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# A new approach for the solutions of the fractional generalized Casson fluid model described by Caputo fractional operator

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### Abstract

The fractional Casson fluid model has been considered in this paper in the context of the Goodman boundary conditions. A new approach for getting the solutions of the Casson fluid models have been proposed. There is the Double integral method and the Heat balance integral method. These two methods constitute the integral balance method. In these methods, the exponent of the approximate solutions is an open main problem, but this issue is intuitively solved by using the so-called matching method. The graphical representations of the solutions of the fractional Casson fluid model support the main results that have been presented. In our investigations, the Caputo derivative has been used.

**Keywords:** Fractional Casson fluid models; Fractional Heat equation; Integral balance methods.

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### 1. Introduction

The numerical and analytical solutions are subject to many investigations in the literature of differential and fractional differential equations. Many models in the literature admit numerical schemes. Thus the numerical approximations are more useful for proposing the solutions of the differential equations. As numerical schemes, we can cite the implicit scheme, the explicit scheme, the predictor-corrector method, the Adams Bashford method, and others. The analytical solutions are not all time possible due to the

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complexities in the structure of many differential equations. These last years their exist many methods that emerged in the literature like the Fourier transform, the homotopy analysis, the homotopy perturbation method, and many others. In the problems of solving the diffusion equations and the fractional diffusion equations, we also note the numerical schemes, the Fourier transform, and the double Laplace transform. Expected the numerical schemes where we have all time the approximation of the solutions, there exist some new methods for proposing the analytical solution of the fractional diffusion equations and some class of fluids models as second-grade fluid and Casson fluid, stokes problems[25], Railegh problems[12], and others. These recent years, we have noted the emergency of the fractional operators in all the fields of physics, mathematical modeling, heat equations, and others[2] and many applications of the fractional operators in real-world problems continue to be provided, see investigation in [20, 28, 30, 32, 33, 34]. As fractional operators, we notice the derivatives with singular kernels and the derivatives with non-singular kernels[4, 3, 7, 11, 14, 19, 30].

The fluids models and the diffusion equations with integer derivative or non-integer derivatives occupy a great part of the literature nowadays. Many papers in the literature address the fluid models[31, 35]. In [29], Sene proves the integral balance method is compatible with the second-grade fluids models; the solution has been proposed in this paper. In [26], Sene experiments the integral balance method for solving the Stokes's first problems. And some interesting results related to the compatibility of this method with the Stokes problem have been established. In [8], Ghosh et al. present the investigations on Casson fluid model with over an exponentially stretching permeable sheet, validate their results by comparing their results with the results in the literature. In [9], Hamid proposes the numerical solutions of the fluids models in the context of fractional order derivative. In [23], Sheikh et al. propose a review of the theoretical aspect of the nanofluids models. In [22], Saqib experiments the fractional derivative with non-singularity, namely Atangana-Baleanu derivative to nanofluid model over an inclined plate. In [1], Aman et al. apply the fractional operator in fluids models by considering operator with the singular kernel. In [24], Sheikh proposes the analytical solution of Casson fluid defined with fractional order derivative using classical Fourier sine transform. Many investigations related to fluids model, heat equations and fractional diffusion equations exist in the literature, see more investigations in [5, 13, 27, 28].

In this paper, a new method for getting the analytical approximations of the solutions of the fractional diffusion equations has been proposed. The technique used in this paper uses the physical concepts, namely the penetration depth. We utilize the integral balance method. We introduce, in particular, the heat balance method and the double integral method for solving the equations into the Casson fluid model. The integral balance method is compatible with heat equations, second-grade fluid, stokes problem. This paper provides it is also compatible with Casson fluids models. These two methods are based on single integration and double integration through the diffusion equations. The mean idea of this method is we stipulate the approximate solution of the fractional diffusion equation exists, and its form depends on the penetration depth and a specific exponent. The question will be to determine the penetration depth and the exponent explicitly.

In Section 2, we recall the fractional operators and some properties. In Section 3, we propose the fractional model under consideration. In Section 4, we investigate the solutions of the model using integral balance methods. In Section 5, we represent the graphics and make the interpretations. Concluding remarks are given in Section 6.

## 2. Basics calculus tools

These paragraphs focus on the tools used in fractional calculus; we mean the fractional operators. We utilize in this paper the Caputo derivative, the Liouville-Riemann integral, and their associated Simon Laplace. We enumerate the following definitions

**Definition 2.1.** [16] Assume that  $k : [0, +\infty[ \rightarrow \mathbb{R}$ , the representation of the Liouville integral of  $k$  follows that

$$(I^\alpha k)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} k(s) ds, \quad (1)$$

with  $t > 0$ , the order  $\alpha \in (0, 1)$ , and the value of gamma is denoted by  $\gamma(\dots)$ .

**Definition 2.2.** [16] Assume that  $k : [0, +\infty[ \rightarrow \mathbb{R}$ , the representation of the Riemann-Liouville operator of  $k$  follows that

$$D_t^{\alpha,RL}k(t) = \frac{1}{\gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} k(s) ds, \tag{2}$$

with  $t > 0$ , the order  $\alpha \in (0, 1)$ .

**Definition 2.3.** [16] Assume that  $k : [0, +\infty[ \rightarrow \mathbb{R}$ , the representation of the Caputo operator of  $k$  follows that

$$D_t^\alpha k(t) = \frac{1}{\gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} k'(s) ds, \tag{3}$$

with  $t > 0$ , the order  $\alpha \in (0, 1)$ .

Note that in the above equations, all the definitions are reduced into the interval  $\alpha \in (0, 1)$ . Still, these definitions can be extended to arbitrary non-integer order outside of the interval considered in this section. For more information, the authors can refer to the literature. Another remarks are the generalizations of all the above definitions exist; these generalizations can be found in [10]. The discrete versions of all fractional derivatives exist too. The other fractional derivatives are not mentioned in this paper because we don't use them, namely Mittag-Leffler [2] fractional derivative, the exponential fractional derivative[7], the conformable derivative[15], and many others. The list is long. We finish this section by recalling the Laplace transform of the Caputo derivative, which is utilized to solve the differential equations with non-integer order derivatives. We have the following relationship

$$\mathcal{L} \{ (D_c^\alpha k)(t) \} = s^\alpha \mathcal{L} \{ k(t) \} - s^{\alpha-1} k(0), \tag{4}$$

with  $\mathcal{L}$  denotes the classical Simon Laplace transformation.

### 3. Fractional model for generalized Casson fluid

Let us present the fractional model presented in this section. The following equations describe the motions of the fractional generalized Casson fluid model

$$D_t^\alpha u = \left[ 1 + \frac{1}{\beta} \right] \frac{\partial^2 u}{\partial z^2} - Mu + Grv, \tag{5}$$

combined with the fractional diffusion heat equation

$$D_t^\alpha v = \frac{1}{Pr} \frac{\partial^2 v}{\partial z^2}, \tag{6}$$

for innovation we consider the Goodman boundary conditions defined for our problem by the following relationships

$$u(0, t) = v(0, t) = 1, \text{ and } u(\varpi) = v(\kappa) = \frac{\partial u(\varpi, t)}{\partial t} = \frac{\partial v(\kappa, t)}{\partial t}, \quad t \geq 0. \tag{7}$$

The parameters  $\kappa$  and  $\varpi$  denote the penetration depths of the previous model and verify the assumption that for  $z \geq \kappa$  and  $z \geq \varpi$  the temperature and the fluid motion change from the initial temperature and the initial concentration are considered negligible. The parameter  $Gr$  represents the Grashold number,  $Pr$  represents the Prandtl number,  $M$  denotes Hartman number and  $\mu = 1 + \frac{1}{\beta}$  represents the Casson diffusion coefficient. The main novelty in this model is the consideration of the Goodman boundaries conditions. In short, we will focus on the impact of the Goodman boundary conditions on the diffusion processes. Note that in classical physic Eq. (6) is called fractional diffusion equation or fractional heat equation and the Eq. (5) is called fractional diffusion with a reaction term. The combination of these two equations represents as well the fractional generalized Casson fluid model.

#### 4. Solutions algorithms for generalized Casson fluid model

In this section, we describe the algorithm of the solutions. With both equation we adopt a mathematical methods for physics. We method adopted are called integral balance methods: the heat balance integral method and the double integral methods. The methods are described as follows.

For the integral balance methods, according to Goodman proposition we suppose the approximate analytical solution are described by

$$v(z, t) = \left[1 - \frac{z}{\kappa}\right]^n, \tag{8}$$

where  $\kappa$  is the penetration depth as described in the presentation of the model, and  $n$  is the exponent of the approximate solution, which will be determined using the matching method. The second step consists of integrating the fractional differential equation between 0 to  $\kappa$ , taking into account the approximate solution Eq. (8). The last step consists of solving the obtained fractional differential equation after single integration between 0 to  $\kappa$ , which depends eventually on time.

For the double integral methods, according to Goodman proposition, we conserve the approximate analytical solution given in Eq. (8). The second step consists of applying the double integration on the fractional differential equation, the first integration between 0 to  $\kappa$ , and the second integration between  $z$  to  $\kappa$  by taking into account the approximate solution Eq. (8). The last step consists of solving the obtained fractional differential equation after the double integration.

Note that all these methods use the given Goodman boundary conditions. To experiment with these methods, we begin our procedure of solutions by applying the heat balance integral method. Let the fractional diffusion equation defined by Eq. (6), and we integrate between 0 to  $\kappa$ , that is

$$\begin{aligned} \int_0^\kappa D_t^\alpha v dz &= \int_0^\kappa \frac{1}{Pr} \frac{\partial^2 v}{\partial z^2} dz, \\ D_t^\alpha \int_0^\kappa v dz &= \frac{1}{Pr} \int_0^\kappa \frac{\partial^2 v}{\partial z^2} dz. \end{aligned} \tag{9}$$

Using the approximate solution in Eq. (8), we obtain after calculations the following relationships

$$\begin{aligned} D_t^\alpha \int_0^\kappa \left(1 - \frac{z}{\kappa}\right)^n dy &= \frac{1}{Pr} \left[\frac{\partial v}{\partial z}\right]_0^\kappa, \\ D_t^\alpha \left[-\frac{\delta}{n+1} \left(1 - \frac{z}{\kappa}\right)^{n+1}\right]_0^\kappa &= \frac{1}{Pr} \frac{n}{\kappa}, \\ D_t^\alpha \left[\frac{\kappa}{n+1}\right] &= \frac{1}{Pr} \frac{n}{\kappa}, \\ \frac{1}{n+1} D_t^{\alpha, \rho} \kappa &= \frac{1}{Pr} \frac{n}{\kappa}. \end{aligned} \tag{10}$$

Multiplying by the parameter  $(n+1)\kappa$  and applying the Laplace transform of the Caputo derivative to both sides of Eq. (10), we have the following relationships

$$\begin{aligned} s^\alpha \bar{\kappa}^2(s) &= \frac{2n(n+1)}{Prs}, \\ \bar{\kappa}^2(s) &= \frac{2n(n+1)}{Prs^{1+\alpha}}. \end{aligned} \tag{11}$$

Thus, the penetration depth is obtained after the application of the inverse of the Laplace transform to both sides of Eq. (11), we have

$$\kappa^2(t) = \frac{2n(n+1)t^\alpha}{Pr}. \tag{12}$$

According to the the heat balance integral method described above, the approximate analytical solution of Eq. (6) is given by the relationship

$$v(z, t) = \left[ 1 - \frac{z}{\sqrt{\frac{2n(n+1)t^\alpha}{Pr}}} \right]^n. \tag{13}$$

The exponent  $n$  is important in our investigations, to propose its value we continue our resolution by proposing the approximate solution using the Double integral method. Applying the double integral between 0 to  $\kappa$  and between  $z$  to  $\kappa$  on the fractional diffusion Eq. (6), we have

$$\begin{aligned} \int_0^\kappa \int_z^\kappa D_t^\alpha v dz dz &= \int_0^\kappa \int_z^\kappa \frac{1}{Pr} \frac{\partial^2 v}{\partial z^2} dz dz, \\ D_t^\alpha \int_0^\kappa \int_z^\kappa v dz dz &= \frac{1}{Pr} \int_0^\kappa \int_z^\kappa \frac{\partial^2 v}{\partial z^2} dz dz. \end{aligned} \tag{14}$$

Using the approximation described in Eq. (8), and calculating the integration between  $z$  to  $\kappa$ , we obtain the following calculations

$$\begin{aligned} D_t^\alpha \left[ -\frac{\kappa^2}{(n+1)(n+2)} \left( 1 - \frac{z}{\kappa} \right)^{n+1} \right]_0^\kappa &= -\frac{1}{Pr} \int_0^\kappa \frac{\partial v}{\partial z} dz, \\ D_t^\alpha \left[ \frac{\kappa^2}{(n+1)(n+2)} \right] &= \frac{1}{Pr} v(0, t), \\ \frac{1}{(n+1)(n+2)} [D_t^{\alpha, \rho} \kappa^2] &= \frac{1}{Pr}. \end{aligned} \tag{15}$$

Applying the Laplace transform to both sides of Eq. (15), we get the following relationship

$$\begin{aligned} s^\alpha \bar{\kappa}^2(s) &= \frac{(n+1)(n+2)}{Prs}, \\ \bar{\kappa}^2(s) &= \frac{(n+1)(n+2)}{Prs^{1+\alpha}}. \end{aligned} \tag{16}$$

Thus, the penetration depth is obtained after the application of the inverse of the Laplace transform to both sides of Eq. (16), we have

$$\kappa^2(t) = \frac{(n+1)(n+2)t^\alpha}{Pr}. \tag{17}$$

According to the Double integral method described above, the approximate analytical solution of Eq. (6) is given by the relationship

$$v(z, t) = \left[ 1 - \frac{z}{\sqrt{\frac{(n+1)(n+2)t^\alpha}{Pr}}} \right]^n. \tag{18}$$

We can observe the exponent  $n$  appears again in the approximate solution with the Double integral method. It is natural to stipulate these approximations are the same. It is called the matching method. Therefore, considering Eq. (13) and Eq. (18), we obtain the following value of the exponent

$$\begin{aligned} \left[ 1 - \frac{z}{\sqrt{\frac{2n(n+1)t^\alpha}{Pr}}} \right]^n &= \left[ 1 - \frac{z}{\sqrt{\frac{(n+1)(n+2)t^\alpha}{Pr}}} \right]^n, \\ n &= 2. \end{aligned} \tag{19}$$

Finally, the approximate solution of the fractional heat equation is obtained by considering the exponent  $n = 2$ , there is

$$v(z, t) = \left[ 1 - \frac{z}{2\sqrt{\frac{3t^\alpha}{Pr}}} \right]^2. \tag{20}$$

The second part of this resolution consists of applying the heat balance integral method and the double integral method in Eq. (5). In other words, we have to solve the fractional differential equation described by the equation

$$D_t^\alpha u = \mu \frac{\partial^2 u}{\partial z^2} - Mu + Gr \left[ 1 - \frac{z}{\kappa} \right]^n. \tag{21}$$

We repeat the same resolutions. First, we apply the heat balance integral method. We apply the single integral between 0 to  $\varpi$ , that is

$$\begin{aligned} \int_0^\varpi D_t^\alpha u dz &= \mu \int_0^\varpi \frac{\partial^2 u}{\partial z^2} dz - M \int_0^\varpi u dz + Gr \int_0^\varpi \left[ 1 - \frac{z}{\kappa} \right]^n dz, \\ D_t^\alpha \int_0^\varpi v dz &= \mu \int_0^\varpi \frac{\partial^2 u}{\partial z^2} dz - M \int_0^\varpi u dz + Gr \int_0^\varpi \left[ 1 - \frac{z}{\kappa} \right]^n dz. \end{aligned}$$

Replacing  $u$  by its supposed approximation, we get after integration the following relationship

$$D_t^\alpha \left[ \frac{\varpi}{n+1} \right] = \frac{\mu n}{\varpi} - \frac{M\varpi}{n+1} + \frac{Gr\kappa}{n+1} - \frac{Gr\kappa}{n+1} \left( 1 - \frac{\varpi}{\kappa} \right). \tag{22}$$

Let rearranging the last terms and multiplying by the parameter  $(n+1)\varpi$ , we have to solve the fractional differential equation described by the following form

$$D_t^\alpha \varpi^2 = 2\mu n(n+1) - 2M\varpi^2 + 2Gr\varpi^2. \tag{23}$$

Applying the Laplace transform of the Caputo derivative to both sides of Eq. (23), we have the following relationships

$$\begin{aligned} s^\alpha \bar{\varpi}^2(s) &= \frac{2\mu n(n+1)}{s} + [-2M + 2Gr] \bar{\varpi}^2(s), \\ \bar{\varpi}^2(s) &= \frac{2\mu n(n+1)}{[1 + 2M - 2Gr] s^{1+\alpha}}. \end{aligned} \tag{24}$$

Thus, the penetration depth is obtained after the application of the inverse of the Laplace transform to both sides of Eq. (24), we have

$$\varpi^2(t) = \frac{2\mu n(n+1)t^\alpha}{[1 + 2M - 2Gr]}. \tag{25}$$

According to the heat balance integral method described above, the approximate analytical solution of Eq. (5) is given by the relationship

$$u(z, t) = \left[ 1 - \frac{z}{\sqrt{\frac{2\mu n(n+1)t^\alpha}{[1+2M-2Gr]}}} \right]^n. \tag{26}$$

The second method consists of applying the double integral method. We Apply the double integral between 0 to  $\kappa$  and between  $z$  to  $\varpi$  on the fractional diffusion Eq. (5), we have

$$\begin{aligned} \int_0^\varpi \int_z^\varpi D_t^\alpha u dz dz &= \mu \int_0^\varpi \int_z^\varpi \frac{\partial^2 u}{\partial z^2} dz dz - M \int_0^\varpi \int_z^\varpi u dz dz + Gr \int_0^\varpi \int_z^\varpi \left[ 1 - \frac{z}{\kappa} \right]^n dz dz, \\ D_t^\alpha \int_0^\varpi \int_z^\varpi v dz dz &= \mu \int_0^\varpi \int_z^\varpi \frac{\partial^2 u}{\partial z^2} dz dz - M \int_0^\varpi \int_z^\varpi u dz dz + Gr \int_0^\varpi \int_z^\varpi \left[ 1 - \frac{z}{\kappa} \right]^n dz dz. \end{aligned}$$

We assume the approximation described in Eq. (8) with  $u$ , and calculating the integration, we obtain the following calculations

$$\begin{aligned}
 D_t^\alpha \left[ -\frac{\varpi^2}{(n+1)(n+2)} \left(1 - \frac{z}{\varpi}\right)^{n+1} \right]_0^\varpi &= -\mu \int_0^\varpi \frac{\partial u}{\partial z} dz - \frac{M\varpi^2}{(n+1)(n+2)} + \frac{Gr\varpi^2}{(n+1)(n+2)}, \\
 D_t^\alpha \left[ \frac{\varpi^2}{(n+1)(n+2)} \right] &= \mu v(0, t) - \frac{M\varpi^2}{(n+1)(n+2)} + \frac{Gr\varpi^2}{(n+1)(n+2)}, \\
 \frac{1}{(n+1)(n+2)} [D_t^\alpha \varpi^2] &= \mu - \frac{M\varpi^2}{(n+1)(n+2)} + \frac{Gr\varpi^2}{(n+1)(n+2)}. \tag{27}
 \end{aligned}$$

Applying the Laplace transform to both sides of Eq. (27), we get the following relationship

$$\begin{aligned}
 s^\alpha \bar{\varpi}^2(s) &= \frac{\mu(n+1)(n+2)}{Pr} - \frac{M\bar{\varpi}^2(s)}{(n+1)(n+2)} + \frac{Gr\bar{\varpi}^2(s)}{(n+1)(n+2)}, \\
 [1 + 2M - 2Gr] \bar{\varpi}^2(s) &= \frac{\mu(n+1)(n+2)}{s^{1+\alpha}}. \tag{28}
 \end{aligned}$$

Thus, the penetration depth is obtained after the application of the inverse of the Laplace transform to both sides of Eq. (28), we have

$$\varpi^2(t) = \frac{\mu(n+1)(n+2)t^\alpha}{[1 + 2M - 2Gr]}. \tag{29}$$

According to the double integral method described above, the approximate analytical solution of Eq. (5) is given by the relationship

$$u(z, t) = \left[ 1 - \frac{z}{\sqrt{\frac{\mu(n+1)(n+2)t^\alpha}{[1+2M-2Gr]}}} \right]^n. \tag{30}$$

We can observe the exponent  $n$  appears again in the approximate of the penetration depth with the double integral method. It is natural to stipulate these approximations are the same. Therefore, considering Eq. (25) and Eq. (29), we obtain the following value of the exponent

$$\begin{aligned}
 \frac{2\mu n(n+1)t^\alpha}{[1 + 2M - 2Gr]} &= \frac{\mu(n+1)(n+2)t^\alpha}{[1 + 2M - 2Gr]}, \\
 n &= 2. \tag{31}
 \end{aligned}$$

Finally, the approximate solution of the fractional heat equation with reaction term is obtained with considering the exponent  $n = 2$ , there is

$$u(z, t) = \left[ 1 - \frac{z}{2\sqrt{\frac{3\mu t^\alpha}{[1+2M-2Gr]}}} \right]^2. \tag{32}$$

### 5. Graphics and interpretations of the solutions

This section is the main section of this work. We will analyze the impact of the fractional-order derivative into the diffusion process, particularly into the temperature and the effect of the fractional-order into the velocity of the fluid in  $z$ -direction. The impact of the Prandtl number  $Pr$ , the Hartman number  $M$ , the Grashold number  $Gr$ , and the Casson diffusion coefficient  $\mu$  into the velocity of the fluid and its temperature.

Let us begin with the fractional order  $\alpha$  into the temperature. We fix the Prandtl number to  $Pr = 0.25$  and the time  $t = 0.3$ . The fractional-order derivative varies into  $(0, 1)$ . In Figure 1, we notice all the trajectories decrease according to the increase of the direction  $z$ . We note when the order  $\alpha$  increases, then all the velocities decrease following the direction indicated by the arrow in Figure 1. We conclude

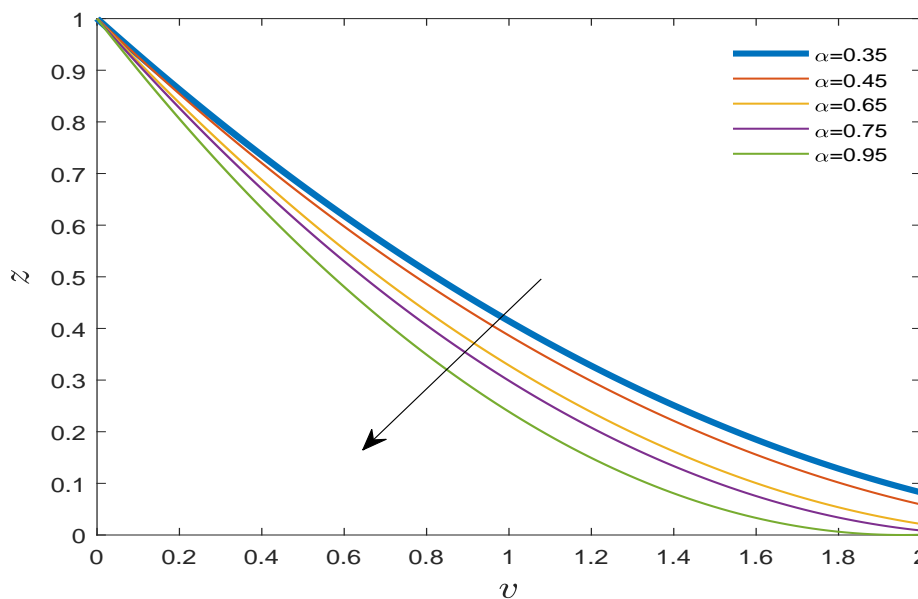


Figure 1: Dynamics of the temperature with different orders  $\alpha$ .

the fractional-order  $\alpha$  accelerate the non-increasing of the velocity. Thus the fractional-order  $\alpha$  has an acceleration effect.

We analyze the impact of the fractional-order  $\alpha$  into the temperature. As in the previous analysis, here we fix the Prandtl number to  $Pr = 10$ , the time  $t = 0.3$ , the Hartman number  $M = 15$ , the Grashold number  $Gr = 10$  and the diffusion coefficient to  $\beta = 10$ . The fractional-order derivative varies into the interval  $(0, 1)$ . In Figure 2, we notice the velocity decrease according to the increase of the state  $z$ . We note when the order  $\alpha$  increases, then all the velocity decrease too as well following the direction indicated by the arrow in Figure 2. We conclude the fractional-order  $\alpha$  accelerates the decrease of the velocity. Thus the fractional-order  $\alpha$  also has an acceleration effect into the diffusion process.

We continue by analyzing the impact of the Prandtl number  $Pr$  into the temperature. We fix the order  $\alpha = 0.95$ , the time  $t = 0.3$  into the temperature diffusion. In Figure 3, we notice when the Prandtl number increase, all the temperatures decrease as well according to the increase of the direction  $z$ . To be more precise, we note when the Prandtl number  $Pr$  increases the temperature decreases more fastly following the direction of the arrow in Figure 3. Finally, we conclude the increase of the Prandtl number  $Pr$  accelerates the decrease of the temperature to zero.

We now consider the velocity of the fluid. The Hartman number  $M$  is fixed to  $Mr = 15$ , the diffusion coefficient to  $\mu = 5$ , the order  $\alpha = 0.95$ , the Grashold number take different value. In Figure 4, we note the general behaviors of the velocities are they decrease, but when the Grashold number  $Gr$  increases, then all the velocities increase to one another. Here the Grashold number  $Gr$  has also acceleration effect but accelerate the intensity of the velocity to a value different to zero, see the direction of the arrow indicated in Figure 4.

We assume the velocity of the fluid. The Grashold number  $Gr$  is fixed to  $Gr = 10$ , the diffusion coefficient to  $\mu = 6$ , the order  $\alpha = 0.95$ , and the Hartman number  $M$  take different value. In Figure 5, we note the general behaviors of the velocities are they decrease, but when the Hartman number  $M$  increases, then all the velocities decrease to one another. Here the Hartman number  $M$  generate has also acceleration effect but accelerates the intensity of the velocity to zero; see the direction of the arrow indicated in Figure 5. Finally, the Hartman number  $M$  and the Prandtl number  $Pr$  have the same effect in the diffusion process of the velocity.

The Casson coefficient  $\mu = 1 + \frac{1}{\beta}$  also generates the same behavior, when we fix Grashold number  $Gr = 10$ ,



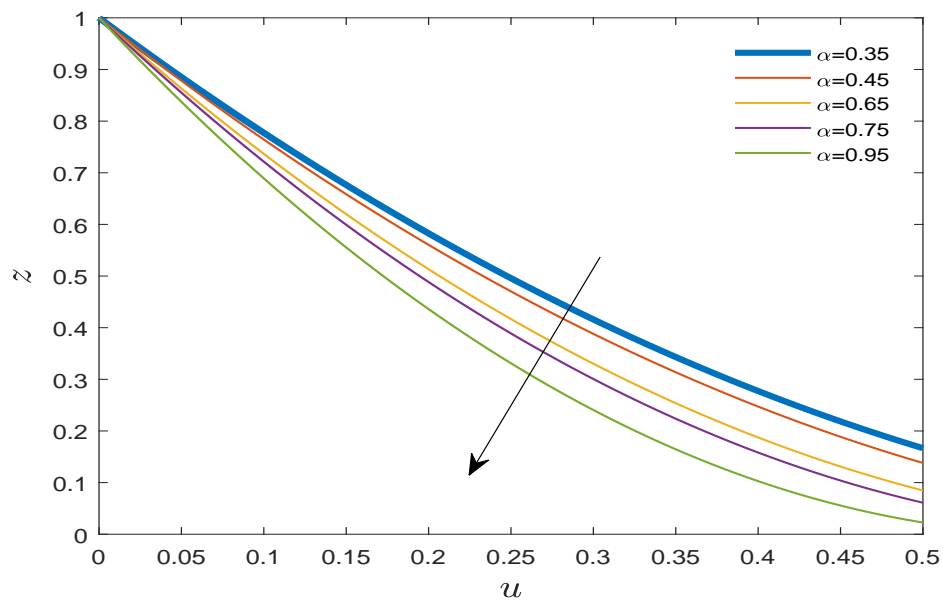


Figure 2: Dynamics of the velocity of the fluid with different orders  $\alpha$ .

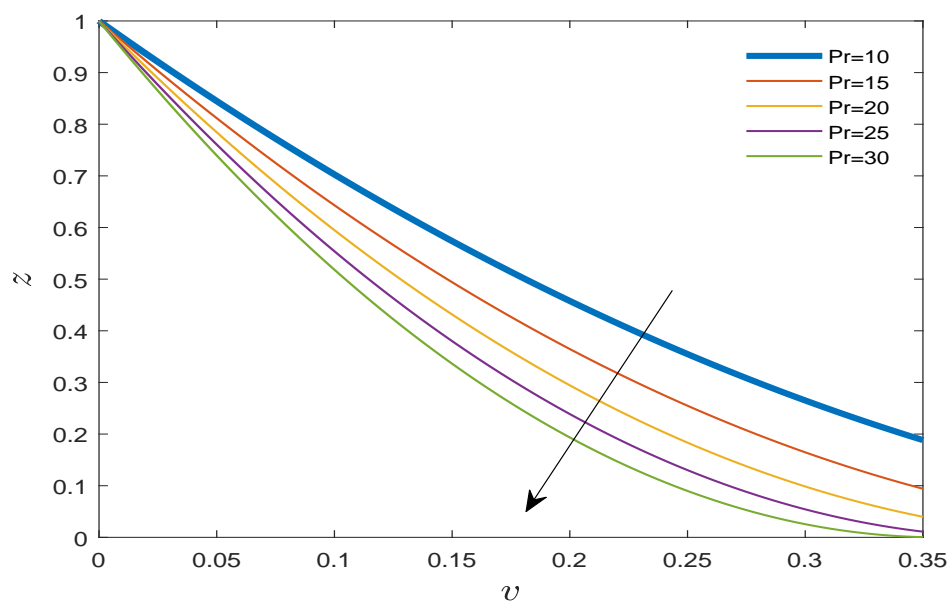


Figure 3: Dynamics of the temperature with different values  $Pr$ .

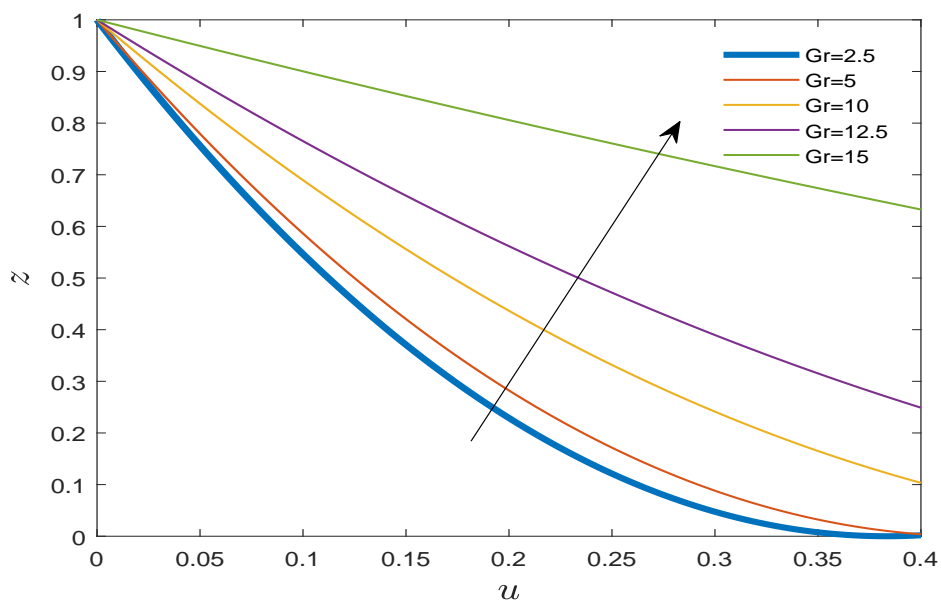


Figure 4: Dynamics of the velocity with different values  $Gr$ .

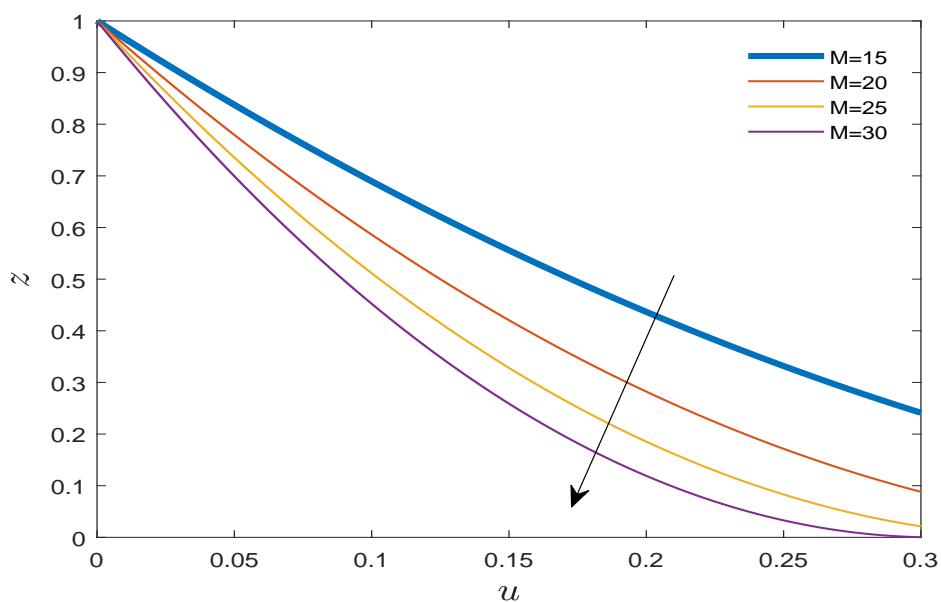


Figure 5: Dynamics of the velocity with different values  $M$ .

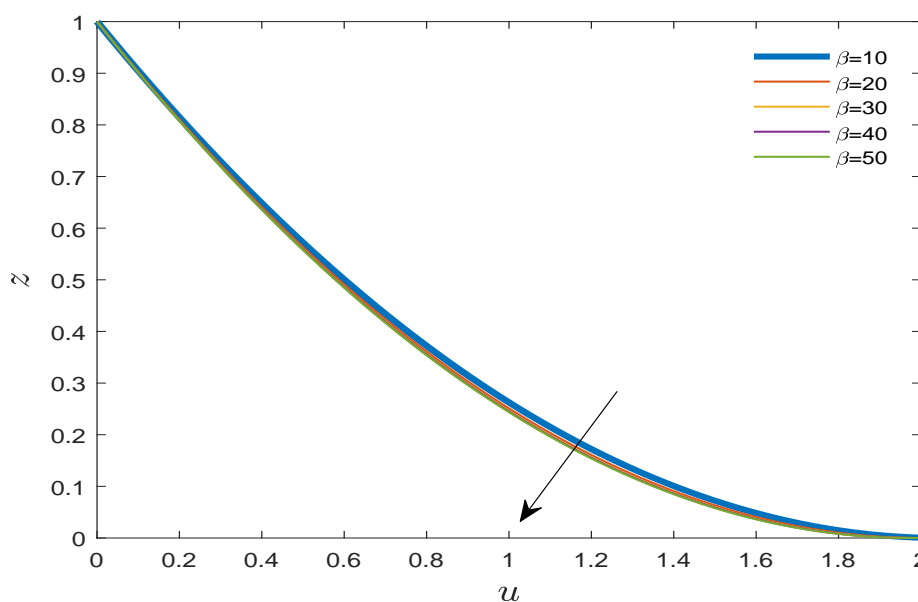


Figure 6: Dynamics of the velocity with different values  $\beta$ .

the order  $\alpha = 0.5$  and the Hartman number  $M = 0.5$ . See the direction of the velocity in Figure 6. We also remark the diffusion coefficient has the same impact as the Hartman number  $M$ .

## 6. Concluding remarks

Some new methods as the Double integral and the heat balance integral methods have been experienced in this paper. The conclusion is these methods are suitable for getting the solutions of the fractional Casson fluid models. The main difference between these methods with the methods existing in the literature is, these methods are possible when the Goodman boundary conditions are considered. For short, we have also experimented in this paper the tolerability of the Goodman boundary condition into the diffusion processes. The graphical representations support the main results of this paper. This works, and the method experienced in this paper will open a new door in the methods for approximating the solutions of the fluid models.

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