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Variational inequalities with the duality operator in Banach spaces

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Abstract

We study variational inequality by way of metric projection in Banach spaces. The main method is to use a topological degree theory for the class of operators of monotone type in Banach spaces. More precisely, some variational inequality associated with the duality operator is considered. As applications, the problem is discussed in the Lebesgue spaces L^p and the Sobolev spaces $W^{1,2}$.

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1. Introduction

One of the most important applications of metric projection operators in Hilbert spaces may be solving variational inequalities.

Alber [1] constructed some metric projection operators in Banach spaces using the Young-Fenchel transformation and established a relationship between variational inequalities and operator equations involving metric projection.

Let X be a real reflexive separable Banach space and K a closed convex set in X. We consider a variational inequality of the form

$$\langle Ju + Tu, v - u\rangle \geq \langle f, v - u\rangle \quad \text{for all } v \in K, \tag{1.1}$$

where $J : X \to X^*$ is the duality operator, $T : X \to X^*$ is a bounded continuous monotone operator, and $f \in X^*$.

The goal of this paper is to investigate variational problem (1.1) via the corresponding operator equation associated with a metric projection in Banach spaces. The main tool for solving the operator equation is a

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topological degree theory for the class of operators of monotone type related the operator T due to Berkovits [3]; see also [8]. The degree theory is based on the Browder degree for operators of monotone type in [5, 6].

The variational problem can be reduced to the operator equation

$$Ju + Tu = f \qquad \text{in } X. \tag{1.2}$$

Moreover, some applications of (1.1) and (1.2) are discussed in the Lebesgue spaces $X = L^p(\Omega)$ and the Sobolev spaces $X = W^{1,2}(\Omega)$, respectively.

2. Degree theory

Let X be a real Banach space with dual space X^*. The symbol $\langle \cdot, \cdot \rangle_X$ denotes the dual pairing between X^* and X in this order. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

Definition 2.1. An operator $F : \Omega \subset X \to X^*$ is said to be:

(1) of class (S_+) if for any sequence (u_n) in Ω such that $u_n \rightharpoonup u$ in X and

$$\limsup_{n\to\infty} \langle Fu_n, u_n - u\rangle \le 0,$$

we have $u_n \to u$ in X.

(2) *quasimonotone* if for any sequence (u_n) in Ω such that $u_n \rightharpoonup u$ in X, we have

$$\limsup_{n\to\infty} \langle Fu_n, u_n - u\rangle \ge 0.$$

(3) *monotone* if $\langle Fu - Fv, u - v\rangle \ge 0$ for all $u, v \in \Omega$.

Notice that the collection of operators of class (S_+) is stable under quasimonotone perturbations and any monotone operator on each weakly closed set is quasimonotone.

Let X be a real reflexive Banach space with dual space X^*. Identifying the bidual space X^{**} with X, we write $\langle x, y\rangle_{X^*}$ for $x \in X$ and $y \in X^*$.

Definition 2.2. Let $T : \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$. An operator $F : \Omega \subset X \to X$ is said to be of class $(S_+)_T$ if for any sequence (u_n) in Ω such that $u_n \rightharpoonup u$ in X, $Tu_n \rightharpoonup y$ in X^*, and $\limsup_{n\to\infty}\langle Fu_n, Tu_n - y\rangle \le 0$, we have $u_n \to u$ in X.

Actually, it can be shown in [3, 8] that in infinite dimensional Hilbert spaces the collection of operators of class $(S_+)_T$ is larger than that of operators of class (S_+).

Given a nonempty set Ω in X, let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω in X, respectively. Let $B_r(a)$ denote the open ball in X of positive radius r centered at a.

We consider the following classes of operators:

$$\mathcal{F}_1(\Omega) := \{F \colon \Omega \subset X \to X^* \mid F \text{ is bounded, continuous, and of class } (S_+)\},$$
$$\mathcal{F}_T(\Omega) := \{F \colon \Omega \subset X \to X \mid F \text{ is bounded, demicontinuous, and of class } (S_+)_T\},$$

for any $\Omega \subset D_F$ and any $T \in \mathcal{F}_1(\Omega)$, where D_F denotes the domain of F. Here, T is called an *essential inner map* to F.

We present an example of abstract Hammerstein operator which belongs to $\mathcal{F}_T$; see [3, Lemma 2.2].

Lemma 2.3. *Let G be any bounded open set in a real reflexive Banach space X. Suppose that $T \in \mathcal{F}_1(\overline{G})$ and $S \colon D_S \subset X^* \to X$ is a bounded demicontinuous quasimonotone operator such that $T(\overline{G}) \subset D_S$. Then the operator $I + S \circ T$ belongs to $\mathcal{F}_T(\overline{G})$, where I denotes the identity operator.*

We consider affine homotopy $H\colon [0,1]\times\overline{G}\to 2^X$ defined by

$$H(t,u) := (1-t)Fu + tSu \qquad \text{for } (t,u)\in[0,1]\times\overline{G}.$$

If $F, S \in \mathcal{F}_T(\overline{G})$ with $T\in\mathcal{F}_1(\overline{G})$, then $H(t,\cdot)$ belongs to $\mathcal{F}_T(\overline{G})$ for all $t\in[0,1]$. We say that affine homotopy H has common essential inner map T.

Berkovits [3] introduced a topological degree theory for the class $\mathcal{F}_T$ with an elliptic super-regularization method due to Browder and Ton [7]. This will be a key tool of our main result in the next section.

Theorem 2.4. *Let X be a real separable reflexive Banach space such that both X and X^* are locally uniformly convex. Let G be any bounded open set in X. For every $F\in\mathcal{F}_T(\overline{G})$, where $T\in\mathcal{F}_1(\overline{G})$, there exists a unique degree function d such that the following properties are satisfied:*

(a) (Existence) If $d(F,G,h)\neq 0$, then the equation $Fu=h$ has a solution in G.

(b) (Additivity) If G_1 and G_2 are two disjoint open subsets of G such that $h\notin F(\overline{G}\setminus(G_1\cup G_2))$, then we have
$$d(F,G,h) = d(F,G_1,h) + d(F,G_2,h).$$

(c) (Homotopy invariance) Suppose that $H\colon[0,1]\times\overline{G}\to X$ is an affine homotopy of class $(S_+)_T$ with a common essential inner map $T\in\mathcal{F}_1(\overline{G})$. If $h:[0,1]\to X$ is a continuous map such that $h(t)\notin H(t,\partial G)$ for all $t\in[0,1]$, then the value of $d(H(t,\cdot),G,h(t))$ is constant for all $t\in[0,1]$.

(d) (Normalization) For any $h\in G$, we have $d(I,G,h)=1$.

Proof. The existence proof is mainly based on the (S_+)-degree; see [2, 4] for more details on the (S_+)-degree. It is shown in [3] that the value of $d(F,G,h)$ is independent of the choice of essential inner map T. Moreover, the uniqueness of the degree function d follows from the uniqueness of the (S_+)-degree. □

For our aim, we need Borsuk's theorem for operators of class $(S_+)_T$; see [3, Theorem 8.1].

Theorem 2.5. *Let G be a bounded open set in X that is symmetric with respect to the origin $0\in G$. Suppose that F belongs to $\mathcal{F}_T(\overline{G})$ and is odd on ∂G with $0\notin F(\partial G)$, where $T\in\mathcal{F}_1(\overline{G})$. Then $d(F,G,0)$ is an odd number.*

Let X be a real reflexive Banach space such that both X and X^* are locally uniformly convex. Let $J: X\to X^*$ denote the (normalized) duality operator determined by

$$\langle Ju,u\rangle = \|u\|^2 \quad\text{and}\quad \|Ju\| = \|u\| \qquad \text{for all } u\in X.$$

It is known in [5, 9] that J is a bounded continuous odd operator of class (S_+). A typical example of the duality operator in uniformly convex Banach spaces is $J: L^p\to L^q,\ Ju = \|u\|^{2-p}|u|^{p-2}u$, where $p,q\in(1,\infty)$ and $p^{-1}+q^{-1}=1$.

3. Main Result

In this section, we study the variational inequality by way of metric projection in Banach spaces. The method is to use the degree theory for the class $\mathcal{F}_T$ in the previous section.

In what follows, let X be a real separable Banach space such that both X and X^* are uniformly convex and let K be a nonempty closed convex set in X. Let $V: X^*\times X\to\mathbb{R}$ be defined by the formula

$$V(\varphi,\xi) := \|\varphi\|^2 - 2\langle\varphi,\xi\rangle + \|\xi\|^2.$$

Let $P_K: X^*\to K$ be the metric projection as follows:

$$P_K\varphi := \tilde{\varphi} \quad \text{for } \varphi\in X^*$$

induced by

$$V(\varphi, \tilde{\varphi}) = \inf_{\xi \in K} V(\varphi, \xi).$$

It is known in [1] that the projection $P_K : X^* \to K$ is bounded, continuous, and monotone.

To begin with, we establish the relationship between variational inequality and the corresponding operator equation in terms of the metric projection in Banach spaces; see [1, Theorem 8.1].

Lemma 3.1. *Suppose that $A : X \to X^*$ is an operator, α is a positive number, and $f \in X^*$. Then $u_0 \in K$ is a solution of the variational inequality*

$$\langle Au - f, \xi - u \rangle \geq 0 \quad \textit{for all } \xi \in K$$

if and only if $u_0 \in X$ is a solution of the operator equation

$$u = P_K(Ju - \alpha(Au - f)),$$

where $J : X \to X^*$ is the duality operator and $P_K : X^* \to K$ is defined as above.

We consider a variational inequality of the form

$$\langle Ju + Tu, v - u \rangle \geq \langle f, v - u \rangle \quad \text{for all } v \in K, \tag{3.1}$$

where $J : X \to X^*$ is the duality operator, $T : X \to X^*$ is an operator, and $f \in X^*$.

According to Lemma 3.1, $u \in K$ is a solution of (3.1) if and only if $u \in K$ is a solution of the operator equation

$$u = P_K(Ju - \alpha(Ju + Tu - f)), \tag{3.2}$$

where α is a positive number and P_K is the metric projection.

We are now in a position to prove the main result in Banach spaces, where the degree theory is applied together with Borsuk's theorem. For the Hilbert space case, we refer to [3, Example 8.3].

Theorem 3.2. *Let K be a closed convex set in X with $0 \in K$ that is symmetric with respect to 0. Suppose that $J : X \to X^*$ is the duality operator and $T : X \to X^*$ satisfies the following conditions:*

(a) T is bounded, continuous, and quasimonotone and take $\alpha > 1$;

(b)

$$\liminf_{\|u\| \to \infty} \frac{\langle Tu, u \rangle}{\|u\|} > -\infty;$$

(c) there is a positive number R_0 such that

$$T(-u) = -Tu \quad \textit{for all } u \in X \textit{ with } \|u\| \geq R_0.$$

Then for any $f \in X^*$, variational inequality (3.1) has at least one solution in K.

Proof. Let $f \in X^*$ be given. Set

$$F := I + P_K(\alpha T + (\alpha - 1)J - \alpha f) \quad \text{and} \quad T_\alpha := \alpha T + (\alpha - 1)J.$$

Note by hypothesis (a) that $T_\alpha : X \to X^*$ is bounded, continuous, and of class (S_+), and $P_K : X^* \to K$ is bounded, continuous, monotone, and odd. Lemma 2.3 implies that $F \in \mathcal{F}_{T_\alpha}(\overline{G})$ for any bounded open set G in X.

To apply the degree theory, we have to show that there exists a constant R with $R \geq R_0$ such that

$$u + P_K(T_\alpha u - t\alpha f) \neq 0 \quad \text{for all } t \in [0, 1] \text{ and all } u \in \partial B_R(0). \tag{3.3}$$

Assume to the contrary that for a sequence (R_n) in $\mathbb{R}$ with $R_n \geq R_0$ and $R_n \to \infty$ as $n \to \infty$, there exist sequences (t_n) in $[0,1]$ and (u_n) in X with $\|u_n\| = R_n$ such that

$$u_n + P_K(T_\alpha u_n - t_n \alpha f) = 0 \qquad \textit{for each } n \in \mathbb{N}.$$

This implies that

$$P_K(T_\alpha u_n - t_n \alpha f) = -u_n \in K \quad \text{and} \quad P_K(J0) = 0.$$

It is known in [1] that the following relation holds for all $\varphi_1, \varphi_2 \in X^*$:

$$\langle \varphi_1 - \varphi_2, P_K(\varphi_1) - P_K(\varphi_2)\rangle \geq 2R^2 \delta \left(\frac{\|P_K(\varphi_1) - P_K(\varphi_2)\|}{2R} \right), \tag{3.4}$$

where $R = \sqrt{(\|P_K(\varphi_1)\|^2 + \|P_K(\varphi_2)\|^2)/2}$. Here, δ denotes the modulus of convexity of the space X and $0 \leq \delta < 1$.

Putting $\varphi_1 = T_\alpha u_n - t_n \alpha f$ and $\varphi_2 = J0$ in (3.4), we get

$$\langle T_\alpha u_n - t_n \alpha f, -u_n \rangle \geq \delta\Big(\frac{1}{\sqrt{2}}\Big) \|u_n\|^2$$

and hence

$$-\frac{\alpha \langle T u_n, u_n\rangle}{\|u_n\|} + \|\alpha f\| \geq \left(\alpha - 1 + \delta\Big(\frac{1}{\sqrt{2}}\Big)\right) \|u_n\|. \tag{3.5}$$

By hypothesis (b), we have

$$\limsup_{\|u\| \to \infty} -\frac{\langle Tu, u\rangle}{\|u\|} < +\infty.$$

Hence it follows from (3.5) and $\alpha > 1$ that

$$\limsup_{n\to\infty} \|u_n\| \leq \frac{1}{\alpha - 1} \limsup_{n \to \infty} \left[-\frac{\alpha \langle T u_n, u_n\rangle}{\|u_n\|} + \|\alpha f\| \right] < +\infty,$$

which contradicts our assumption that $\|u_n\| \to \infty$ as $n \to \infty$. Therefore, assertion (3.3) must be true.

Now we obtain from the homotopy invariance property of the degree d in Theorem 2.4 with (3.3) that

$$d(F, B_R(0), 0) = d(I + P_K \circ T_\alpha, B_R(0), 0). \tag{3.6}$$

Note by hypothesis (c) that the operator $I + P_K \circ T_\alpha$ belongs to $\mathcal{F}_{T_\alpha}(\overline{B_R(0)})$ and it is odd on $\partial B_R(0)$. Borsuk's Theorem 2.5 says that $d(I + P_K \circ T_\alpha, B_R(0), 0)$ is an odd number, which implies by (3.6) that

$$d(F, B_R(0), 0) \neq 0.$$

By the existence property of the degree d, equation (3.2) has a solution u in K. We conclude that the point u is a solution of variational inequality (3.1). This completes the proof. □

Corollary 3.3. *Suppose that $J : X \to X^*$ is the duality operator and $T : X \to X^*$ satisfies conditions (a)–(c) of Theorem 3.2. Then for every $f \in X^*$, the operator equation*

$$Ju + Tu = f$$

has a solution in X.

Proof. Apply Theorem 3.2 with $K = X$. □

4. Applications

We are concerned with the following variational inequality

$$\langle Ju + Tu, v - u\rangle \geq \langle f, v - u\rangle \quad \text{for all } v \in K,$$

where $J : X \to X^*$ is the duality operator on a real separable uniformly convex Banach space X and K is a closed convex set in X, and $f \in X^*$.

Concerning the operator T, we suppose that $g : \mathbb{R} \to \mathbb{R}$ is a real-valued function such that

(g1) g is continuous and odd on $\mathbb{R}$;

(g2) there exist numbers $a, c \in \mathbb{R}$ with $a \geq 0$ and $c > 0$ such that

$$|g(u)| \leq a + c|u|^{p-1} \quad \text{for all } u \in \mathbb{R};$$

(g3) g is monotonically increasing on $\mathbb{R}$, that is,

$$g(u) \leq g(v) \quad \text{for all } u, v \in \mathbb{R} \text{ with } u \leq v;$$

(g4) there exists a positive number r such that

$$g(u)u \geq 0 \quad \text{for all } u \in \mathbb{R} \text{ with } |u| \geq r.$$

Theorem 4.1. *Let $X = L^p(\Omega)$ and $b \in L^q(\Omega)$, where Ω is a nonempty measurable set in $\mathbb{R}^N$ with $\operatorname{meas}\Omega < \infty$ and $p, q \in (1, \infty)$ with $p^{-1} + q^{-1} = 1$. Let K be a closed convex set in X with $0 \in K$ that is symmetric with respect to 0. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a real-valued function that satisfies conditions (g1)–(g4). Let $J, T : X \to X^*$ and $f : X \to \mathbb{R}$ be defined by*

$$\begin{aligned}
\langle Ju, v\rangle &= \|u\|^{2-p} \int_\Omega |u|^{p-2} uv \, dx, \\
\langle Tu, v\rangle &= \int_\Omega g(u) v \, dx, \\
\langle f, v\rangle &= \int_\Omega bv \, dx,
\end{aligned}$$

for $u, v \in X$. Then the variational inequality

$$\langle Ju + Tu, v - u\rangle \geq \langle f, v - u\rangle \quad \textit{for all } v \in K \tag{4.1}$$

has a solution in K.

Proof. It is clear that X and X^* are separable uniformly convex Banach spaces, $J : X \to X^*$ is the duality operator, and f is a bounded linear functional on X. Under hypotheses (g1)–(g3), the operator T is continuous, bounded, monotone, and odd. Let $A = \{x \in \mathbb{R}^N : |u(x)| \geq r\}$. Then it follows from (g2) and (g4) that the following estimate holds for all $u \in X$:

$$\begin{aligned}
\langle Tu, u\rangle &= \int_{\Omega\setminus A} g(u)u \, dx + \int_A g(u)u \, dx \\
&\geq \int_{\Omega\setminus A} g(u)u \, dx \\
&\geq -\int_\Omega (a + cr^{p-1}) r \, dx,
\end{aligned}$$

which implies, in view of $\operatorname{meas}\Omega < \infty$, that

$$\liminf_{\|u\|\to\infty} \frac{\langle Tu, u\rangle}{\|u\|} > -\infty;$$

Applying Theorem 3.2, variational inequality (4.1) has a solution u in K. □

We close this section with the solvability of operator equation in Hilbert spaces.

Theorem 4.2. *Let $X = W^{1,2}(\Omega)$ and $b \in L^2(\Omega)$, where Ω is a bounded region in $\mathbb{R}^N$. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a real-valued function that satisfies conditions (g1)–(g4) with $p = 2$. Let $J, T : X \to X^*$ and $f : X \to \mathbb{R}$ be defined by*

$$\langle Ju, v\rangle = \int_\Omega \left(uv + \sum_{i=1}^N D_i u D_i v \right) dx,$$
$$\langle Tu, v\rangle = \int_\Omega g(u) v \, dx,$$
$$\langle f, v\rangle = \int_\Omega bv \, dx,$$

for $u, v \in X$. Then the operator equation

$$Ju + Tu = f \tag{4.2}$$

has a solution in X.

Proof. Obviously, X is a separable uniformly convex Banach space, J is the duality operator, and f is a bounded linear functional. As before, T is continuous, bounded, monotone, odd, and

$$\liminf_{\|u\|\to\infty} \frac{\langle Tu, u\rangle}{\|u\|} > -\infty.$$

Applying Corollary 3.3, equation (4.2) has a solution in X. □

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