DIFFERENTIAL TRANSFORM METHOD FOR SOLVING A CLASS OF SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract: This paper applies the differential transform method to search for semi numerical-analytical solutions of a class of singular two-point boundary value problems. To make clear and illustrate the features and capabilities of the presented method, four examples are carried out. The results demonstrate the computational efficiency and reliability of the method.

Keywords: Singular point, Singular two-point boundary value problems, Differential transform method

Mathematics Subject Classifications (2000): 65L10, 34B05

BİR TÜR TEKİL İKİ-NOKTA SINIR DEĞER PROBLEMLERİNİN DİFERANSİYEL DÖNÜŞÜM METODU İLE ÇÖZÜMÜ

Özet: Bu makale, bir tür tekil iki-nokta sınır değer problemlerinin yarı sayısal-analitik çözümlerinin diferansiyel dönüşüm yöntemi ile elde edilişini ele almaktadır. Yöntemin özelliklerini açığa vurmak ve işlerliğini örneklendirmek için dört örnek ele alındı. Elde edilen sonuçlar, metodun hesaplanabilirlik açısından güvenilirliğini ve etkinliğini ortaya koymaktadır.

Anahtar Kelimeler: Tekil nokta, Tekil iki-nokta sınır değer problemleri, Diferansiyel dönüşüm yöntemi

1. INTRODUCTION

In the present paper, differential transform method (ZHOU 1986) is used to solve singular two-point boundary value problems as in the following form which are assumed to have a unique solution in the interval of integration (AGARWAL 1986).

$$y''(x) + \frac{k}{x}y'(x) + b(x)y(x) = c(x), \quad 0 < x < 1, \tag{1}$$

with boundary conditions

$$y'(0) = \alpha, \quad y(1) = \beta. \tag{2}$$

Problems of this form are encountered in the study of electrohydrodynamics and the theory of thermal explosions.

Singular boundary value problems have been studied by several authors (CHAWLA & KATTI 1984, RAVI KANTH & REDDY 2003, GUSTAFSSON 1973, JAMET 1970). In (ÇAĞLAR N & ÇAĞLAR H 2006), they have considered the use of B-splines solutions for solving these type of problems. In (RAVI KANTH & REDDY 2005), the authors have discussed cubic splines for such problems. A higher order finite difference method has been described by the same authors for the problems considered in the present paper in (RAVI KANTH & REDDY 2004). There are also some studies in the literature about such problems (ERIKSON & THOMEE 1984, IYENGAR & JAIN 1987, RUSSELL & SHAMPINE 1975, ASCHER et al.1998).

In the present work, we introduce differential transform method for solving (1) and (2). The differential transform is a semi numerical-analytical method for solving differential equations. The concept of differential transform was introduced by Zhou (ZHOU 1986), who solved linear and non-linear initial value problems in electric circuit analysis. The differential transform method gives an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method takes computationally long time for large orders. The present method reduces the size of computational domain and is applicable to many problems easily.

The paper is organized as follows. In the differential transform method section of this paper , the definitions and operations of differential transform are introduced. In the numerical results section of our paper, we present four examples to illustrate the efficiency and simplicity of the method. Conclusions will be presented finally.

2. DIFFERENTIAL TRANSFORM METHOD

Differential transform of function $y(x)$ is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0}. \tag{3}$$

In Eq. (3), $y(x)$ is the original function and $Y(k)$ is the transformed function, which is called the T-function. Differential inverse transform of $Y(k)$ is defined as

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x-x_0)^k. \tag{4}$$

From Eqs. (3) and (4), we obtain

$$y(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0}. \tag{5}$$

Eq. (5) implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which are described by the transformed equations of the original function. In this study, we use the lower case letter to represent the original function and upper case letter to represent the transformed function.

In actual applications, function $y(x)$ is expressed by a finite series and Eq. (4) can be written as

$$y(x) = \sum_{k=0}^{n} Y(k)(x - x_0)^k. \tag{6}$$

and Eq. (6) implies that $\sum_{k=n+1}^{\infty} Y(k)(x - x_0)^k$ is negligibly small. In fact, n is decided by the convergence of natural frequency in this study.

From the definitions (3) and (4), it is easily proved that the transformed functions comply with the basic mathematics operations shown in Table 1.

Table 1 The fundamental operations of differential transform method

Original function	Transformed function
$y(x) = u(x) \pm v(x)$	$Y(k) = U(k) \pm V(k)$
$y(x) = cw(x)$	$Y(k) = cW(k)$
$y(x) = dw(x)/dx$	$Y(k) = (k+1)\, W(k+1)$
$y(x) = d^j w(x)/dx^j$	$Y(k) = (k+1)\,(k+2)\ldots(k+j)W(k+j)$
$y(x) = u(x)v(x)$	$Y(k) = \sum_{r=0}^{k} U(r)V(k-r)$
$y(x) = x^j$	$Y(k) = \delta(k-j) = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$

3. NUMERICAL RESULTS

In this section, we have presented the numerical experiments with four problems, which have been widely discussed in the literature.

Example 1. Consider the following boundary value problem (RAVI KANTH & REDDY 2005, ÇAĞLAR N & ÇAĞLAR H 2006, IYENGAR & JAIN 1987)

$$y''(x) + \frac{1}{x} y'(x) + y(x) = 0, \quad 0 < x < 1, \tag{7}$$

with boundary conditions

$$y'(0) = 0, \quad y(1) = 1. \tag{8}$$

The exact solution for this problem is

$$y(x) = \frac{J_0(x)}{J_0(1)}. \tag{9}$$

where, $J_0(x)$ is Bessel function of order zero.

Taking differential transform of (7) and using the fundamental operations of differential transform method, we obtain the following recurrence relation:

$$Y(k+1) = -\frac{1}{(k+1)^2} \sum_{l=0}^{k} \delta(l-1)Y(k-l). \tag{10}$$

By using Eqs. (3) and (8), the boundary conditions at $x_0 = 0$ can be transformed as follows:

$$Y(1) = 0, \quad \sum_{k=0}^{n} Y(k) = 1. \tag{11}$$

Using Eqs. (10) and (11), and by taking $n = 4$, the following linear equation is obtained:

$$\frac{49a}{64} = 1, \tag{12}$$

where $a = y(0) = Y(0)$ is the missing boundary condition. Solving (12) and using the inverse transform rule in Eq. (6), we get

$$y(x) = 1.306122 - 0.326531x^2 + 0.020408x^4 + O(x^6). \tag{13}$$

By continuing the same procedure, for $n = 8$, we get the following series solution

$$y(x) = 1.306852 - 0.326713x^2 + 0.020420x^4$$
$$- 0.000567x^6 + 0.000009x^8 + O(x^{10}). \tag{14}$$

It is clear that in the the limit case $n \to \infty$, the series solution obtained by differential transform method converges to the series expansion of the exact solution (9).

Example 2. Consider the boundary value problem (RAVI KANTH & REDDY 2005, ÇAĞLAR N & ÇAĞLAR H 2006, RUSSELL & SHAMPINE 1975)

$$y''(x) + \frac{2}{x} y'(x) - 4y(x) = -2, \quad 0 < x < 1, \tag{15}$$

subject to the boundary conditions

$$y'(0) = 0, \quad y(1) = 5.5. \tag{16}$$

The exact solution for this problem is

$$y(x) = 0.5 + \frac{5 \sinh 2x}{x \sinh 2}. \tag{17}$$

One can see that the differential transform of Eq. (15) can be evaluated by using the fundamental operations of differential transform method as follows:

$$Y(k+1) = \frac{1}{(k+1)(k+2)} \left(4 \sum_{l=0}^{k} \delta(l-1)Y(k-l) - 2\delta(k-1) \right). \tag{18}$$

and we apply the differential transform at $x_0 = 0$, therefore, the boundary conditions are transformed as follows:

$$Y(1) = 0, \quad \sum_{k=0}^{n} Y(k) = 5.5. \tag{19}$$

Using Eq. (18) and (19), and by taking $n = 6$, the following linear equation is obtained:

$$-\frac{128}{315} + \frac{571a}{315} = 5.5, \tag{20}$$

where $a = y(0) = Y(0)$ is the missing boundary condition. Solving (20) and using the inverse transform rule in Eq. (6), we get the following series solution:

$$y(x) = 3.258319 + 1.838879x^2 + 0.367776x^4 + 0.035026x^6 + O(x^8). \qquad (21)$$

Taking $n = 12$ and applying the same procedure, we get

$$y(x) = 3.257206 + 1.838137x^2 + 0.367627x^4 + 0.035012x^6$$
$$+ 0.001945x^8 + 0.000071x^{10} + 0.000002x^{12} + O(x^{14}). \qquad (22)$$

As the number of terms involved increases, one can observe that the series solution obtained by differential transform method converges to the series expansion of the exact solution (17).

Example 3. Consider the boundary value problem (RAVI KANTH & REDDY 2005, ÇAĞLAR N & ÇAĞLAR H 2006, ERIKSON & THOMEE 1984)

$$-y''(x) - \frac{2}{x}y'(x) + (1 - x^2)y(x) = x^4 - 2x^2 + 7, \quad 0 < x < 1, \qquad (23)$$

with boundary conditions

$$y'(0) = 0, \quad y(1) = 0. \qquad (24)$$

The exact solution for this problem is

$$y(x) = 1 - x^2. \qquad (25)$$

Applying the fundamental operations of differential transform method, Eq. (23) is transformed as follows:

$$Y(k+1) = \frac{1}{(k+1)(k+2)} \times$$
$$\left(\sum_{l=0}^{k} \delta(l-1)Y(k-l) - \sum_{l=0}^{k} \delta(l-3)Y(k-l) - \delta(k-5) + 2\delta(k-3) - 7\delta(k-1) \right). \qquad (26)$$

By choosing $x_0 = 0$, the boundary conditions in Eq. (24) transform to

$$Y(1) = 0, \quad \sum_{k=0}^{n} Y(k) = 0. \qquad (27)$$

Using Eqs. (26) and (27), for $n = 2$, the following linear equation is obtained:

$$-\frac{7}{6} + \frac{7a}{6} = 0, \qquad (28)$$

where $a = y(0) = Y(0)$ is the missing boundary condition. Solving Eq. (28) and using the inverse transform rule in Eq. (6), we get the following series solution:

$$y(x) = 1 - x^2. \qquad (29)$$

For $n > 2$, note that one evaluates the same solution, which is the exact solution (25).

Example 4. Consider the following boundary value problem ((RAVI KANTH & REDDY, 2005), (ÇAĞLAR N & ÇAĞLAR H 2006) and (ASCHER et al., 1988))

$$y''(x) + \frac{1}{x}y'(x) = \left(\frac{8}{8 - x^2} \right)^2, \quad 0 < x < 1, \qquad (30)$$

with boundary conditions

$$y'(0) = 0, \quad y(1) = 0. \tag{31}$$

The exact solution for this problem is

$$y(x) = 2\log\frac{7}{8-x^2}. \tag{32}$$

Using the fundamental operations of differential transform method and taking differential transform of Eq. (30), we obtain

$$Y(k) = \frac{1}{k^2} \cdot \begin{cases} \dfrac{\dfrac{k}{2}}{2^{\frac{3k}{2}-3}}, & k = \text{even}, \\ 0, & k = \text{odd}. \end{cases} \tag{33}$$

The boundary conditions in Eq. (31) can be transformed at $x_0 = 0$ as

$$Y(1) = 0, \quad \sum_{k=0}^{n} Y(k) = 0. \tag{34}$$

Choosing $n = 6$ and using Eqs. (33) and (34), we get the following linear equation

$$\frac{205}{768} + a = 0. \tag{35}$$

Solving Eq.(35) for a, then appliying inverse transform rule in Eq.(6), we get the following series solution:

$$y(x) = -0.266927 + 0.250000x^2 + 0.015625x^4 + 0.001302x^6 + O(x^8). \tag{36}$$

Repeating the same procedure, for $n = 12$, the following serşies solution is obtained:

$$y(x) = -0.267063 + 0.250000x^2 + 0.015625x^4 + 0.001302x^6$$
$$+ 0.000122x^8 + 0.000012x^{10} + 0.000001x^{12} + O(x^{14}). \tag{37}$$

It is clearly seen that as $n \to \infty$ the series solution obtained by differential transform method converges to the series expansion of the exact solution (32).

4. CONCLUSIONS

In this study, we implemented differential transform method to the solution of the singular two-point boundary value problems. Numerical examples that are commonly encountered in the literature are carried out. It is observed that differential transform method is an effective and reliable tool for the solution of the singular boundary value problems considered in the present paper.

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