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CEVA, MENELAUS AND STEWART THEOREMS FOR GEODESIC TRIANGLES ON THE HYPERBOLIC UNIT SPHERE H_0^2

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Abstract: In this study, we give Ceva, Menelaus and Stewart Theorems for geodesic triangles on the hyperbolic unit sphere H_0^2.

Keywords: Geodesic triangle, Lorentz space, timelike vector.

H_0^2 HİPERBOLİK BİRİM KÜRESİ ÜZERİNDEKİ GEODEZİK ÜÇGENLER İÇİN CEVA, MENELAUS VE STEWART TEOREMLERİ

Özet: Bu çalışmada H_0^2 hiperbolik birim küresi üzerindeki geodezik üçgenler için Ceva, Menelaus ve Stewart teoremleri verilmektedir.

Anahtar Kelimeler: Geodezik üçgen, Lorentz uzayı, timelike vector.

1. INTRODUCTION

In plane Lorentzian geometry points, timelike, spacelike and lightlike lines, triangles, etc are studied.(BIRMAN & NOMIZO 1984). On the hyperbolic sphere, there are points, but there are no straight lines, at least not in the usual sense. However, straight timelike lines in the Lorentzian plane are characterized by the fact that they are the shortest paths between points. The curves on the hyperbolic sphere with the same property are hyperbolic circles. Thus it is natural to use these circles as replacements for timelike lines.

The formulas for the sine and cosine rules are given for the Euclidean sphere S^2 (AYRES 1954, BELL&THOMAS 1943, YAŞAYAN&HEKİMOĞLU 1982) and hyperbolic sphere (BRONSTEIN vd. 1995). The formulas which are related with the spacelike angles and hyperbolic angles corresponding to the sides of geodesic triangles on hyperbolic unit sphere H_0^2 are given in (BIRMAN & NOMIZO 1984). The hyperbolic sine rule and the hyperbolic cosine rules I-II are given in (ÖZDEMİR&KAZAZ 2005). In this study we give Ceva, Menelaus and Stewart Theorems for geodesic triangles on the hyperbolic unit sphere H_0^2.

2. BASIC CONCEPTS

In this section, we give a brief summary of the theory of Lorentzian concepts.

Let $IR^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in IR\}$ be three-dimensional vector space, $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in IR^3. The Lorentz scalar product of x and y is defined by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3 .$$

$\left(IR^3, \langle , \rangle\right)$ is called three-dimensional Lorentz space or Minkowski 3-space and denoted by IR_1^3. The vector $x \in IR_1^3$ is called a spacelike, timelike and null (lightlike) vector if $\langle x, x \rangle > 0$ or $x = 0$, $\langle x, x \rangle < 0$ and $\langle x, x \rangle = 0$ for $x \neq 0$, respectively. The norm of $x \in IR_1^3$ is denoted by $\|x\|$ and defined as $\|x\| = \sqrt{|\langle x, x \rangle|}$ and x is called a unit vector if $\|x\| = 1$.

Let $e = (0, 0, 1) \in IR^3$. A timelike vector $x = (x_1, x_2, x_3)$ is called *future pointing* (resp. *past pointing*) if $\langle x, e \rangle < 0$ (resp. $\langle x, e \rangle > 0$). Thus a timelike vector $x = (x_1, x_2, x_3)$ is future pointing if and only if $x_1^2 + x_2^2 < x_3^2$ and $x_3^2 > 0$.

The set of all timelike unit vectors is called *hyperbolic unit sphere* and denoted by

$$H_0^2 = \left\{ x = (x_1, x_2, x_3) \in IR_1^3 : \langle x, x \rangle = -1 \right\} .$$

There are two components of the hyperbolic unit sphere H_0^2**.** The components of H_0^2 through $(0,0,1)$ and $(0,0,-1)$ are called *the future pointing hyperbolic unit sphere* and *the past pointing hyperbolic unit sphere* and denoted by H_0^{2+} and H_0^{2-} respectively. Thus we have

$$H_0^{2+} = \left\{ x = (x_1, x_2, x_3) \in IR_1^3 : x \text{ is a future pointing timelike vector} \right\}$$

and

$$H_0^{2-} = \left\{ x = (x_1, x_2, x_3) \in IR_1^3 : x \text{ is a past pointing timelike vector} \right\} .$$

Henceforth, we will use the notation H_0^2 instead of H_0^{2+}.

For any vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in IR_1^3, in the meaning Lorentz vector product of x and y is defined by

$$x \times y = (x_3 y_2 - x_2 y_3, x_1 y_3 - x_3 y_1, x_1 y_2 - x_2 y_1) .$$

The hyperbolic sine rule and the hyperbolic cosine rules I-II are given in (ÖZDEMİR&KAZAZ 2005) as fallows:

Lemma 2.1.(The Hyperbolic Sine Rule) *Let ABC be a hyperbolic geodesic triangle on the hyperbolic unit sphere* H_0^2*. Then the hyperbolic sine rule is given by*

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$$\frac{\sinh a}{\sin\alpha}=\frac{\sinh b}{\sin\beta}=\frac{\sinh c}{\sin\gamma}.$$

Lemma 2.2.(The Hyperbolic Cosine Rule I) *Let ABC be a hyperbolic geodesic triangle on the hyperbolic unit sphere* H_0^2**.** *Then the hyperbolic cosine rule I is given by*

$$\cos\alpha=\frac{\cosh b\cosh c-\cosh a}{\sinh b\sinh c},$$
$$\cos\beta=\frac{\cosh c\cosh a-\cosh b}{\sinh c\sinh a},$$
$$\cos\gamma=\frac{\cosh a\cosh b-\cosh c}{\sinh a\sinh b}.$$

Lemma 2.3.(The Hyperbolic Cosine Rule II). *Let ABC be a hyperbolic geodesic triangle on* H_0^2**.** *Then the hyperbolic cosine rule II is given by*

$$\cosh a=\frac{\cos\beta\cos\gamma+\cos\alpha}{\sin\beta\sin\gamma},$$
$$\cosh b=\frac{\cos\alpha\cos\gamma+\cos\beta}{\sin\alpha\sin\gamma},$$
$$\cosh c=\frac{\cos\alpha\cos\beta-\cos\gamma}{\sin\alpha\sin\beta}.$$

where a, b and c are hyperbolic angles between timelike vectors i and j, j and k, k and i, respectively.

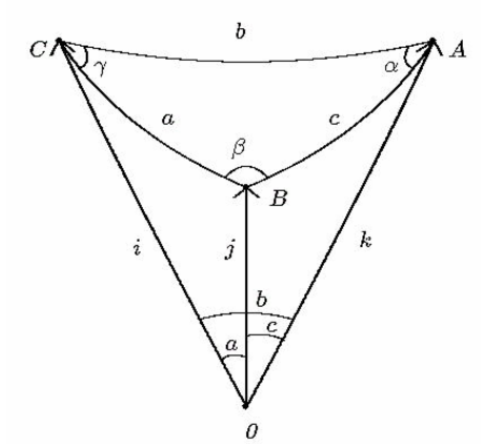

Figure 1

3. CEVA, MENELAUS AND STEWART THEOREMS FOR GEODESIC TRIANGLES ON THE HYPERBOLIC UNIT SPHERE H_0^2

Let ABC be a geodesic triangle and AD, BE and CF be geodesic curves on the hyperbolic unit sphere H_0^2 where D, E and F are points on the geodesic curves BC, AC and AB, respectively. Let the geodesic curves AD, BE and CF intersect at a point P as shown in Figure 2. Then we can give Ceva theorem for geodesic triangles on the hyperbolic unit sphere H_0^2 as follows:

Theorem 3.1.(Ceva Theorem): There is a trigonometric relation between the parts of the geodesic curves which construct the hyperbolic geodesic triangle ABC such that

$$\frac{\sinh a_1}{\sinh a_2}\frac{\sinh b_1}{\sinh b_2}\frac{\sinh c_1}{\sinh c_2}=1.$$

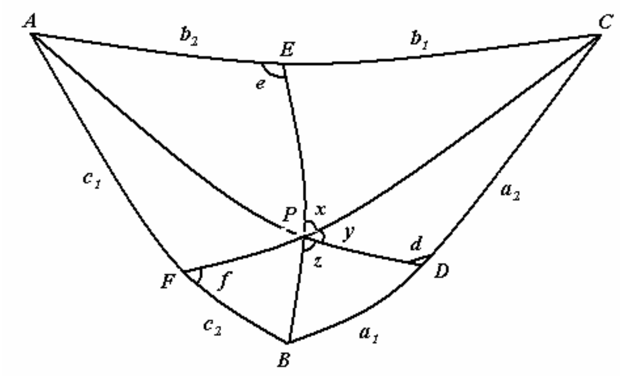

Figure 2.

Proof: If we use hyperbolic sine rule for the geodesic triangles BPD, DPC, CPE, EPA, APF and FPB, respectively, in Figure 2, we obtain

$$\sinh a_1 = \frac{\sin z}{\sin d}\sinh PB, \tag{1}$$

$$\sinh a_2 = \frac{\sin y}{\sin d}\sinh PC, \tag{2}$$

$$\sinh b_1 = \frac{\sin x}{\sin e}\sinh PC, \tag{3}$$

$$\sinh b_2 = \frac{\sin z}{\sin e}\sinh PA, \tag{4}$$

$$\sinh c_1 = \frac{\sin y}{\sin f}\sinh PA, \tag{5}$$

$$\sinh c_2 = \frac{\sin x}{\sin f}\sinh PB. \tag{6}$$

Dividing (1) by (2), (3) by (4) and (5) by (6) respectively, we have

$$\frac{\sinh a_1}{\sinh a_2} = \frac{\sin z}{\sin y}\frac{\sinh PB}{\sinh PC}, \tag{7}$$

$$\frac{\sinh b_1}{\sinh b_2} = \frac{\sin x}{\sin z}\frac{\sinh PC}{\sinh PA}, \tag{8}$$

$$\frac{\sinh c_1}{\sinh c_2} = \frac{\sin y}{\sin x}\frac{\sinh PA}{\sinh PB}. \tag{9}$$

If we multiple (7), (8) and (9) we get

$$\frac{\sinh a_1}{\sinh a_2}\frac{\sinh b_1}{\sinh b_2}\frac{\sinh c_1}{\sinh c_2}=1. \tag{10}$$

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That completes the proof.

Conversely, if D, E, F are the points on the sides a, b, c respectively, and if (10) holds then, the geodesic curves AD, BE and CF intersect at one point. To show this, assume that AD and BE intersect at a point P'. Then the geodesic curve CP' intersects the side c at a point F'. So, F' divides c into two parts as c_1' and c_2'. Thus from Ceva theorem we can write

$$\frac{\sinh a_1}{\sinh a_2}\frac{\sinh b_1}{\sinh b_2}\frac{\sinh c_1'}{\sinh c_2'}=1.$$

Since (10) holds we get

$$\frac{\sinh c_1}{\sinh c_2}=\frac{\sinh c_1'}{\sinh c_2'}.$$

By using the fact that $c_1+c_2=c=c_1'+c_2'$ we obtain $c_1'=c_1$ and $c_2'=c_2$. So that the point F must coincide with the point F'. It means that the geodesic curves AD, BE and CF intersect at one point.

Let ABC and BDF be geodesic triangles on H_0^2 where BC and FB are geodesics on the geodesics BD and AB respectively. Then Menelaus theorem can be given as follows:

Theorem 3.2.(Menelaus Theorem): There is a trigonometric relation for the geodesic triangles ABC and BDF on H_0^2 such that

$$\frac{\sinh BF}{\sinh AF}=\frac{\sinh AE}{\sinh CE}=\frac{\sinh CD}{\sinh BD}=-1. \tag{11}$$

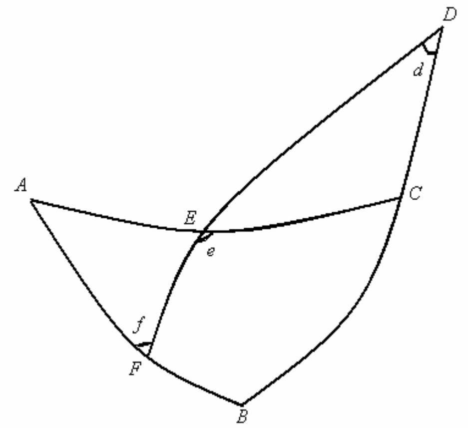

Figure 3.

Proof: Consider the geodesic triangles BDF, AEF ve CDE in Figure 3. By using hyperbolic sine rule for this triangles we obtain

$$\frac{\sinh BF}{\sinh BD}=\frac{\sin d}{\sin f}, \tag{12}$$

$$\frac{\sinh AE}{\sinh AF}=\frac{\sin f}{\sin e}, \tag{13}$$

$$\frac{\sinh CD}{\sinh CE}=\frac{\sin e}{\sin d}. \tag{14}$$

If we take

$$BF=c_1,\ FA=c_2,\ c_1+c_2=c, \tag{15}$$

$$AE=b_1,\ EC=b_2,\ b_1+b_2=b, \tag{16}$$

$$CD=-a_1,\ BD=a_2,\ -a_1+a_2=a, \tag{17}$$

then by multiplying (12), (13) and (14) we find

$$\frac{\sinh c_1}{\sinh a_2}=\frac{\sinh b_1}{\sinh c_2}=\frac{-\sinh a_1}{\sinh b_2}=1.$$

Thus

$$\frac{\sinh BF}{\sinh AF}=\frac{\sinh AE}{\sinh CE}=\frac{\sinh CD}{\sinh BD}=-1.$$

Let now ABC be a geodesic triangle on the hyperbolic unit sphere H_0^2 **.** If a point D on the geodesic curve BC is connected to vertex A then we have Figure 4. So Stewart theorem for geodesic triangles can be given.

Theorem 3.3.(Stewart Theorem): Let ABC be a geodesic triangle on the hyperbolic unit sphere H_0^2 as shown in Figure 4. Then Stewart theorem is given by

$$\sinh f\cosh c+\sinh e\cosh b=\cosh d\sinh a. \tag{18}$$

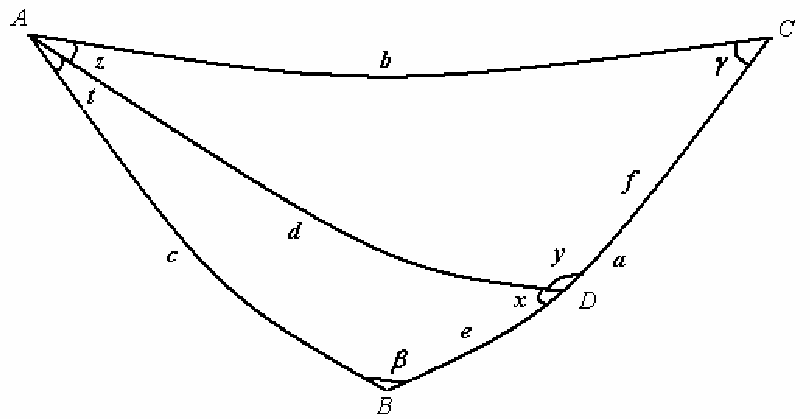

Figure 4.

Proof: From the hyperbolic cosine rule I for the geodesic triangles ABD and ACD we write

$$\cos x=\frac{\cosh d\cosh e-\cosh c}{\sinh d\sinh e},$$

$$\cos y=\frac{\cosh d\cosh f-\cosh b}{\sinh d\sinh f}.$$

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Since $x+y=180^0$ we can write

$$\cos x+\cos y=0, \tag{19}$$

and so we have

$$\frac{\cosh d\cosh e-\cosh c}{\sinh d\sinh e}+\frac{\cosh d\cosh f-\cosh b}{\sinh d\sinh f}=0. \tag{20}$$

Equation (20) can be written as

$$\cosh d(\sinh f\cosh e+\sinh e\cosh f)=\sinh f\cosh c+\sinh e\cosh b. \tag{21}$$

Since $e+f=a$ and

$$\sinh(e+f)=\sinh e\cosh f+\cosh e\sinh f=\sinh a \tag{22}$$

(21) can be written as

$$\cosh d\sinh a=\sinh f\cosh c+\sinh e\cosh b$$

That is desired.

4. CONCLUSIONS

In this paper, Ceva Menelaus and Stewart Theorems have been given for geodesic triangles on the hyperbolic unit sphere H_0^2**.** One can easily see that these theorems are very similar to those which are given for spherical triangles in Euclidean sphere S^2 (YAŞAYAN&HEKİMOĞLU, 1982).

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