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DUAL TIMELIKE NORMAL AND DUAL TIMELIKE SPHERICAL CURVES IN DUAL MINKOWSKI SPACE D_1^3

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Abstract: In this paper, we give characterizations of dual timelike normal and dual timelike spherical curves in the dual Minkowski 3-space D_1^3 and we show that every dual timelike normal curve is also a dual timelike spherical curve.

Keywords: Normal curves, Dual Minkowski 3-Space, Dual Timelike curves.

Mathematics Subject Classifications (2000): 53C50, 53C40.

D_1^3 DUAL MINKOWSKI UZAYINDA DUAL TIMELIKE NORMAL VE DUAL TIMELIKE KÜRESEL EĞRİLER

Özet: Bu çalışmada, D_1^3 dual Minkowski 3-uzayında dual timelike normal ve dual timelike küresel eğrilerin karakterizasyonları verildi ve her dual timelike normal eğrinin aynı zamanda bir dual timelike küresel eğri olduğu gösterildi.

Anahtar Kelimeler: Normal eğriler, Dual Minkowski 3-uzayı, Dual timelike eğriler.

1. INTRODUCTION

In the Euclidean space E^3, to each regular unit speed curve $\alpha : \mathrm{I} \subset IR \to E^3$, with at least four continuous derivatives, it is possible to associate three mutually orthogonal unit vector fields T, N and B, called respectively the tangent, the principal normal and the binormal vector fields. The planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are known as the osculating plane, the rectifying plane and the normal plane, respectively. The curves $\alpha : \mathrm{I} \subset IR \to E^3$ for which the position vectors α always lie in their rectifying plane, are for simplicity called *rectifying curves* (EKMEKÇİ&İLARSLAN 1998). The characterizations of rectifying curves in Minkowski 3-space are given in (İLARSLAN v.d. 2003). Similarly, the curves for which the position vector α always lies in their osculating plane are for simplicity called *osculating curves;* and finally, the curves for which the position vector α always lies in their normal plane are for simplicity called *normal curves.* By definition, for a normal curve, the position vector α satisfies

$$\alpha(s) = \lambda(s)N(s) + \mu(s)B(s)$$

for some differentiable functions $\lambda(s)$ and $\mu(s)$ (İLARSLAN 2005). Spacelike normal curves are given in (İLARSLAN 2005).

In this study, we will give characterizations of dual timelike normal and dual timelike spherical curves in the dual Minkowski 3-space D_1^3.

2. PRELIMINARIES

Minkowski space-time E_1^3 is an Euclidean space E^3 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system in E_1^3. Since g is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three causal characters; it can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null(lightlike) if $g(v,v) = 0$ and $v \neq 0$.

Dual numbers had been introduced by W. K. Clifford (1845-1879). A *dual number* has the form $\bar{a} = a + \varepsilon a^*$ where a and a^* are real numbers and ε is dual unit with $\varepsilon^2 = 0$. We denote the set of dual numbers by *D:*

$$D = \left\{ \bar{a} = a + \varepsilon a^* : a, a^* \in IR,\ \varepsilon^2 = 0 \right\}.$$

Now let *f* be a differentiable function. Then the Maclaurin series generated by *f* is given by

$$f(\bar{x}) = f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x),$$

where $f'(x)$ is derivative of $f(x)$. Then we have

$$\sin(x + \varepsilon x^*) = \sin(x) + \varepsilon x^* \cos(x),$$
$$\cos(x + \varepsilon x^*) = \cos(x) - \varepsilon x^* \sin(x),$$
$$\sinh(x + \varepsilon x^*) = \sinh(x) + \varepsilon x^* \cosh(x),$$
$$\cosh(x + \varepsilon x^*) = \cosh(x) + \varepsilon x^* \sinh(x),$$
$$\sqrt{x + \varepsilon x^*} = \sqrt{x} + \varepsilon \frac{x^*}{2\sqrt{x}},\ \ (x > 0).$$

Let D^3 be the set of all triples of dual numbers, i.e.,

$$D^3 = \left\{ \tilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) : \bar{a}_i \in D,\ i = 1,2,3 \right\}.$$

The elements of D^3 are called as *dual vectors*. A dual vector $\tilde{a}$ may be expressed in the form $\tilde{a} = a + \varepsilon a^*$, where a and a^* are the vectors of IR^3.

The *Lorentzian inner product* of two dual vectors $\tilde{a} = a + \varepsilon a^*$ and $\tilde{b} = b + \varepsilon b^*$ is defined by

$$g(\tilde{a}, \tilde{b}) = g(a,b) + \varepsilon\left(g(a,b^*) + g(a^*,b)\right)$$

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where $g(a,b)$ is the Lorentzian inner product of the vectors *a* and *b* in the Minkowski 3-space E_1^3. Thus, a dual vector $\tilde{v} = v + \varepsilon v^*$ is called *dual spacelike vector* if $g(\tilde{v},\tilde{v}) > 0$ or $\tilde{v} = 0$, *dual timelike vector* if $g(\tilde{v},\tilde{v}) < 0$ and *dual null(lightlike) vector* if $g(\tilde{v},\tilde{v}) = 0$ and $\tilde{v} \neq 0$.

The set of dual timelike, spacelike and lightlike vectors is called *dual Minkowski 3-space* and it is denoted by D_1^3, i.e.

$$D_1^3 = \left\{ \tilde{a} = a + \varepsilon a^* : a, a^* \in E_1^3 \right\}.$$

An arbitrary curve $\tilde{\alpha} = \tilde{\alpha}(s)$ in D_1^3 can locally be dual spacelike, dual timelike or dual null, if all of its velocity vectors $\tilde{\alpha}'(s)$ are, respectively, dual spacelike, dual timelike or dual null. Also, recall that the norm of a dual vector $\tilde{v} = v + \varepsilon v^*$ is given by $\|\tilde{v}\| = \sqrt{|g(\tilde{v},\tilde{v})|}$ where $g(\tilde{v},\tilde{v}) = g(v,v) + 2\varepsilon g(v,v^*)$. Therefore, $\tilde{v}$ is a dual unit vector if $g(\tilde{v},\tilde{v}) = \pm 1 + \varepsilon 0$. Next, vectors $\tilde{v}, \tilde{w}$ in D_1^3 are said to be dual orthogonal if $g(\tilde{v},\tilde{w}) = 0 + \varepsilon 0$. The velocity of the dual curve $\tilde{\alpha}(s)$ is given by $\|\tilde{\alpha}'(s)\|$.

The dual Lorentzian space with center $\tilde{c} = (c_1, c_2, c_3) \in D_1^3$ and radius $\overline{r} \in D$ in dual space-time D_1^3 is dual hyper-quadratic

$$\tilde{S}_1^2(\overline{r}) = \left\{ \tilde{a} = (a_1, a_2, a_3) \in D_1^3 : g(\tilde{a} - \tilde{c}, \tilde{a} - \tilde{c}) = \overline{r}^2 \right\},$$

with dimension 2 and index 1.

Denote by $\left\{ \tilde{T}(s), \tilde{N}(s), \tilde{B}(s) \right\}$ the moving dual Frenet frame along the dual curve $\tilde{\alpha}(s)$ in the dual Minkowski space-time D_1^3. Then $\tilde{T}, \tilde{N}, \tilde{B}$ are the dual tangent, the dual principal normal and the dual binormal vector fields, respectively. Dual spacelike or dual timelike curve $\tilde{\alpha}(s)$ is said to be parametrized by arclength function s, if $\tilde{\alpha}'(s)$ is dual spacelike or dual timelike. In particular, a dual null curve $\tilde{\alpha}(s)$ is said to be parametrized by a pseudo-arclength function s, if $\tilde{\alpha}''(s)$ is unit.

Let $\tilde{\alpha}(s)$ be a dual timelike curve in the dual Minkowski space-time D_1^3 parametrized by arclength function s. Then for the curve $\tilde{\alpha}$ the Frenet formulae are given by

$$\begin{bmatrix} \tilde{T}' \\ \tilde{N}' \\ \tilde{B}' \end{bmatrix} = \begin{bmatrix} 0 & \tilde{k}_1 & 0 \\ \tilde{k}_1 & 0 & \tilde{k}_2 \\ 0 & -\tilde{k}_2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{T} \\ \tilde{N} \\ \tilde{B} \end{bmatrix} \qquad (1)$$

where $g(\tilde{T},\tilde{T}) = -1$, $g(\tilde{N},\tilde{N}) = g(\tilde{B},\tilde{B}) = 1$, $g(\tilde{T},\tilde{N}) = g(\tilde{T},\tilde{B}) = g(\tilde{N},\tilde{B}) = 0$ and the functions $\tilde{k}_1(s) = k_1(s) + \varepsilon k_1^*(s)$ and $\tilde{k}_2(s) = k_2(s) + \varepsilon k_2^*(s)$ are called dual curvature and dual torsion of $\tilde{\alpha}$ respectively (YÜCESAN vd. 2002).

3. THE DUAL TIMELIKE NORMAL CURVES IN D_1^3

In this section, we will give characterize dual timelike normal curves in dual Minkowski space D_1^3. Let now give the definition of dual timelike normal curve:

Definition 3.1. Let $\tilde{\alpha}(s)$ be a unit speed dual timelike curve in D_1^3. $\tilde{\alpha}(s)$ is called dual timelike normal curve if the position vector $\tilde{\alpha}(s)$ satisfies the following condition

$$\tilde{\alpha}(s) = \tilde{\lambda}(s)\tilde{N}(s) + \tilde{\mu}(s)\tilde{B}(s),$$

where $\tilde{\lambda}(s)$ and $\tilde{\mu}(s)$ are dual differentiable functions of pseudo arclength parameter s.

Theorem 3.1. Let $\tilde{\alpha}(s)$ be a unit speed dual timelike normal curve in D_1^3 with curvatures $\tilde{k}_1(s) > 0,\ \tilde{k}_2(s) \neq 0$. Then the following statements hold:

i) The curvatures $\tilde{k}_1(s)$ and $\tilde{k}_2(s)$ satisfy the following equality

$$\frac{1}{\tilde{k}_1(s)} = \overline{c}_1 \cos\left(\int \tilde{k}_2(s)ds\right) + \overline{c}_2 \sin\left(\int \tilde{k}_2(s)ds\right)$$

(ii) The principal normal and binormal components of the position vector of the curve are given respectively by

$$g(\tilde{\alpha}(s), \tilde{N}) = \overline{c}_1 \cos\left(\int \tilde{k}_2(s)ds\right) + \overline{c}_2 \sin\left(\int \tilde{k}_2(s)ds\right)$$

$$g(\tilde{\alpha}(s), \tilde{B}) = -\overline{c}_1 \sin\left(\int \tilde{k}_2(s)ds\right) + \overline{c}_2 \cos\left(\int \tilde{k}_2(s)ds\right),$$

where $\overline{c}_1 = c_1 + \varepsilon c_1^*,\ \overline{c}_2 = c_2 + \varepsilon c_2^* \in D$ and $c_1, c_1^*, c_2, c_2^* \in IR$.

Conversely, if $\tilde{\alpha}(s)$ is a unit speed dual timelike curve in D_1^3 with the curvatures $\tilde{k}_1(s) > 0,\ \tilde{k}_2(s) \neq 0$ and one of the statements (i) and (ii) holds, then $\tilde{\alpha}$ is a normal curve or congruent to a normal curve.

Proof: Assume that $\tilde{\alpha}(s)$ is a unit speed dual timelike curve in D_1^3, where s is pseudo arclength parameter. Then, by Definition 3.1, we have

$$\tilde{\alpha}(s) = \tilde{\lambda}(s)\tilde{N}(s) + \tilde{\mu}(s)\tilde{B}(s),$$

where $\tilde{\lambda}(s)$ and $\tilde{\mu}(s)$ are dual differentiable functions of pseudo arclength parameter s. Differentiating this with respect to s and by applying the Frenet equations (1), we obtain

$$\tilde{\lambda}\tilde{k}_1 = 1,\quad \tilde{\lambda}' - \tilde{\mu}\tilde{k}_2 = 0,\quad \tilde{\lambda}\tilde{k}_2 + \tilde{\mu}' = 0 \qquad (2)$$

From the first and second equations in (2), we get

$$\tilde{\lambda} = \frac{1}{\tilde{k}_1},\ \tilde{\mu} = \frac{1}{\tilde{k}_2}\left(\frac{1}{\tilde{k}_1}\right)'. \qquad (3)$$

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Thus

$$\tilde{\alpha}(s) = \frac{1}{\tilde{k}_1}\tilde{N} + \frac{1}{\tilde{k}_2}\left(\frac{1}{\tilde{k}_1}\right)'\tilde{B} \qquad (4)$$

Further, from the third equation in (2) and using (3) we find the following differential equation

$$\left[\frac{1}{\tilde{k}_2}\left(\frac{1}{\tilde{k}_1}\right)'\right]' + \frac{\tilde{k}_2}{\tilde{k}_1} = 0 \qquad (5)$$

Putting $\tilde{y}(s) = \frac{1}{\tilde{k}_1}$ and $\tilde{z}(s) = \frac{1}{\tilde{k}_2}$, equation (5) can be written as

$$\left(\tilde{z}(s)\tilde{y}'(s)\right)' + \frac{\tilde{y}(s)}{\tilde{z}(s)} = 0 .$$

If we change variables in the above equation as $\tilde{t} = \int \frac{1}{\tilde{z}(s)}ds = \int \tilde{k}_2(s)ds$ then we get

$$\frac{d^2\tilde{y}}{d\tilde{t}^2} + \tilde{y} = 0 .$$

The solution of this differential equation is

$$\tilde{y} = \overline{c}_1 \cos(\tilde{t}) + \overline{c}_2 \sin(\tilde{t})$$

where $\overline{c}_1, \overline{c}_2 \in D$. Therefore

$$\frac{1}{\tilde{k}_1(s)} = \overline{c}_1 \cos\left(\int \tilde{k}_2(s)ds\right) + \overline{c}_2 \sin\left(\int \tilde{k}_2(s)ds\right) \qquad (6)$$

Thus the statement (i) is proved. Next, substituting (6) into (3) and (4) we get

$$\begin{aligned} \tilde{\lambda} &= \overline{c}_1 \cos\left(\int \tilde{k}_2(s)ds\right) + \overline{c}_2 \sin\left(\int \tilde{k}_2(s)ds\right), \\ \tilde{\mu} &= -\overline{c}_1 \sin\left(\int \tilde{k}_2(s)ds\right) + \overline{c}_2 \cos\left(\int \tilde{k}_2(s)ds\right) \end{aligned} \qquad (7)$$

and

$$\tilde{\alpha} = \left[\overline{c}_1 \cos\left(\int \tilde{k}_2(s)ds\right) + \overline{c}_2 \sin\left(\int \tilde{k}_2(s)ds\right)\right]\tilde{N} + \left[-\overline{c}_1 \sin\left(\int \tilde{k}_2(s)ds\right) + \overline{c}_2 \cos\left(\int \tilde{k}_2(s)ds\right)\right]\tilde{B} \quad (8)$$

From (8) we find

$$\begin{aligned} g(\tilde{\alpha},\tilde{\alpha}) &= \overline{c}_1^2 + \overline{c}_2^2 , & (9)\\ g(\tilde{\alpha},\tilde{N}) &= \overline{c}_1 \cos\left(\int \tilde{k}_2(s)ds\right) + \overline{c}_2 \sin\left(\int \tilde{k}_2(s)ds\right), & (10)\\ g(\tilde{\alpha},\tilde{B}) &= -\overline{c}_1 \sin\left(\int \tilde{k}_2(s)ds\right) + \overline{c}_2 \cos\left(\int \tilde{k}_2(s)ds\right). & (11) \end{aligned}$$

Consequently, we have proved (ii).

Conversely, suppose that statement (i) holds. Then we have

$$\frac{1}{\tilde{k}_1(s)} = \overline{c}_1 \cos\left(\int \tilde{k}_2(s)ds\right) + \overline{c}_2 \sin\left(\int \tilde{k}_2(s)ds\right), \qquad \overline{c}_1,\ \overline{c}_2 \in D .$$

Differentiating this with respect to s we get

$$\left[\frac{1}{\tilde{k}_2}\left(\frac{1}{\tilde{k}_1}\right)'\right]' = -\frac{\tilde{k}_2}{\tilde{k}_1}.$$

By applying Frenet equations, we obtain

$$\frac{d}{ds}\left[\tilde{\alpha}(s) - \frac{1}{\tilde{k}_1}\tilde{N} - \frac{1}{\tilde{k}_2}\left(\frac{1}{\tilde{k}_1}\right)'\tilde{B}\right] = 0.$$

Consequently, $\tilde{\alpha}$ is congruent to a normal curve. Next, assume that statement (ii) holds. Then (9) and (10) are satisfied. Differentiating (9) with respect to s and using (10) we find $g(\tilde{\alpha},\tilde{T}) = 0$, which means that $\tilde{\alpha}$ is normal curve, which proves the theorem.

Corollary 3.1. *The real and dual parts of the equation* (6) *are, respectively, given by*

$$\frac{1}{k_1(s)} = c_1 \cos\left(\int k_2(s)ds\right) + c_2 \sin\left(\int k_2(s)ds\right), \qquad c_1, c_2 \in IR,$$

and

$$-\frac{k_1^*(s)}{k_1^2(s)} = \int k_2^*(s)ds\left(-c_1 \sin\left(\int k_2(s)ds\right) + c_2 \cos\left(\int k_2(s)ds\right)\right)$$
$$+ c_1^* \cos\left(\int k_2(s)ds\right) + c_2^* \sin\left(\int k_2(s)ds\right).$$

Similarly, the real and dual parts of the equations (10) *and* (11) *are, respectively, given by*

$$g(\alpha, N) = c_1 \cos\left(\int k_2(s)ds\right) + c_2 \sin\left(\int k_2(s)ds\right)$$
$$g(\alpha, N^*) + g(\alpha^*, N) = \int k_2^*(s)ds\left(-c_1 \sin\left(\int k_2(s)ds\right) + c_2 \cos\left(\int k_2(s)ds\right)\right)$$
$$+ c_1^* \cos\left(\int k_2(s)ds\right) + c_2^* \sin\left(\int k_2(s)ds\right)$$

and

$$g(\alpha, B) = -c_1 \sin\left(\int k_2(s)ds\right) + c_2 \cos\left(\int k_2(s)ds\right)$$
$$g(\alpha, B^*) + g(\alpha^*, B) = -\int k_2^*(s)ds\left(c_1 \cos\left(\int k_2(s)ds\right) + c_2 \sin\left(\int k_2(s)ds\right)\right)$$
$$- c_1^* \sin\left(\int k_2(s)ds\right) + c_2^* \cos\left(\int k_2(s)ds\right)$$

where α, N, B *and* α^*, N^*, B^* *are reel and dual parts of* $\tilde{\alpha}$, $\tilde{N}$ *and* $\tilde{B}$ *respectively. Here, real parts of* (6), (10) *and* (11) *are the conditions for a unit speed timelike curve* $\alpha = \alpha(s)$ *with Frenet frame* $\{T, N, B\}$ *and curvatures* k_1 *and* k_2 *to be a timelike normal curve in Minkowski space-time* E_1^3.

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Also, we see that $g(\alpha, N^*) + g(\alpha^*, N) = -\dfrac{k_1^*}{k_1^2}$, i.e., dual part of equation (6) is equal to dual part of equation (10). So, we can give the following corollary:

Corollary 3.2. *Let $\tilde{\alpha}(s)$ be a unit speed dual timelike curve in D_1^3 with curvatures $\tilde{k}_1(s) > 0,\ \tilde{k}_2(s) \neq 0$. Then $\tilde{\alpha}(s)$ is a dual timelike normal curve if and only if*

$$\frac{1}{\tilde{k}_1(s)} = \frac{1}{k_1(s)} + \varepsilon\Big(g(\alpha(s), N^*(s)) + g(\alpha^*(s), N(s))\Big).$$

Theorem 3.2. Let $\tilde{\alpha} = \tilde{\alpha}(s)$ be a unit speed dual timelike normal curve in D_1^3 with curvatures $\tilde{k}_1(s) > 0,\ \tilde{k}_2(s) \neq 0$. Then there holds

$$\frac{1}{\tilde{k}_1(s)} = \pm\sqrt{\overline{r}^2 - \overline{c}_2^2}\,\cos\Big(\int \tilde{k}_2(s)ds\Big) + \overline{c}_2 \sin\Big(\int \tilde{k}_2(s)ds\Big) \tag{12}$$

Proof: Since $\tilde{\alpha}(s)$ is a dual timelike normal curve the position vector $\tilde{\alpha}$ is spacelike. Then $g(\tilde{\alpha}, \tilde{\alpha}) = \overline{r}^2,\ \ \overline{r} \in D$. Substituting this into (9), we get $\overline{c}_1 = \pm\sqrt{\overline{r}^2 - \overline{c}_2^2}$. By using this last equation and (6) we obtain that (12) holds.

4. DUAL TIMELIKE SPHERICAL CURVES

In this section, we characterize dual timelike curves which lie on dual Lorentzian sphere $\tilde{S}_1^2(\overline{r})$ with radius $\overline{r} \in D$.

Theorem 4.1. *Let $\tilde{\alpha}(s)$ be a unit speed dual timelike curve. Then $\tilde{\alpha}$ lies on $\tilde{S}_1^2(\overline{r})$ if and only if*

$$\overline{r}^2 = \left(\frac{1}{\tilde{k}_1}\right)^2 + \left[\left(\frac{1}{\tilde{k}_1}\right)' \frac{1}{\tilde{k}_2}\right]^2. \tag{13}$$

Proof: Assume that $\tilde{\alpha}$ lies on $\tilde{S}_1^2(\overline{r})$ which we may assume to have centre at the origin 0. Then

$$g(\tilde{\alpha}, \tilde{\alpha}) = \overline{r}^2.$$

Differentiations of this give first

$$g(\tilde{\alpha}, \tilde{T}) = 0, \tag{14}$$

and then

$$g(\tilde{\alpha}, \tilde{N}) = \frac{1}{\tilde{k}_1}. \tag{15}$$

and the derivation of the last equality gives us

$$g(\tilde{\alpha},\tilde{B}) = \left(\frac{1}{\tilde{k}_1}\right)'\frac{1}{\tilde{k}_2} \tag{16}$$

Then, from (15) and (16) we can write

$$\tilde{\alpha} = \frac{1}{\tilde{k}_1}\tilde{N} + \left(\frac{1}{\tilde{k}_1}\right)'\frac{1}{\tilde{k}_2}\tilde{B}\,. \tag{17}$$

Since we have that the radius of the sphere is $\overline{r} = \|\tilde{\alpha} - 0\|$ we obtain that

$$\overline{r}^2 = \left(\frac{1}{\tilde{k}_1}\right)^2 + \left[\left(\frac{1}{\tilde{k}_1}\right)'\frac{1}{\tilde{k}_2}\right]^2,$$

which completes the proof.

Conversely assume that the regular C^4-curve $\tilde{\alpha}(s)$ satisfies the conditions *(i)* and *(ii)* of the theorem. Let us consider the parametrized curve $\tilde{\alpha} = \tilde{c}(s)$ defined by

$$\tilde{c}(s) = (\tilde{\alpha} - \frac{1}{\tilde{k}_1}\tilde{N} - \left(\frac{1}{\tilde{k}_1}\right)'\frac{1}{\tilde{k}_2}\tilde{B})(s)\,, \tag{18}$$

and the function $\tilde{r}(s)$ defined by

$$[\tilde{r}]^2 \equiv [\tilde{\alpha} - \tilde{c}]^2 = \left(\frac{1}{\tilde{k}_1}\right)^2 + \left[\left(\frac{1}{\tilde{k}_1}\right)'\frac{1}{\tilde{k}_2}\right]^2. \tag{19}$$

If we differentiate (18) and (19) and make use of Frenet formulae, the result is $\tilde{c}' = 0,\ \tilde{r}' = 0$. Therefore, the parametrized curve $\tilde{\alpha} = \tilde{c}(s)$ reduces to a point $\tilde{c}$ and the function $\tilde{r}(s)$ is a constant $\overline{r}$. Hence by (19), $\tilde{\alpha}(s)$ lies on $\tilde{S}_1^2(\overline{r})$ with center $\tilde{c}$ and radius $\overline{r}$.

Corollary 4.1. *The reel and dual parts of* (13) *are given by*

$$r^2 = \left(\frac{1}{k_1}\right)^2 + \left[\left(\frac{1}{k_1}\right)'\frac{1}{k_2}\right]^2$$

and

$$rr^* = -\frac{k_1^*}{k_1^3} - \left(\left(\frac{1}{k_1}\right)'\right)^2\frac{k_2^*}{k_2^3} + \left(\frac{1}{k_1}\right)'\left(\frac{k_1^*}{k_1^2}\right)'\frac{1}{k_2^2},$$

respectively. Here, r^2 which is the reel part of (13), characterizes a unit speed timelike curve $\alpha = \alpha(s)$ with Frenet frame $\{T, N, B\}$ and curvatures k_1 and k_2 which lies on Lorentzian sphere $S_1^2(r)$ with radius r in Minkowski space-time E_1^3.

Equation (17) shows that $\tilde{\alpha}(s)$ is a dual timelike normal curve. So, we can give the following corollary:

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Corollary 4.2. *Let $\tilde{\alpha}(s)$ be a unit speed dual timelike curve. Then $\tilde{\alpha}(s)$ lies on $\tilde{S}_1^2(\overline{r})$ if and only if $\tilde{\alpha}(s)$ is a dual timelike normal curve.*

Also, by Theorem 3.1, Theorem 3.2, Corollary 3.2 and Corollary 4.2 we have the following corollary:

Corollary 4.3. *Let $\tilde{\alpha}(s)$ be a unit speed dual timelike curve. Then $\tilde{\alpha}$ lies on $\tilde{S}_1^2(\overline{r})$ if and only if there are constants $\overline{c}_1, \overline{c}_2, \overline{r} \in D$ such that*

$$\begin{aligned} \frac{1}{\tilde{k}_1(s)} &= \overline{c}_1 \cos\left(\int \tilde{k}_2(s)ds\right) + \overline{c}_2 \sin\left(\int \tilde{k}_2(s)ds\right) \\ &= \frac{1}{k_1(s)} + \varepsilon\left(g(\alpha(s), N^*(s)) + g(\alpha^*(s), N(s))\right) \qquad (20) \\ &= \pm\sqrt{\overline{r}^2 - \overline{c}_2^2} \cos\left(\int \tilde{k}_2(s)ds\right) + \overline{c}_2 \sin\left(\int \tilde{k}_2(s)ds\right). \end{aligned}$$

Reel part of (20) which is given in Corollary 3.1 by

$$\frac{1}{k_1(s)} = c_1 \cos\left(\int k_2(s)ds\right) + c_2 \sin\left(\int k_2(s)ds\right), \qquad c_1, c_2 \in IR,$$

characterizes a unit speed timelike curve $\alpha(s)$ with curvatures $k_1(s) > 0,\ k_2(s) \neq 0$ which lies on Lorentzian sphere $S_1^2(r)$ with radius r in Minkowski space-time E_1^3. So we can give the following corollary:

Corollary 4.4. *Let $\alpha(s)$ be a unit speed timelike curve in E_1^3 with curvatures $k_1(s) > 0,\ k_2(s) \neq 0$. Then α lies on $S_1^2(r)$ if and only if there are constants $c_1, c_2 \in IR$ such that*

$$\frac{1}{k_1(s)} = c_1 \cos\left(\int k_2(s)ds\right) + c_2 \sin\left(\int k_2(s)ds\right).$$

5. CONCLUSIONS

In this study, the characterizations of dual timelike normal and dual timelike spherical curves have been given in dual Minkowski 3-space. Also, it was observed that every dual timelike normal curve is a dual timelike spherical curve.

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