\mathbb{Z}\boldsymbol{G})$ displays the group of units which are normalized in the integral group ring $\mathbb{Z}\boldsymbol{G}$ of the group $\boldsymbol{G}$. In 1960s, Zassenhaus did explicitly much powerful some conjectures on units in integral group ring $\mathbb{Z}\boldsymbol{G}$ of a finite group $\boldsymbol{G}$. First of these conjectures (ZC1) says that an arbitrary torsion unit in $\mathbb{Z}\boldsymbol{G}$ must be rationally conjugated to an element in the group $\boldsymbol{G}$.

ZC1 has been solved for metacyclic groups which can be considered as a split extension $\langle a\rangle \rtimes \langle x\rangle$ of two cyclic groups with $(o(a), o(x)) = 1$ by Milies, Ritter and Sehgal in [4].

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On the other hand, the conjecture has been advanced by Sehgal. He showed that if $H$ is a subgroup of finite order in $\mathcal{U}_1(\mathbb{Z}G)$ of a nilpotent class two group $G$, then there exists $\alpha \in \mathbb{Q}G$ such that $H^{\alpha} = \alpha H \alpha^{-1} \subseteq G$. In [3], Bhandari and Luthar proved ZC1 in the sense of metacyclic groups which have order $pq$ where $p$ and $q$ are distinct primes. Interested readers can meet so many recent studies for more on Zassenhaus conjectures in [1, 2, 7, 9, 11, 12, 15].

One of famous results on ZC1 belongs to Hertweck who verified the conjecture on account of finite groups of the structure $G = AX$ where $A$ is a cyclic and normal subgroup of the group $G$ and also $X$ is an Abelian subgroup of $G$ [12]. Bächle et al. identified all small groups of order $\leq 288$ in GAP in order that Zassenhaus conjecture is satisfied [1].

On the other hand, one can see some counterexamples to Zassenhaus conjectures in [10] and [15]. Besides, many studies can be found on Zassenhaus conjectures (especially ZC1) for non-abelian groups. In [7], Gildea had a result which proves that the groups $PSL(2,8)$ and $PSL(2,17)$ satisfy ZC1. A vulnerable study on alternating group $A_6$ has been introduced by Hertweck [11]. Ari described torsion units in the integral group ring of $A_4$ which is the alternating group of order 24 with respect to ZC1 [8]. Herman and Singh reported that the method of Luthar-Passi under some restrictions are enough to provide the conjecture via a computer software for the whole groups of order $\leq 96$, except for one group of order 48 which is non-split and covering group of $S_4$ and one of order 72 of the form $(C_3 \times C_3) \rtimes C_8$ [2].

In the next section, we introduce a characterization of non-trivial and torsion ones of units in the integral group ring $\mathbb{Z}T_3$ of the binary dihedral group

$$T_3 = \langle a, b : a^6 = 1, a^3 = b^2, bab^{-1} = a^{-1} \rangle$$

taking inspiration from [12].

## 2. Material and Method: Representations of Torsion Units and the Main Result

Let $\mathcal{GL}(2, \mathbb{Q}(\zeta))$ be the group of all 2×2-matrices whose entries are from the extension field $\mathbb{Q}(\zeta)$. Here $\zeta$ is the third primitive root of 1 and $I_n$ be the set of indices up to $n$. Then, we can constitute the following faithful irreducible representation of $T_3$.

$$\begin{aligned} \rho: \quad T_3 &\longrightarrow \mathcal{GL}(2, \mathbb{Q}(\zeta)) \\ a &\mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix} \\ b &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

Extending $\rho$ linearly over $\mathbb{Z}$ to integral group rings, we attain a representation of $\mathbb{Z}T_3$ as $\sigma: \mathbb{Z}T_3 \rightarrow \mathcal{M}_2(\mathbb{Q}(\zeta))$ by

$$\sigma(\sum_{g \in T_3} \gamma_g g) = \sum_{g \in T_3} \gamma_g \rho(g)$$

where $\mathcal{M}_2(\mathbb{Q}(\zeta))$ is the ring of all 2×2-matrices defined over $\mathbb{Q}(\zeta)$. Hence, the image of an element $\gamma \in \mathbb{Z}T_3$ is stated by the representation

$$\sigma(\gamma) = \begin{bmatrix} \sum_{i=0}^{5} \alpha_i \zeta^i & \sum_{j=0}^{2} (\beta_j + \beta_{j+3}) \zeta^{2j} \\ \sum_{i=0}^{5} \beta_i \zeta^i & \sum_{j=0}^{2} (\alpha_j + \alpha_{j+3}) \zeta^{2j} \end{bmatrix}$$

Now, we can recall the following lemmas which are very useful to characterize torsion units.

**Lemma 2.1** *Consider $G$ as a metacyclic group. Let $N$ become its commutator subgroup and*

$$\begin{aligned} \phi: \mathbb{Q}G &\rightarrow \mathbb{Q}(G/N) \\ \sum \gamma_g g &\mapsto \sum \gamma_g gN \end{aligned}$$

*be the natural ring epimorphism. Provided that a unit $\gamma \in \mathcal{U}_1(\mathbb{Z}G)$ is a rationally conjugate to $\exists g \in G$, we get $\phi(\gamma) = gN$ [17].*

**Lemma 2.2** *Let $\sigma$ be the representation of $\mathbb{Z}G$ obtained by a representation $\rho$ of the group $G$ extending linearly. If $\gamma \in \mathcal{U}_1(\mathbb{Z}G)$ is rational conjugate to an element $g \in G$, then $|\sigma(\gamma)| = |\rho(g)|$ [17].*

Let $G$ become a finite group and also $\gamma = \gamma_e + \sum_{e\neq g\in G} \gamma_g g$ be a torsion unit which has order $o(\gamma)$ in $\mathcal{U}_1(\mathbb{Z}G)$. If $\gamma_e \neq 0$, then $\gamma = \pm 1$ as stated in [19]. As an immediate result, a torsion unit in $\mathbb{Z}G$ which is rationally conjugate to a central element of the group $G$ has to be trivial.

Let $C_g$ show the conjugacy class of $g \in G$. By using ZC1, we can recall that a torsion unit in $\mathcal{U}_1(\mathbb{Z}T_3)$ is conjugate of a representative in one of the elements in

$$\{C_e, C_a, C_{a^2}, C_{a^3}, C_b, C_{ba}\}$$

**Corollary 2.3** *If $\gamma \in \mathcal{U}_1(\mathbb{Z}T_3)$ is a torsion unit which is conjugate to a representative in $C_e$ or $C_{a^3}$, then $\gamma \in T_3$.*

We can conclude from ZC1 that if a torsion unit $\gamma \in \mathcal{U}_1(\mathbb{Z}G)$ is of rational conjugate form to $g \in G$, then $o(g) = o(\gamma)$. Since torsion units of order 2 in $\mathcal{U}_1(\mathbb{Z}T_3)$ are trivial, we focus on torsion units of order 3,4 or 6. Keeping in mind that considering $N$ as the center of $T_3$, we can obtain the homomorphic image

$$\phi(\gamma) = \sum_{j=0}^{2} (\alpha_j + \alpha_{j+3})a^jN + (\beta_j + \beta_{j+3})ba^jN$$

Now, we have motivated to phrase the main theorem of the study.

**Theorem 2.4** *Let $\gamma_{o(\gamma),g}$ denote a torsion unit with order $o(\gamma)$ and conjugate to $g \in G$. Non-trivial torsion units in $\mathcal{U}_1(\mathbb{Z}T_3)$ can be written in sense of 3 free integer parameters as follows:*

$$\gamma_{3,a^2} = a^2 + \sum_{j=0}^{2} \lambda_j(ba^j - ba^{j+3})$$

$$\gamma_{4,b} = b + \sum_{j=1}^{2} \delta_j(a^j - a^{j+3}) + k\sum_{j=0}^{5} (-1)^j ba^j$$

$$\gamma_{4,ba} = ba + \sum_{j=1}^{2} \mu_j(a^j - a^{j+3}) + l\sum_{j=0}^{5} (-1)^j ba^j$$

$$\gamma_{6,a} = a + \sum_{j=0}^{2} \eta_j(ba^j - ba^{j+3})$$

where $\lambda_j$, $\delta_j$, $\mu_j$, $\eta_j$, $k$ and $l$ are free integer parameters.

*Proof.* Firstly, let $o(\gamma) = 3$. Lemma 2.1. implies that

$$\phi(\gamma_{3,a^2}) = \textstyle\sum_{j=0}^{2} (\alpha_j + \alpha_{j+3})a^jN + (\beta_j + \beta_{j+3})ba^jN = a^2N \quad (2.1)$$

if and only if

$$\alpha_j + \alpha_{j+3} = \begin{cases} 0, & j = 0,1 \\ 1, & j = 2 \end{cases}$$

and $\beta_j + \beta_{j+3} = 0$ for $j = 0,1,2$. We can note that $\alpha_0 = \alpha_3 = 0$ from Berman-Higman [16]. On the other hand, we obtain the representation of $\gamma_{3,a^2}$ as

$$\sigma(\gamma_3) = \begin{bmatrix} \left(\alpha_1 - \alpha_2 + \frac{1}{2}\right) + \left(\alpha_1 + \alpha_2 - \frac{1}{2}\right)\sqrt{-3} & 0 \\ -(2\beta_0 + \beta_1 - \beta_2) - (\beta_1 + \beta_2)\sqrt{-3} & -\frac{1}{2} - \frac{1}{2}\sqrt{-3} \end{bmatrix}.$$

Lemma 2.2. implies that $|\sigma(\gamma_{3,a^2})| = |\rho(a^2)| = 1$. Thus,

$$|\sigma(\gamma_3)| = (\frac{-X}{2} + \frac{3Y}{2}) + (\frac{-Y}{2} + \frac{-X}{2})\sqrt{-3} = 1 \qquad (2.2)$$

with $X = \alpha_1 - \alpha_2 + \frac{1}{2}$ and $Y = \alpha_1 + \alpha_2 - \frac{1}{2}$.
(2.2) is provided if and only if $\alpha_1 = 0$ and $\alpha_2 = 1$. Hence, we obtain all the parameters of $\gamma_{3,a^2}$ as follows

$$\alpha_j = \begin{cases} 0, & j \in I_5 \backslash \{2\} \\ 1, & j = 2 \end{cases}$$

and $\beta_{j+3} = -\beta_j$ for $j = 0,1,2$. Therefore, the parameters of $\gamma_{3,a^2}$ can be rearranged by these relations and thus $\gamma_{3,a^2}$ can be expressed as claimed in the current theorem (2.4).
Now, let $o(\gamma) = 4$. Then, $\gamma_{4,b}$ is rational conjugate to $b$ or $ba$. If $\gamma_{4,b}$ is torsion unit which is conjugate to $b$, we can say from Lemma 2.1. that

$$\phi(\gamma_{4,b}) = bN. \qquad (2.3)$$

Clearly, (2.3) is operative if and only if $\alpha_j + \alpha_{j+3} = 0$ for $j = 0,1,2$ and

$$\beta_j + \beta_{j+3} = \begin{cases} 0, & j = 1,2 \\ 1, & j = 0 \end{cases}$$

Moreover, the representation of $\gamma_{4,b}$ under $\sigma$ is

$$\sigma(\gamma_{4,b}) = \begin{bmatrix} (\alpha_1 - \alpha_2) + (\alpha_1 + \alpha_2)\sqrt{-3} & 1 \\ (-2\beta_0 - \beta_1 + \beta_2 + 1) - (\beta_1 + \beta_2)\sqrt{-3} & 0 \end{bmatrix}$$

and $|\sigma(\gamma_{4,b})| = 1$ by Lemma 2.2. Thus, we obtain

$$|\sigma(\gamma_{4,b})| = (2\beta_0 + \beta_1 - \beta_2 - 1) + (\beta_1 + \beta_2)\sqrt{-3} = 1 \qquad (2.4)$$

and conclude that $\beta_2 = -\beta_1$, $\beta_1 = 1 - \beta_0$. Replacing the parameters according to free ones, we immediately obtain the phrase of $\gamma_{4,b}$ as in the theorem (2.4). Let us consider $\gamma_{4,ba}$ which is conjugate to $ba$. In this case, we know that

$$\phi(\gamma_{4,ba}) = baN \qquad (2.5)$$

and (2.5) satisfied if and only if $\alpha_j + \alpha_{j+3} = 0$ for $j = 0,1,2$ and

$$\beta_j + \beta_{j+3} = \begin{cases} 0, & j = 0,2 \\ 1, & j = 1 \end{cases}$$

as well. Necessary and sufficient conditions for (2.5) and Lemma 2.2. imply that

$$\sigma(\gamma_{4,ba}) = \begin{bmatrix} (\alpha_1 - \alpha_2) + (\alpha_1 + \alpha_2)\sqrt{-3} & -\frac{1}{2} + \frac{1}{2}\sqrt{-3} \\ (-2\beta_0 - \beta_1 + \beta_2 + \frac{1}{2}) + (\frac{1}{2} - \beta_1 + \beta_2)\sqrt{-3} & 0 \end{bmatrix}$$

and

$$|\sigma(\gamma_{4,ba})| = (-\beta_0 - 2\beta_1 - \beta_2 + 1) + (\beta_0 - \beta_2)\sqrt{-3} = -1 \qquad (2.6)$$

respectively. Therefore, one can notice that necessary and sufficient condition on parameters in $\gamma_{4,ba}$ for (2.6) is as follows

$$\beta_0 = \beta_2 = 1 - \beta_1. \qquad (2.7)$$

Thus, rearranging the coefficients of $\gamma_{4,ba}$ with respect to the conditions in (2.5) and (2.7) introduces a canonical statement of $\gamma_{4,ba}$ as asserted in the Theorem 2.4. Finally, let $o(\gamma) = 6$. As proceeded above similarly,

$$\phi(\gamma_{6,a}) = aN. \qquad (2.8)$$

Thus, (2.8) is ensured if and only if

$$\alpha_j + \alpha_{j+3} = \begin{cases} 0, & j = 0,2 \\ 1, & j = 1 \end{cases}$$

and $\beta_j + \beta_{j+3} = 0$ for $j = 0,1,2$. Using these conditions and the representation of $\gamma_{6,a}$, we obtain the following rearranged representation

$$\sigma(\gamma_{6,a}) = \begin{bmatrix} (\alpha_1 - \alpha_2 - \frac{1}{2}) + (\alpha_1 + \alpha_2 - \frac{1}{2})\sqrt{-3} & 0 \\ (-2\beta_0 - \beta_1 + \beta_2) + (-\beta_1 - \beta_2)\sqrt{-3} & -\frac{1}{2} + \frac{1}{2}\sqrt{-3} \end{bmatrix}.$$

As another implementation of Lemma 2.2., we deduce that

$$|\sigma(\gamma_{6,a})| = (1 - 2\alpha_1 - \alpha_2) - \alpha_2\sqrt{-3} = -1. \quad (2.9)$$

Hence, $\alpha_1 = 1$ and $\alpha_2 = 0$. To sum up, the parameters of $\gamma_{6,a}$ are obtained as

$$\alpha_j = \begin{cases} 0, & j \in I_5 \backslash \{1\} \\ 1, & j = 1 \end{cases}$$

and $\beta_{j+3} = -\beta_j$ for $j = 0,1,2$. Hence, this gives a canonical form of $\gamma_{6,a}$ as stated in the theorem.

## 3. Results and Disscussion

In this paper, we have first investigated that dicyclic groups satisfy the first of Zassenhaus conjecture according to the result of Hertweck [12]. Thus, we have correlated the torsion units of integral group ring $\mathbb{Z}\boldsymbol{T_3}$ with torsion units in the matrix ring $\boldsymbol{\mathcal{M}_2}(\mathbb{Q}(\boldsymbol{\zeta}))$ such that $\boldsymbol{\zeta}$ is the third primitive root of 1 and we characterized the parametric structures of torsion units in $\mathbb{Z}\boldsymbol{T_3}$ using ZC1. Interestingly, it is proved that all the non-trivial torsion units of order 3,4 or 6 can be indicated by three free integer parameters.

As a future work, using ZC1, the structure of torsion and non-trivial ones of units in the integral group ring $\mathbb{Z}T_\text{n}$ of generalized dicyclic group $T_n$ which has order $4n$ may be declared.

## Author's Contributions

Ömer KÜSMÜŞ performed all the original theoretical results and applicational calculations with final version of the study as the only author of the paper.

**Statement of Conflicts of Interest**

No potential conflict of interest was reported by the author.

**Statement of Research and Publication Ethics**

The author declares that this study complies with Research and Publication Ethics.

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