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Stability in Retarded Functional Equations

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Abstract

This article deals with the stability behavior of a scalar linear retarded equation. Useful exponential estimations and stability criteria of the solutions were established. Finally, two examples are given for the stability of the zero solution of the retarded equation.

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1. Introduction

The this study, we investigate the stability behavior of all solutions of the linear difference retarded functional equation

$$x(s)=\int_{-1}^{0}x(s-u(\alpha))dq(\alpha), \quad s\geq 0 \tag{1.1}$$

in equation (1.1), $x(s)\in\mathbb{R}$, $q(\alpha)$ is a increasing continuous real-valued function on $[-1,0]$. Also, $u(\alpha)$ is a positive continuous real-valued function on $[-1,0]$ and the integral is Riemann-Stieltjes integral.
Considering the value $\|u\|=\max\{u(\alpha):-1\leq\alpha\leq 0\}$, by a solution of equation (1.1) it is meant a continuous function $x:[-\|u\|,\infty)\to\mathbb{R}$ that satisfies (1.1) for every $s\geq 0$. Jointly the retarded equation (1.1), it is customary to identify an *initial condition* of the form

$$x(s)=\varphi(s), \quad -\|u\|\leq s\leq 0 \tag{1.2}$$

in equation (1.2), we give the initial function φ that is a continuous real-valued function on the initial interval $[-\|u\|,0]$.
If we examine the literature (see for example [1]), we can see the most common general linear retarded functional equation as

$$x(s)=\int_{-u}^{0}x(s+\alpha)d\eta(\alpha),$$

in this equation $u\in\mathbb{R}^{+}$ and η is a function of bounded variation on atomic at zero $[-1,0]$. In the literature, it is known that this equation is the most common general linear delay functional equation. Our preference for equation (1.1) takes into account the possibility of understanding delays more clearly on the role of functional retarded equations in stability behaviors.
Ferreira and Pinelas [2-5] have established the oscillatory criteria for the retarded functional equations of form (1.1).
The stability theory of delay equations has been of great interest in the last three decades, as seen in the textbooks [1, 6-11] and the references therein. However, there may be some gaps in the literature on functional retarded equations, except for certain results that can be obtained through some studies on discrete difference systems and neutral type differential systems. For example, as a mathematical model, [12] and [13] can examine the issues given for vintage human capital issues. In addition, they can look at the article [14].
This article deals with the stability of linear retarded equations. An exponential boundary and a stability criterion of the solutions were obtained. We obtain our results by using a real root corresponded to the solution of characteristic equation. The application of our results to special delay equations leads to an improved version of some of the results given by Driver in [15] and also by Pituk in [16]. In order to obtain our results, we use the techniques which is a combination of the methods used in [17-20].
If $x(s)=e^{\lambda s}$ for $s\in\mathbb{R}$ is obtained as a solution oation (1.1), it can be seen that λ will be a root of the *characteristic equation*

$$1=\int_{-1}^{0}e^{-\lambda u(\alpha)}dq(\alpha). \tag{1.3}$$

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In this section, some definitions will be given (see, for example, [1]). The trivial solution of (1.1) is said to be *stable* if for every $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that, for any initial function φ with

$$\|\varphi\| = \max_{-\|u\| \le s \le 0} |\varphi(s)| < \delta$$

the solution x of (1.1)-(1.2) satisfies

$$\|x(s)\| < \varepsilon, \quad \text{for all} \quad s \in [-\|u\|, \infty).$$

Otherwise, the trivial solution of (1.1) is said to be *unstable*. Furthermore, the trivial solution of (1.1) is called *asymptotically stable* if it is stable in the above sense and in addition there exists a number $\ell_0 > 0$ such that, for any initial function φ with $\|\varphi\| < \ell_0$, the solution x of (1.1)-(1.2) satisfies

$$\lim_{s \to \infty} x(s) = 0.$$

2. Main Results

Now, it will be given a lemma, that plays a very important role in obtaining main results.

Lemma 2.1. *Let the above hypotheses on $u(\alpha)$ and $q(\alpha)$ functions hold. Then, in the interval $(-\infty, \infty)$, the characteristic equation (1.3) has a unique real root.*

Proof. Set

$$F(\lambda) = 1 - \int_{-1}^{0} e^{-\lambda u(\alpha)} dq(\alpha) \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

Then, it is easy to show that $F(-\infty) = -\infty$, $F(\infty) = 1$. Furthermore, for all $\lambda \in \mathbb{R}$

$$F'(\lambda) = \int_{-1}^{0} u(\alpha) e^{-\lambda u(\alpha)} dq(\alpha) > 0.$$

So, F is strictly increasing on $(-\infty, \infty)$ and hence, there exists a unique real root λ_0 of the equation $F(\lambda) = 0$ in this interval. The proof is now complete. □

In this section, we obtain Theorem 2.2 below and its stability criteria.

Theorem 2.2. *Let λ_0 be a real root of the characteristic equation (1.3). Then, there exists a number $N(\lambda_0; \varphi)$ such that, for any*

$$\varphi \in C([-\|u\|, 0], \mathbb{R}) \quad \text{with} \quad \max_{-\|u\| \le s \le 0} |e^{-\lambda s} \varphi(s)| < N(\lambda_0; \varphi),$$

the solution x of (1.1)-(1.2) satisfies

$$|x(s)| < N(\lambda_0; \varphi) e^{\lambda_0 s}, \quad \text{for all} \quad s \ge -\|u\|. \tag{2.1}$$

Furthermore, the trivial solution of (1.1) is stable if $\lambda_0 = 0$ and asymptotically stable if $\lambda_0 < 0$.

Proof. Let now x be the solution of (1.1)-(1.2). Define

$$y(s) = e^{-\lambda_0 s} x(s) \quad \text{for} \quad s \in [-\|u\|, \infty).$$

Then, for every $s \ge 0$, we have

$$y(s) = \int_{-1}^{0} e^{-\lambda_0 u(\alpha)} y(s - u(\alpha)) dq(\alpha). \tag{2.2}$$

Furthermore, the initial condition (1.2) can be equivalently written

$$y(s) = e^{-\lambda_0 s} \varphi(s), \quad -\|u\| \le s \le 0. \tag{2.3}$$

Then, in view of (2.3), we have

$$|y(s)| < N(\lambda_0; \varphi), \quad -\|u\| \le s \le 0. \tag{2.4}$$

It will be shown that $N(\lambda_0; \varphi)$ is a bound of y on the whole interval $[-\|u\|, \infty)$, namely

$$|y(s)| < N(\lambda_0; \varphi), \quad \text{for all} \quad s \in [-\|u\|, \infty). \tag{2.5}$$

Otherwise, by (2.4), there exists a $s_0 > 0$ such that

$$|y(s)| < N(\lambda_0; \varphi), \quad \text{for} \quad -\|u\| \le s < s_0, \quad \text{and} \quad |y(s_0)| = N(\lambda_0; \varphi).$$

Then, using (1.3), from (2.2), it is obtained

$$\begin{aligned} N(\lambda_0;\varphi) = |y(s_0)| &= |\int_{-1}^{0} e^{-\lambda_0 u(\alpha)} y(s_0 - u(\alpha)) dq(\alpha)| \\ &\leq \int_{-1}^{0} e^{-\lambda_0 u(\alpha)} |y(s_0 - u(\alpha))| dq(\alpha) \\ &< N(\lambda_0;\varphi) \int_{-1}^{0} e^{-\lambda_0 u(\alpha)} dq(\alpha) = N(\lambda_0;\varphi). \end{aligned}$$

which leads to a contradiction. Thus, our claim is true. That is, it follows that (2.5) is always satisfied. As a result, take into account the definition of y, it is obtained

$$|x(s)| < N(\lambda_0;\varphi) e^{\lambda_0 s}, \quad \text{for all} \quad s \geq -\|u\|.$$

So, the proof of the first part of this theorem is completed. Now, the second part (stability criteria) of the theorem needs to prove.
Suppose that $\lambda_0 = 0$ and let $\varphi \in C([-\|u\|,0],\mathbb{R})$ be an any initial function and let x be the solution of (1.1)-(1.2). Then (2.1) is provided and from here

$$|x(s)| < N(\lambda_0;\varphi) \quad \text{for all} \quad s \geq -\|u\|.$$

For any $\varepsilon > 0$, it is choosen $\delta = \varepsilon$ as $N(\lambda_0;\varphi) < \delta$, it is obtained that $\|\varphi\| < \delta$. After then

$$|x(s)| < N(\lambda_0;\varphi) < \delta = \varepsilon.$$

From the last inequality, we obtain that the trivial solution of (1.1)-(1.2) is stable. Furthermore, if $\lambda_0 < 0$, then again from the Theorem 2.2, it is concluded that the above inequality for each $\varphi \in C([-\|u\|,0],\mathbb{R})$ is true, i.e.

$$|x(st)| < N(\lambda_0;\varphi) \quad \text{for all} \quad s \geq -\|u\|.$$

So, as above, it was concluded that the trivial solution of (1.1)-(1.2) is stable. Besides, the inequality (2.1) guarantees that

$$\lim_{s\to\infty} x(s) = 0.$$

Thus, for $\lambda_0 < 0$ the trivial solution of (1.1)-(1.2) is asymptotically stable. We proved the second part of the theorem. Hence, the proof of Theorem 2.2 is completed.

□

3. Examples

It will be applied the stability criteria of the Theorem 2.2 in the following examples.
Example 3.1. Take into account the equation (1.1) for $u(\alpha) = 2-\alpha$ and $q(\alpha) = -\alpha^2$. In this example, it is applied the characteristic equations (1.3). That is,

$$1 = \int_{-1}^{0} e^{-\lambda(2-\alpha)} d(-\alpha^2)$$

or

$$1 = -2\int_{-1}^{0} e^{\lambda(\alpha-2)} \alpha d(\alpha). \tag{3.1}$$

It can be seen easily that $\lambda = 0$ is a root of (3.1). Therefore, the zero solution of (1.1) is stable.
Example 3.2. Take into account the equation (1.1) for $u(\alpha) = 1-\frac{\alpha^2}{2}$ and $q(\alpha) = -\frac{\alpha^2}{2(e-\sqrt{e})}$. In this example, it is applied the characteristic equations (1.3). That is,

$$1 = \int_{-1}^{0} e^{-\lambda(1-\frac{\alpha^2}{2})} d(-\frac{\alpha^2}{2(e-\sqrt{e})}) = -\frac{1}{(e-\sqrt{e})}\int_{-1}^{0} e^{\lambda(\frac{\alpha^2-2}{2})} \alpha d\alpha$$

or

$$e-\sqrt{e} = \lambda^{-1}(e^{-\frac{\lambda}{2}} - e^{-\lambda}). \tag{3.2}$$

We obtain that $\lambda = -1$ is a root of (3.2). So, the zero solution of (1.1) is asymptotically stable.

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