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SECOND GAUSSIAN CURVATURE OF B-SCROLL SURFACES IN MINKOWSKI 3-SPACE

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Abstract: In this paper, we have studied the second Gaussian curvature K_{II} of B-Scroll surfaces and extension of the B-Scroll surfaces in the Minkowski 3-space R_1^3 (R_1^3, $d^2x + d^2y - d^2z$) . Furthermore, we calculated the second Gaussian curvature and the mean curvature for B-scroll and extension B-Scroll surfaces in the Minkowski 3-space.

Key words: Gauss Curvature, Ruled Surfaces, Minkowski space**,** B-Scroll Surface

Mathematics Subject Classifications (2000): 51B20, 53A04, 53A05.

MİNKOWSKİ 3-UZAYINDA B-SCROLL YÜZEYLERİNİN İKİNCİ GAUSSİAN EĞRİLİĞİ

Özet: Bu çalışmada, Minkowski 3-uzayı R_1^3 (R_1^3, $d^2x + d^2y - d^2z$) de B-Scroll yüzeylerinin genişletilmişi ve B-Scroll yüzeylerinin, K_{II} ikinci Gaussian eğriliği çalışıldı. Daha sonra da Minkowski 3-uzayında B-Scroll yüzeylerinin genişletilmişi ve B-Scroll yüzeylerinin ikinci Gaussian eğrliği ve ortalama eğriliği hesaplandı.

Anahtar kelimeler: Gauss Eğriliği, Regle yüzeyler, Minkowski uzayı, B-Scroll Yüzeyi

1. INTRODUCTION

The study of a ruled surfaces which are surfaces swept out by a straight line L moving along a curve, is an interesting research area in the theory of surfaces in Euclidean geometry (NASSAR & FATHI 2000). Moreover, there is a different approach to this topic by using dual spherical curves (SPIVAK 1979).

In the case of an indefinite metric, since the curve and the line L may have different casual characters, some difficulties may arise. If the surface is Euclidean, then the geometry is similar to the classical one. If the surface is timelike or degenerate, the study is quite different (CHOI 1995, O'NEILL 1983, ALTIN 1999, BLAIR & KOUFOGIORGOS 1992).
B-Scroll surface was first introduced by Graves (BLAIR & KOUFOGIORGOS 1992) and used to classify the codimension one isometric immersion between Lorentz

surfaces. Then some authors studied and developed the geometry of B-Scroll surface (SPIVAK 1979).

2. PRELIMINARIES

Let R_1^3 be a 3-dimensional Lorentzian space with pseudo metric $d^2s=d^2x + d^2y- d^2z$, if <X,Y>=0, *X* and *Y* are called perpendicular in the sense of Lorentz, where < , > is the induced inner product in R_1^3. The norm of $X \in R_1^3$ is denoted by $\|X\|$ and defined as

$$\|X\| = \sqrt{|\langle X, X\rangle|}$$

Let X is a vector. If <X,X> < 0, then X is called timelike, if <X,X> >0 and X=0, then X is called spacelike and if <X,X> =0 and $X \neq 0$, then X is called lightlike (or null) vector, respectively. Observe that, a timelike curve corresponds to the path of an observer moving at less than the speed of light while the spacelike curves faster and the null curves are equal of the speed of light (ÇÖKEN & AYYILDIZ 2003).
A regular curve

$$\alpha(t) = I \to R_1^3, \qquad I \subset R$$

is said to be a spacelike, timelike or null curve if the velocity vector *α'(s)* is a spacelike, timelike, or null vector, respectively.
A surface in the 3-dimensional Lorentz space is called a timelike surface if the induced metric on the surface is a Lorentz metric, i.e, the normal on the surface is a spacelike vector.
The null frame, also called Cartan frame, of a null curve $\alpha=\alpha(s) \in R_1^3$ parametrized by natural parametrization, is a frame field $\{e_1, e_2, e_3\}$, having properties

$$\begin{aligned} \langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0, \quad \langle e_3, e_3\rangle = 1 \\ \langle e_1, e_3\rangle = \langle e_2, e_3\rangle = 0, \quad \langle e_1, e_2\rangle = -1 \end{aligned} \tag{2.1}$$

and

$$e_1 \times e_2 = e_3, \quad e_1 \times e_3 = e_1, \quad e_3 \times e_2 = e_2$$
$$\det(e_1, e_2, e_3) = 1.$$

The matrix form of null frame is

$$\begin{bmatrix} e_1' \\ e_2' \\ e_3' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \kappa \\ 0 & 0 & \tau \\ \tau & \kappa & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

The infinitesimal displacement of null frame is given as

$$\begin{aligned} \alpha'(s) = e_1 \qquad & e_1'(s) = \kappa(s) e_3 \\ e'_2(s) = \tau(s) e_3 \qquad & e_3'(s) = \tau(s) e_1 + \kappa(s) e_2 \end{aligned} \tag{2.2}$$

where $e_1(s)$ is the tangent null vector of curve *α(s),* $e_2(s)$ is the principal normal vector field and $e_3(s)$ is the binormal vector field. The functions κ(s) and τ(s) are the curvature and torsion of the curve *α(s),* respectively.

3. B-SCROLL SURFACES

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Let *α=α(s)* be a null curve in R_1^3 with the null frame and the infinitesimal displacement. The immersion

$$\varphi : U \subset R^2 \to R_1^3$$
$$(s,v) \to \varphi(s,v) = \alpha(s) + v e_2(s)$$

defines a ruled surface generated by a principal normal of the null curve *α=α(s)*, $\forall (s,v) \in U$. This ruled surfaces is called a B-scroll, which has introduced by Graves (BLAIR & KOUFOGIORGOS 1992). Let's denote B-Scroll as BS.
Now, we consider a ruled surface in R_1^3 generated by null generator $\widetilde{L}(s)$ moving with Cartan frame (null frame) of a null curve *α=α(s)*. Then,

$$\widetilde{L}(s) = \sum_{i=1}^{3} l_i(s) e_i(s)$$

where component $l_i = l_i(s), (i = 1,2,3)$ are scalar functions of the parameter of arc lenght of the null curve *α=α(s)*. Thus, if $\widetilde{L}(s)$ moves with Cartan frame, the constructed ruled surface is given by the following parametrization

$$M : \varphi(s,v) = \alpha(s) + v e_2(s), \forall (s,v) \in U \subset \mathrm{R}^2 \qquad (3.1)$$

Let *α=α(s)* be a null curve in R_1^3 and let $\kappa(s)$ and $\tau(s)$ denote the curvature and torsion of α(s), respectively. Consider the Cartan frame $\{ e_1(s), e_2(s), e_3(s) \}$ attached to the curve *α=α(s)* such that $e_2 = e_2(s)$ is the principal normal vector field null and $e_3 = e_3(s)$ is the binormal vector field (spacelike type).

A surface *M* in a three-dimensional Euclidean space E^3 with positive Gaussian curvature *K* possesses a positive definite second fundamental form *II* if appropriately orientated. Therefore, the second fundamental form defines a new Riemannian metric on *M*. In turn, we can consider the Gaussian curvature K_{II} of the second fundamental form which is regarded as a Riemannian metric. If a surface has non-zero Gaussian curvature everywhere, K_{II} can be defined formally and it is the curvature of the Riemannian or pseudo-Riemannian manifold (*M, II*).
Naturally, we can extend such a notion into that of surfaces in a three-dimensional Minkowski space R_1^3

Definition 3.1. Given a B-Scroll surface *M*, we define the Second Gaussian Curvature as

$$K_{II} = \frac{1}{\left(|eg| - f^2\right)} \left\{ \begin{vmatrix} -\frac{1}{2}e_{vv} + f_{sv} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_v \\ f_v - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_v & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_v & \frac{1}{2}g_s \\ \frac{1}{2}e_v & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}$$

The coefficient of the first fundamental form, *E, F* and *G* are given by

$$E = \langle \varphi_s, \varphi_s \rangle, \qquad F = \langle \varphi_s, \varphi_v \rangle, \qquad G = \langle \varphi_v, \varphi_v \rangle \qquad (3.2)$$

and the coefficient of the second fundamental form *e, f, g* are given as

$$e=\frac{\langle\varphi_{ss},\varphi_s\times\varphi_v\rangle}{D},\qquad f=\frac{\langle\varphi_{sv},\varphi_s\times\varphi_v\rangle}{D},\qquad g=\frac{\langle\varphi_{vv},\varphi_s\times\varphi_v\rangle}{D} \tag{3.3}$$

where < , > denotes the scalar product of R_1^3 and

$$D=\begin{cases}\sqrt{EG-F^2}, & if\ M\ is\ spacelike\\ \sqrt{F^2-EG}, & if\ M\ is\ timelike\end{cases}$$

Note that, since the surface is timelike we have $D=\sqrt{F^2-EG}$.

Calculating partial derivative of (3.1) with respect to v and s, we get

$$\begin{aligned}\varphi_s &= e_1+v\tau e_3\\ \varphi_{ss} &= v\tau^2 e_1+v\tau\kappa e_2+(\kappa+v\tau')e_3\\ \varphi_{sv} &= \tau e_3\\ \varphi_v &= e_2\\ \varphi_{vv} &= 0\end{aligned}$$

Using (3.3), we obtain that $E=v^2\tau^2$, $F=-1$, $G=0$ and $D=1$. Also,

$$\begin{aligned}&e=(\kappa+v\tau')-v^2\tau^3,\\ &f=\tau,\\ &g=0\end{aligned}$$

Substituting *e, f* and *g* into Second Gaussian Curvature matrix form K_{II}, we have

$$K_{II}=\tau^3$$

Now, to calculate the mean curvature *H*, we first need to find the coefficients

$$M=\langle\varphi_{sv},n\rangle,\quad N=\langle\varphi_{vv},n\rangle\ \text{and}\quad L=\langle\varphi_{ss},n\rangle\quad\text{where}\quad n=\frac{\varphi_s\times\varphi_v}{\|\varphi_s\times\varphi_v\|}.$$

After necessary calculation, we get $M=\tau$, $N=0$ and $L=-v^2\tau^3+\kappa+v\tau'$. As a result,

$$H=\frac{1}{2}\left(\frac{EN+GL-2MF}{F^2-EG}\right)=\tau$$

Finally, we obtain the relation between K_{II} and H as,

$$K_{II}=H^3.$$

Teorem 3.1 : Let *m, n* be natural numbers. B-scrolls over null curves are the only null scroll with non-degenerate second fundamental form in a 3-dimensional Lorentz-Minkowski space satisfying the condition $K_{II}=K^mH^n$ along each ruling (DAE 2006).

4. EXTENSION B-SCROLL SURFACES

Let $\widetilde{L}$ moves with Cartan frame (2.2). The constructed ruled surface is given by the following parametrization

$$\begin{aligned}&\varphi: U\subset R^2\to R_1^3\\ &(s,v)\to\varphi(s,v)=\alpha(s)+v\widetilde{L}(s)\end{aligned} \tag{4.1}$$

where

$$\langle\widetilde{L}(s),\widetilde{L}(s)\rangle=l_3^2-2l_1l_2=0.$$

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This ruled surface is called an extension B-scroll surface, which is denoted by **EBS**. The striction curve *β=β(s)* for the extension B-scroll surface is given by

$$\beta(s)=\alpha(s)+\frac{l_2'+l_3\kappa}{\left\|\widetilde{L}(s)\right\|}\widetilde{L}(s)$$

Moreover the base curve *α=α(s)* is the striction curve *β=β(s)* if and only if $l_2'+l_3\kappa=0$. Calculating partial derivate of (4.1) with respect to *v* and *s*, we obtain

$$\begin{aligned}\varphi_s&=e_1+v\widetilde{L}'(s)\\ \varphi_{ss}&=\kappa e_3+v\widetilde{L}''(s)\\ \varphi_{sv}&=\widetilde{L}'(s)\\ \varphi_v&=\widetilde{L}(s)\\ \varphi_{vv}&=0.\end{aligned}$$

It is known that the extension B-scroll surface is a timelike ruled surface (SPIVAK 1979). So, we have $D=l_2$. Similar calculation to the case of BS surface, we find the coefficients of K_{II} and *H* as follows;

$$\begin{aligned}e&=\frac{\left\langle \kappa e_3+v\widetilde{L}''(s),e_1+v\widetilde{L}'(s)\times\widetilde{L}\right\rangle}{D}\\ &=\kappa v\left(l_1'l_2+l_2l_3\tau\right)+v^2\left[\left(l_3''+2l_1'\kappa+l_1\kappa+l_2\tau'\right).\left(l_1'l_2+l_2l_3\tau\right)\right]\\ &\quad+v^2\left[\left(-l_1''-l_3'\tau-l_3\tau'-\tau l_1\kappa-\tau l_3-\tau^2l_3\right).\left(l_1l_2\kappa+l_3'l_2+l_2^2\tau\right)\right]\end{aligned}$$

f=0, *g=0* and $E=-2v\left(l_2'+l_3\kappa\right)+v^2\left\|\widetilde{L}'\right\|$, $L=\left(\kappa+v\tau'\right)-v^2\tau^3$, $F=-l_2$, $M=l_2\left(l_3'+l_2\tau+l_1\kappa\right)$, *G=0, N=0.*

Substituting these result into the equations of K_{II} and *H*, we get

$$K_{II}=0 \text{ and } H=-\frac{1}{\lambda}$$

where $\lambda=\dfrac{l_2}{l_1\kappa+l_2'+l_2\tau}$ and the unit normal vector field *n*,

$$n=\frac{\alpha'\times\widetilde{L}+v\widetilde{L}'\times\widetilde{L}}{l_2}$$

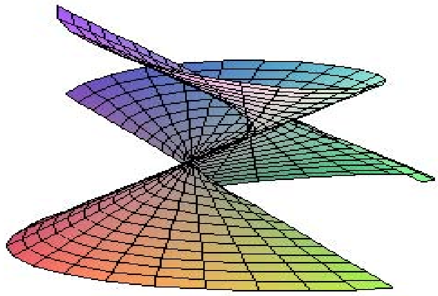
The result is confirmed by the following example.

Example 4.1: Consider the EBS surface *M* in R_1^3 parametrized by

$$\varphi(s,v)=\left(\cos s+v\sin s,\sin s+v\cos s,s+v\right)$$

This parametrization defines a null ruled surfaces in R_1^3 with a null base curve $\alpha(s)=(\cos s,\sin s,s)$ and null generator $\widetilde{L}(s)=(\sin s,\cos s,1)$. It is straightforward computation to see that $K_{II}=0$ and $H=-\dfrac{1}{\lambda}$, as desired.

The EBS surface is illustrated in the following figure.



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