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On the Moduli Space of Flat Tori Having Unit Area

İsmail Sağlam*

(Dedicated to the memory of **Prof. Dr. Aurel BEJANCU (1946 - 2020)**)

ABSTRACT

Inspiring from Thurston's asymmetric metric on Teichmüller spaces, we define and study a natural (weak) metric on the Teichmüller space of the torus. We prove that this weak metric is indeed a metric: it separates points and it is symmetric. Our main strategy to do this is to compute the metric explicitly. We relate this metric with the hyperbolic metric on the upper half-plane. We define another metric which measures how much length of a closed geodesic changes when we deform a flat structure on the torus. We show that these two metrics coincide.

Keywords: *weak metric, asymmetric metric, Teichmüller space, torus, flat metric, hyperbolic metric*

AMS Subject Classification (2020): *Primary: 51F99 ; Secondary: 57M50*

1. Introduction

Teichmüller space of a surface of finite type is homeomorphic to a Euclidean space of appropriate dimension. It parametrizes several important structures on the surface such as conformal structures and hyperbolic structures. There are numerous metrics on Teichmüller spaces. Among them, we can count Teichmüller's metric, the Weil-Peterson metric, Thurston's asymmetric metric, the Caretheodary, the Kobayashi and the Bergman metrics.

Let $S_{\mathfrak{g},\mathfrak{p}}$ be a surface of genus $\mathfrak{g}$ with $\mathfrak{p}$ punctures. Take two conformal structures g and h on this surface. Let $\mathcal{Q}$ be the set of all quasiconformal homeomorphisms

$$f : (S_{\mathfrak{g},\mathfrak{p}}, g) \to (S_{\mathfrak{g},\mathfrak{p}}, h)$$

that are isotopic to the identity map. Let $K(f)$ be the quasiconformal dilatation of f. Then

$$\tau(g,h) = \frac{1}{2} \inf_{f \in \mathcal{Q}} \log K(f)$$

is called Taichmüller's metric. Let us denote the Teichmüller spce of $S_{\mathfrak{g},\mathfrak{p}}$ by $\mathbb{T}(S_{\mathfrak{g},\mathfrak{p}})$. It is known that Teichmüller's metric is a complete Finsler metric.

One can also consider $\mathbb{T}(S_{\mathfrak{g},\mathfrak{p}})$ as the set of equivalence classes of complete hyperbolic metrics on $S_{\mathfrak{g},\mathfrak{p}}$ which have finite area if $\chi(S_{\mathfrak{g},\mathfrak{p}}) < 0$, where $\chi(S_{\mathfrak{g},\mathfrak{p}})$ is is the Euler characteristic of $S_{\mathfrak{g},\mathfrak{p}}$. Let Diff$_0(S_{\mathfrak{g},\mathfrak{p}})$ be the group of diffeomorphisms of $S_{\mathfrak{g},\mathfrak{p}}$ which are isotopic to the identity map. Now fix two hyperbolic metrics g_1 and g_2 on $S_{\mathfrak{g},\mathfrak{p}}$. If $f \in$ Diff$_0(S_{\mathfrak{g},\mathfrak{p}})$, we set

$$K'(f) = \sup_{x \in S_{\mathfrak{g},\mathfrak{p}}} \left(\frac{\sup\{\|df_x(u)\| : u \in T_x S_{\mathfrak{g},\mathfrak{p}}, \|u\| = 1\}}{\inf\{\|df_x(u)\| : u \in T_x S_{\mathfrak{g},\mathfrak{p}}, \|u\| = 1\}} \right)$$

where df_x is the differential of f and the norm of $df_x(u)$ and u are measured with respect to g_2 and g_1, respectively. In that case, Teichmüller's metric is given by

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*** Corresponding author***

$$\tau(g_1, g_2) = \frac{1}{2} \inf\{\log K'(f) : f \in \mathrm{Diff}_0(S_{\mathfrak{g},\mathfrak{p}})\}.$$

Thurston's asymmetric metric was introduced by Thurston in [6]. Thurston considered Teichmüller space of a surface of finite type as the set of isotopy classes of complete finite-area hyperbolic structures on the surface. Let $f \in \mathrm{Diff}_0(S_{\mathfrak{g},\mathfrak{p}})$ and g and h be two such hyperbolic structures. We define the Lipschitz constant of the map $f : (S_{\mathfrak{g},\mathfrak{p}}, g) \to (S_{\mathfrak{g},\mathfrak{p}}, h)$ as

$$\mathrm{Lip}(f) = \sup\left\{\frac{d_{g_2}(f(x), f(y))}{d_{g_1}(x,y)} : x, y \in S_{\mathfrak{g},\mathfrak{p}}, x \neq y\right\}$$

Thurston's asymmetric metric is defined as

$$\lambda(g_1, g_2) = \inf\{\log \mathrm{Lip}(f) : f \in \mathrm{Diff}_0(S_{\mathfrak{g},\mathfrak{p}})\}.$$

This metric separates the points of the Teichmüller space, that is, if $g_1 \neq g_2$, then $\lambda(g_1, g_2) \neq 0$. But it is not symmetric: $\lambda(g_1, g_2) \neq \lambda(g_2, g_1)$ in general.

Now consider a surface $S_{\mathfrak{g},0}$ of genus $\mathfrak{g} > 1$ with no punctures and a hyperbolic metric g on it. Let α be an isotopy class of closed curves on $S_{\mathfrak{g}}$ and $a : [0,1] \to S_{\mathfrak{g}}$ be a representative for α. Let

$$l_g(\alpha) = \inf\{\int_0^1 \sqrt{g(\dot{a}, \dot{a})}dt : a \in \alpha\}.$$

Let $\mathcal{S}(S_{\mathfrak{g},0})$ be the set of nontrivial isotopy classes of closed curves in $S_{\mathfrak{g},0}$. Thurston defined another (asymmetric) distance on $\mathbb{T}(S_{\mathfrak{g},0})$:

$$\kappa(g_1, g_2) = \sup_{\alpha \in \mathcal{S}(S_{\mathfrak{g},0})} \log \frac{l_{g_2}(\alpha)}{l_{g_1}(\alpha)}.$$

He proved that

$$\lambda(g_1, g_2) = \kappa(g_1, g_2).$$

In the present paper, we define and study a metric on $\mathbb{T}(T^2)$, Teichmüller space of the torus, whose definition is completely analogous to that of the asymmetric metric defined by Thurston. Before stating our results, we state the results on [1].

In [1], the authors studied Teichmüller space of T^2 as the set of equivalence classes of flat metric on T^2, where T^2 is a surface of genus 1. Let $\mathfrak{F}(T^2)$ be the set of flat metrics on T^2. If we scale a flat metric by a positive scalar, then we get another flat metric. Thus we may define

$$\mathbb{T}(T^2) = \mathfrak{F}(T^2)/(\mathbb{R}_+^* \times \mathrm{Diff}_0(T^2)).$$

This means that instead of arbitrary flat structures, one should consider "normalized" flat structures on T^2. However, there are several ways to normalize a flat structure on T^2. Now we explain the normalization used by the authors of [1].

A marking of the torus is a group isomorphism $\psi : \mathbb{Z}^2 \to \Pi_1(T_2)$. Since $\Pi_1(T_2)$ is abelian, there is nothing to worry about the base point. Given a marked torus, we denote by ϵ the isotopy class corresponding to the generator $(1,0) \in \mathbb{Z}^2$. In [1], the authors defined $\mathbb{T}(T^2)$ as the set of equivalence classes of triples (S, g, ψ) where S is a closed oriented surface of genus 1, g is a flat metric and ψ is a marking of S such that the length of any geodesic in the isotopy class of ϵ is 1. They call two triples (S, g, ψ) and (S', g', ψ') equivalent if there exists a diffeomorphism $f : S \to S'$ such that $f^*g' = g$ and $f_*(\psi) = \psi'$. Briefly, it means taht the normalization is obtained by scaling any metric so that the length of any geodesic in the isotopy class of ϵ is equal to 1. After this normalization, the authors define a weak metric inspiring from Thurston's asymmetric metric. This metric is not symmetric and does not separates points. Also, it is not quasi-isometric to Teichmüller's metric τ on the Teichmuller space of the torus. But one of its natural symmetrizations is equal to the Teichmüller's metric.

There is another natural way to normalize a flat structure on T^2 or any compact surface. Simply, we can scale the metric with a positive scalar so that the area of the resulting surface is equal to 1. In this paper, we use this normalization to study $\mathbb{T}(T^2)$. We define analogues of the distances defined by Thurston. We denote these metrics by λ and κ. We prove the following results.

- λ and κ coincide on $\mathbb{T}(T^2)$.

- $\mathbb{T}(T^2)$ with the metric 2λ is isometric to upper half-plane with the hyperbolic metric $d_{\mathbb{H}}$. In particular,
- λ and κ are symmetric and they separate points.
- The extremal map for λ is induced by an affine map and it coincides with the Teichmüller extremal map between two flat structures on the torus.
  Note that in [3] and [4], the authors considered similar problems. But our approach is significantly different then their approach. See [5], [7] and [8] for more information about flat metrics on surfaces.

2. Teichmüller space of the torus and the weak metrics λ and κ

In this section, we first give two equivalent definitions of $\mathbb{T}(T^2)$. First of all, we define T^2 to be the surface $\mathbb{C}/\mathbb{Z}+i\mathbb{Z}$. Let $\mathfrak{F}_1(T^2)$ be the set of unit area flat metrics on T^2. Note that by a flat metric we mean a Riemannian metric on T^2 with constant curvature equal to 0. The Teichmüller space of T^2 is defined as

$$\mathbb{T}(T^2) = \mathfrak{F}_1(T^2)/\mathrm{Diff}_0(T^2),$$

where, as before, $\mathrm{Diff}_0(T^2)$ is the group of diffeomorphisms of T^2 which are isotopic to the identity map. The action of $\mathrm{Diff}_0(T^2)$ on $\mathfrak{F}_1(T^2)$ is given by the pull-back of the metric.

Now we introduce the other definition. Recall that for a surface S of genus 1, a marking of S is a group isomorphism $\psi : \mathbb{Z}^2 \to \Pi_1(S)$. Consider the set of the tuples (S, g, ψ), where S is a closed surface of genus 1, ψ is a marking of S and g is a flat metric of unit-area on S. Two triples (S, g, ψ) and (S', g', ψ') are equivalent if there is an isomorphism $f : S \to S'$ such that $f^*g' = g$ and $f_*\psi = \psi'$.

We will define two weak metrics λ and κ on $\mathbb{T}(T^2)$. We will then show that λ and κ are metrics. Let us first define what a weak metric is.

Definition. A weak metric on a set X is a map $\eta : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$ the following properties hold:

1. $\eta(x, x) = 0.$
2. $\eta(x, y) \geq 0.$
3. $\eta(x, y) + \eta(y, z) \geq \eta(x, z).$

See [2] for more information about weak metrics.

Let g_1 and g_2 be elements of $\mathfrak{F}_1(T^2)$ and ϕ be a homeomorphism of T^2 which is to the identity map. We define

$$\mathrm{Lip}(\phi) = \sup_{x \neq y} \left(\frac{d_{g_2}(\phi(x), \phi(y))}{d_{g_1}(x, y)} \right)$$

$$\lambda(g_1, g_2) = \inf\{\log \mathrm{Lip}(\phi) : \phi : T^2 \to T^2 \text{ is a homeomorphism isotopic to the identity}\}$$

We see that $\lambda(g_1, g_2)$ is invariant under the action of $\mathrm{Diff}_0(T^2)$. So

$$\lambda : \mathbb{T}(T^2) \times \mathbb{T}(T^2) \to \mathbb{R}$$

is well-defined and it is clear that λ is a weak metric.

Now we define κ. Let g_1 and g_2 be in $\mathfrak{F}_1(T^2)$.

$$\kappa(g_1, g_2) = \sup_{\alpha \in \mathcal{S}(T^2)} \log \left(\frac{l_{g_2(\alpha)}}{l_{g_1(\alpha)}} \right),$$

where $\mathcal{S}(T^2)$ is the set isotopy classes of nontrivial closed loops on T^2. Note that since $\Pi_1(T^2)$ is abelian, $\mathcal{S}(T^2)$ is in one-to-one bijection with the nontrivial elements of $\Pi_1(T^2)$.

It is not difficult to see that κ is invariant under the action of $\mathrm{Diff}_0(T^2)$. Therefore

$$\kappa : \mathbb{T}(T^2) \times \mathbb{T}(T^2) \to \mathbb{R}$$

is well-defined and it is clear that it is a weak metric.

We will later show that $\kappa = \lambda$. Let us first prove the following important inequality.

Lemma 2.1. $\kappa \leq \lambda$.

Proof. This follows from the fact that for any homeomorphism $f : (T^2, g_1) \to (T^2, g_2)$ and any curve $c : [0,1] \to T^2$

$$l_{g_2}(f \circ c) \leq \text{Lip}(f) l_{g_1}(c).$$

□

3. The metric κ

In this section, we show that the weak metric κ is indeed a metric: it separates points and it is symmetric. Actually, we show that the weak metric space $\mathbb{T}(T^2)$ with the metric 2κ is isometric with the upper half-plane with the hyperbolic metric $d_{\mathbb{H}}$. We know that hyperbolic plane and Teichmüller space of the torus with Teichmüller's metric are isometric. See [1]. Thus we explicitly obtain the relation between κ and τ: $2\kappa = \tau$.

Before pursuing, we need to identify $\mathbb{T}(T^2)$ with the upper half-plane $\mathbb{H} = \{x + \imath y \in \mathbb{C} : y > 0\}$. Let $(S, g, \psi) \in \mathbb{T}(T^2)$ and γ_1 and γ_2 be geodesics of S which are in the isotopy class of $\psi_*(1,0)$ and $\psi_*(0,1)$, respectively. Cut S through γ_1 and γ_2. Then you get a disk which is isometric to a parallelogram in $\mathbb{C}$. Since the area of S is 1, the area of the disk is 1, hence we can uniquely choose a parallelogram $P_{x+\imath y}$ with vertices $0, \frac{1}{y}, x + \frac{1}{y} + \imath y, x + \imath y$ so that $x \in \mathbb{R}$ and $y > 0$. Note that the edges $[0, \frac{1}{y}]$ and $[0, x + \imath y]$ of $P_{x+\imath y}$ correspond to γ_1 and γ_2, respectively. Thus we have a map

$$i : \mathbb{T}(T^2) \to \mathbb{H}$$

and we can construct a map

$$i' : \mathbb{H} \to \mathbb{T}(T^2)$$

as follows. For each $x + \imath y \in \mathbb{H}$, consider the parallelogram $P_{x+\imath y}$. Identify its opposite edges accordingly to obtain an oriented flat torus S of area 1. Now let $\psi : \mathbb{Z}^2 \to \Pi_1(S)$ be the marking which sends $(1,0)$ and $(0,1)$ to the oriented edges $[0, \frac{1}{y}]$ and $[0, x + \imath y]$ of S, respectively. It is clear that i and i' are inverses of each other. So $\mathbb{T}(T^2)$ and $\mathbb{H}$ are naturally identified.

Note that there is another way to obtain the flat torus from $P_{x+\imath y}$. The quotient space $\mathbb{C}/\frac{1}{y}\mathbb{Z} + (x + \imath y)\mathbb{Z}$ with the induced flat metric gives the same point in $\mathbb{T}(T^2)$.

Now we find an explicit formula for κ. Since $\mathbb{C}$ is the universal cover of $S = \mathbb{C}/\frac{1}{y}\mathbb{Z} + (x + \imath y)\mathbb{Z}$, any straight line son $\mathbb{C}$ which starts at origin and ends a point in $\frac{1}{y}\mathbb{Z} + (x + \imath y)\mathbb{Z}$ gives us a geodesic in S. In particular, a geodesic corresponding to $(m, n) \in \mathbb{Z}^2$ is represented by a straight line segment from the origin to the point $\frac{m}{y} + n(x + \imath y) = \frac{m}{y} + nx + \imath ny$: the projection $\mathbb{C} \to S$ gives us the desired geodesic. Thus the length of such a geodesic is equal to

$$|\frac{m}{y} + nx + \imath ny| = \sqrt{(\frac{m}{y} + nx)^2 + n^2 y^2}$$

Now assume that the points (S_1, g_1, ψ_1) and (S_2, g_2, ψ_2) correspond the points $\zeta_1 = x_1 + \imath y_1, \zeta_2 = x_2 + \imath y_2 \in \mathbb{H}$. Then

$$\begin{aligned}
\exp(\kappa(g_1, g_2)) = \exp(\kappa(\zeta_1, \zeta_2)) &= \sup_{m,n \in \mathbb{Z}} \frac{\sqrt{(\frac{m}{y_2} + nx_2)^2 + n^2 y_2^2}}{\sqrt{(\frac{m}{y_1} + nx_1)^2 + n^2 y_1^2}} \\
&= \sup_{m,n \in \mathbb{Z}} \frac{\sqrt{(\frac{m}{ny_2} + x_2)^2 + y_2^2}}{\sqrt{(\frac{m}{ny_1} + x_1)^2 + y_1^2}} \\
&= \sup_{z \in \mathbb{R}} \frac{\sqrt{(\frac{z}{y_2} + x_2)^2 + y_2^2}}{\sqrt{(\frac{z}{y_1} + x_1)^2 + y_1^2}} \\
&= \frac{y_1}{y_2} \sup_{z \in \mathbb{R}} \frac{\sqrt{(z + x_2 y_2)^2 + y_2^4}}{\sqrt{(z + x_1 y_1)^2 + y_1^4}}
\end{aligned}$$

Let us write what we get in a single equation:

$$\kappa(g_1, g_2) = \kappa(\zeta_1, \zeta_2) = \log\left(\frac{y_1}{y_2}\sup_{z\in\mathbb{R}}\frac{\sqrt{(z+x_2y_2)^2+y_2^4}}{\sqrt{(z+x_1y_1)^2+y_1^4}}\right) \tag{3.1}$$

We will find this limit explicitly. We need a result from [1]. Let $M : \mathbb{H}\times\mathbb{H}\to\mathbb{R}$ be the function defined by

$$M(\zeta,\zeta') = \sup_{z\in\mathbb{R}}\frac{|\zeta'-z|}{|\zeta-z|}.$$

Lemma 3.1 (Proposition 3, [1])**.**

$$M(\zeta,\zeta') = \frac{|\zeta'-\bar{\zeta}|+|\zeta'-\zeta|}{|\zeta-\bar{\zeta}|}. \tag{3.2}$$

Theorem 3.1.

$$\kappa(\zeta_1,\zeta_2) = \log\left(\frac{\sqrt{(x_1y_1-x_2y_2)^2+(y_1^2+y_2^2)^2}+\sqrt{(x_1y_1-x_2y_2)^2+(y_1^2-y_2^2)^2}}{2y_1y_2}\right) \tag{3.3}$$

Proof. Equation 3.1 indicates that

$$\kappa(\zeta_1,\zeta_2) = \log\left(\frac{y_1}{y_2}\sup_{z\in\mathbb{R}}\frac{\sqrt{(z+x_2y_2)^2+y_2^4}}{\sqrt{(z+x_1y_1)^2+y_1^4}}\right)$$

By the definition of M, we have

$$\kappa(\zeta_1,\zeta_2) = \log\frac{y_1}{y_2}M(\zeta,\zeta'),$$

where $\zeta = -x_1y_1+\imath y_1^2$ and $\zeta' = -x_2y_2+\imath y_2^2$. By Equation 3.2

$$\begin{aligned}\kappa(\zeta_1,\zeta_2) &= \log\frac{y_1}{y_2}M(\zeta,\zeta'),\\ &\log\left(\frac{y_1}{y_2}\frac{|\zeta'-\bar{\zeta}|+|\zeta'-\zeta|}{|\zeta-\bar{\zeta}|}\right)\\ &= \log\left(\frac{y_1}{y_2}\frac{\sqrt{(x_1y_1-x_2y_2)^2+(y_1^2+y_2^2)^2}+\sqrt{(x_1y_1-x_2y_2)^2+(y_1^2-y_2^2)^2}}{2y_1^2}\right)\\ &= \log\left(\frac{\sqrt{(x_1y_1-x_2y_2)^2+(y_1^2+y_2^2)^2}+\sqrt{(x_1y_1-x_2y_2)^2+(y_1^2-y_2^2)^2}}{2y_1y_2}\right)\end{aligned}$$

□

3.1. *κ and $d_{\mathbb{H}}$*

Now let us consider the hyperbolic metric on $\mathbb{H}$. It is induced from the Riemannian metric $ds^2 = \frac{dx^2+dy^2}{y^2}$, and we denote it by $d_{\mathbb{H}}$. Let $w_1 = u_1+\imath v_1, w_2 = u_2+\imath v_2\in\mathbb{H}$. It is well known that

$$d_{\mathbb{H}}(w_1,w_2) = 2\log\left(\frac{\sqrt{(u_2-u_1)^2+(v_2-v_1)^2}+\sqrt{(u_2-u_1)^2+(v_2+v_1)^2}}{2\sqrt{v_1v_2}}\right)$$

Theorem 3.2. *The weak metric space $(\mathbb{T}(T^2), 2\kappa)$ and the metric space $(\mathbb{H}, d_{\mathbb{H}})$ are isometric.*

Proof. We identify $\mathbb{T}(T^2)$ with $\mathbb{H}$. We claim that the map $\mathcal{A} : (\mathbb{H}, 2\kappa)\to(\mathbb{H}, d_{\mathbb{H}})$ given by

$$A(x+\imath y) = xy+\imath y^2$$

is an isometry. Let $\zeta_1 = x_1+\imath y_1, \zeta_2 = x_2+\imath y_2\in\mathbb{H}$. Let $w_1 = \mathcal{A}(\zeta_1) = x_1y_1+\imath y_1^2$ and $w_2 = \mathcal{A}(\zeta_2) = x_2y_2+\imath y_2^2$. Then

$$
\begin{aligned}
&d_{\mathbb{H}}(\mathcal{A}(\zeta_1), \mathcal{A}(\zeta_2)) \\
= 2\log\left(\frac{\sqrt{(x_2y_2 - x_1y_1)^2 + (y_2^2 - y_1^2)} + \sqrt{(x_2y_2 - x_1y_1)^2 + (y_1^2 + y_2^2)}}{2\sqrt{y_1^2y_2^2}}\right) \\
&= 2\kappa(\zeta_1, \zeta_2).
\end{aligned}
$$

□

4. The metric λ

In this section we prove that $\lambda = \kappa$. We know that $\kappa \leq \lambda$. Thus it remains to show that $\lambda \leq \kappa$. Given (S_1, g_1, ψ_1) and (S_2, g_2, ψ_2), we will find a marking preserving homeomorphism ϕ between S_1 and S_2 so that $\log \mathrm{Lip}(\phi) = \kappa(g_1, g_2)$. Then it will follow from the definition of λ that $\lambda \leq \kappa$. Now we proceed.

Let $\zeta_1 = x_1 + \imath y_1$ and $\zeta_2 = x_2 + \imath y_2$ be the points in $\mathbb{H}$ which correspond to (S_1, g_1, ψ_1) and (S_2, g_2, ψ_2), respectively. Let $\tilde{\phi} : \mathbb{C} \to \mathbb{C}$ be the affine map such that

$$\tilde{\phi}(0) = \tilde{\phi}(0),\ \tilde{\phi}(\frac{1}{y_1}) = \tilde{\phi}(\frac{1}{y_2}),\ \tilde{\phi}(\zeta_1) = \tilde{\phi}(\zeta_2).$$

Clearly, $\tilde{\phi}$ induces a map

$$\phi : \mathbb{C}/y_1\mathbb{Z} + (x_1 + \imath y_1)\mathbb{Z} \to \mathbb{C}/y_1\mathbb{Z} + (x_2 + \imath y_2)\mathbb{Z}$$

and $\mathrm{Lip}(\phi) = \mathrm{Lip}(\tilde{\phi})$. We claim that $\exp(\mathrm{Lip}(\tilde{\phi})) = \kappa(\zeta_1, \zeta_2)$.

Observe that the map $\tilde{\phi}$ has the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{y_1}{y_2} & \frac{x_2}{y_1} - \frac{x_1}{y_2} \\ 0 & \frac{y_2}{y_1} \end{pmatrix}$$

Using the complex notation, we see that $\tilde{\phi}(z) = \alpha z + \beta \bar{z}$, where

$$\alpha = \frac{a + \imath c - (b + \imath d)\imath}{2} = \frac{\frac{y_1}{y_2} + \frac{y_2}{y_1} - \imath(\frac{x_2}{y_1} - \frac{x_1}{y_2})}{2}$$

$$\beta = \frac{a + \imath c + (b + \imath d)\imath}{2} = \frac{\frac{y_1}{y_2} - \frac{y_2}{y_1} + \imath(\frac{x_2}{y_1} - \frac{x_1}{y_2})}{2}$$

Now we need a result from [1].

Lemma 4.1 (Lemma 4, [1]). *A map of the form $f(z) = pz + q\bar{z}$, $p, q \in \mathbb{C}$, has Lipschitz constant*

$$\mathrm{Lip}(f) = |p| + |q|.$$

Now one can use the above lemma to show that $\mathrm{Lip}(\tilde{\phi}) = \exp \kappa(\zeta_1, \zeta_2)$. Thus we have proved the following theorem.

Theorem 4.1. $\lambda = \kappa$.

The inequality $\kappa \leq \lambda$ and the fact that κ separates points imply that λ separates points. Now we prove directly from the definition of λ that it separates points.

Proposition 4.1. *λ separates points of the Teichmüller space of the torus.*

Proof. The proof of this proposition is similar to the proof of the Proposition 2.1 in [6]. It uses the fact that the areas of the surfaces that we consider are the same. Assume that g and h are in $\mathbb{T}(T^2)$ so that $\lambda(g, h) \leq 0$. Pick a continuous map $T^2 \to T^2$ with global Lipschitz constant equal to $\lambda' = \exp \lambda(g, h)$. This follows from the fact that T^2 is compact Lipschitz continuous maps form an equicontinuous family. Since the Lipschitz constant of the map is less than or equal to 1, it follows that every small closed disk is sent to a disk of the same radius. Since the areas of the surfaces are equal, each disk is mapped to a disk of the same size, hence the map is an isometry. □

Now assume that between (T^2, g) and (T^2, h) such that $\mathrm{Lip}(f) = 1$. The argument in the previous proposition implies that f is an isometry. Thus we have the following result.

Proposition 4.2. *Let f be a homeomorphism between (T^2, g) and (T^2, h). Then Lip$(f) = 1$ if and only if f is an isometry.*

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References

[1] Belkhirat, A., Papadopoulos, A., Troyanov, M.: *Thurston's weak metric on the Teichmüller space of the torus*. Transactions of the American Mathematical Society, **357(8)**, 3311-3324 (2005).
[2] Busemann, H.: *Local metric geometry*, Trans. Amer. Math. Soc. **56** 200-274 (1944).
[3] Greenfield, M., Ji, L.: *Metrics and compactifications of Teichmüller spaces of flat tori*. Preprint arXiv:1903.10655. (2019).
[4] Greenfield, M.: *A new modular characterization of the hyperbolic plane*. Preprint arXiv:1707.00818. (2017).
[5] Sağlam, İ.: *Complete flat cone metrics on punctured surfaces*. Turkish Journal of Mathematics; **43**, 813-832 (2019).
[6] Thurston, W.P.: *Minimal Stretch maps between hyperbolic surfaces*. Preprint arxiv:9801039. (1985).
[7] Troyanov, M.: *Les surfaces euclidienne a singularites coniques*, L'Enseign. Math. **32** 79–94 (1986).
[8] Troyanov, M.: *On the moduli space of singular Euclidean surfaces*. In: Handbook of Teichmüller Theory, Vol. 1 (European Mathematical Society). 507–540 (2007).

Affiliations

İSMAIL SAĞLAM
ADDRESS: Adana Alparslan Turkes Science and Technology University University, Dept. of Aerospace Engineering Adana-Turkey.
E-MAIL: isaglamtrfr@gmail.com
ORCID ID:0000-0002-1283-6396