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ON THE APPROXIMATION OF SINGULAR INTEGRALS

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Abstract. In this work we present an approximation for singular integrals of Cauchy type using a small technical in order to eliminate the singularity. This approximation is destined to resolve numerically the singular integral equations with Cauchy type kernel on a smooth contour oriented.

Key words. Singular integral, interpolation, Hőlder space and Hőlder condition.

Mathematics Subject Classifications (2000): Primary 45D05, 45E05, 45L05; Secondary 65R20.

INTRODUCTION

The main considerations of the present work concern the construction and foundation of some numerical schemes which are destined for numerical solution of singular integral equations with Cauchy type kernel

$$a(t_0)\varphi(t_0)+\frac{b(t_0)}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt+\int_\Gamma k(t,t_0)\varphi(t)dt = f(t_0) \qquad (1)$$

where under Γ we designate a smooth contour oriented, t and t_0 are points on Γ, $a(t), b(t), k(t,t_0)$ and $f(t)$ are functions given on Γ.

Our schemes describe the quadrature method for the approximation of singular integral operator

$$F(t_0)=\frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt, \quad t,t_0\in\Gamma, \qquad (2)$$

with Cauchy kernel by a sequence of numerical integration operators.

For the existence of the principal value of this integral for a given density $\varphi(t)$, we will need more than mere continuity, in other words, the density $\varphi(t)$ has to satisfy the Hölder condition $H(\mu)$ (MUSKHELISHVILI 1968). So we note that, singular integral operators of the first kind with Cauchy kernel have index zero. In particular, injective singular integral operators of the first kind are bijective and bounded inverse.

The Quadrature

Let $t = t(s) = x(s) + iy(s)$ where $s \in [a,b]$ be the parametric complex equation of the curve Γ with the respect to some parameter s. Consider that N is an arbitrary natural number, generally we take it large enough, and divide the interval $[a,b]$ into N equal subintervals of $[a,b]$

$$[a,b] = \{a = s_0 < s_1 < ..... < s_N = b\},$$

be called I_1 to I_N, so that, we have $I_{\sigma+1} = [s_\sigma, s_{\sigma+1}]$.

$$s_\sigma = a + \sigma \frac{l}{N}, \; l = b - a, \; \sigma = 0,1,2,...., N.$$

Further, fixing a natural number m, and divide each of segments $[s_\sigma, s_{\sigma+1}[$ by points

$$s_{\sigma k} = s_\sigma + hx_k, \; h = \frac{l}{N}, \; k = 0,1,...,m,$$

where the points $\{x_k\}$ represent the increasing sequence belongs to the interval $[0,1[$.
Denoting by

$$t_\sigma = t(s_\sigma), \; t_{\sigma k} = t(s_{\sigma k}); \quad \sigma = 0,1,2,..., N; \; k = 0,1,....,m.$$

Assuming that for the indices $\sigma, \nu = 0,1,2,...., N-1$ the points t and t_0 belong respectively to the arcs $t_\sigma \overset{\frown}{t}_{\sigma+1}$ and $t_\nu \overset{\frown}{t}_{\nu+1}$ where $t_\alpha \overset{\frown}{t}_{\alpha+1}$ designate the smallest arc with ends t_α and $t_{\alpha+1}$ (NADIR 1985, NADIR 2004, SANIKIDZE 1970).
For an arbitrary numbers σ, ν from $0,1,2,..., N-1$, we define the function $\beta_{\sigma\nu}(\varphi; t, t_0)$ dependents of φ, t and t_0 by

$$\beta_{\sigma\nu}(\varphi; t, t_0) = U(\varphi; t, \sigma) - U(\varphi; t_0, \nu) \tag{3}$$

where the expression $U(\varphi; t, \sigma)$, designates the approximation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}[$ of the curve Γ, given by the following formula

$$U(\varphi, t, \sigma) = \frac{t_{\sigma(k+1)} - t}{t_{\sigma(k+1)} - t_{\sigma k}} \varphi(t_{\sigma k}) + \frac{t - t_{\sigma k}}{t_{\sigma(k+1)} - t_{\sigma k}} \varphi(t_{\sigma(k+1)}); \; k = 0,1,.....m-1,$$

where the density φ represents still a given function on the curve Γ of the class $H(\mu)$.
Seeing that, the equality $t - t_0 = 0$ is possible only when $\sigma = \nu$, in this case, the function $\beta_{\sigma\sigma}(\varphi; t, t_0)$ contains $(t - t_0)$ as factor

$$\beta_{\sigma\sigma}(\varphi; t, t_0) = U(\varphi, t, \sigma) - U(\varphi, t_0, \sigma). \tag{4}$$

Putting now the function

$$\psi_{\sigma\nu}(\varphi; t, t_0) = \begin{cases} \varphi(t_0) + \beta_{\sigma\nu}(\varphi; t, t_0), \; t \in t_\sigma t_{\sigma+1}\,; \; t_0 \in t_\nu t_{\nu+1} \\ \sigma = 0,1,..., N-1\,; \; \nu = 0,1,..., N-1 \end{cases}. \tag{5}$$

After this construction, one replaces the singular integral (2)

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$$F(t_0)=\frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt$$

by the following ones

$$S(\varphi,t_0)=\frac{1}{\pi i}\int_\Gamma \frac{\psi_{\sigma\nu}(\varphi;t,t_0)}{t-t_0}dt=\varphi(t_0)+\frac{1}{\pi i}\int_\Gamma \frac{\beta_{\sigma\nu}(\varphi;t,t_0)}{t-t_0}dt. \qquad (6)$$

Theorem

Let Γ *be a smooth contour oriented and let* φ *be a density satisfies the Hölder condition* $H(\mu)$ *then, the following estimation*

$$|F(t_0)-S(\varphi;t_0)|\le \max\left(\frac{C\ln(mN)}{(mN)^\mu},\frac{C}{N^\mu}\right)\ N,\ m>1$$

holds, where the constant C *depends only of the contour* Γ.

Proof

For any $t\in \overset{\frown}{t_\sigma t_{\sigma+1}}$ and $t_0\in \overset{\frown}{t_\nu t_{\nu+1}}$ with $\sigma\neq\nu$, we can write

$$\begin{aligned}\varphi(t)-\psi_{\sigma\nu}(\varphi;t,t_0)&=\varphi(t)-\varphi(t_0)\\ &-\Big\{\frac{t_{\sigma(k+1)}-t}{t_{\sigma(k+1)}-t_{\sigma k}}\varphi(t_{\sigma k})+\frac{t-t_{\sigma k}}{t_{\sigma(k+1)}-t_{\sigma k}}\varphi(t_{\sigma(k+1)})\\ &-\frac{t_{\nu(k+1)}-t_0}{t_{\nu(k+1)}-t_{\nu k}}\varphi(t_{\nu k})-\frac{t_0-t_{\nu k}}{t_{\nu(k+1)}-t_{\nu k}}\varphi(t_{\nu(k+1)})\Big\}.\end{aligned} \qquad (7)$$

If $\sigma=\nu$, we can easily put our expression in the form

$$\begin{aligned}\varphi(t)-\psi_{\sigma\sigma}(\varphi;t,t_0)&=\varphi(t)-\varphi(t_0)\\ &-\frac{t-t_0}{t_{\sigma(k+1)}-t_{\sigma k}}(\varphi(t_{\sigma(k+1)})-\varphi(t_{\sigma k})).\end{aligned} \qquad (8)$$

Taking into account expressions $(7),(8)$ above, we have

$$\begin{aligned}\frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)-\psi_{\sigma\nu}(\varphi;t,t_0)}{t-t_0}dt&=\frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\int_{t_\sigma t_{\sigma+1}}\frac{\varphi(t)-\varphi(t_0)}{t-t_0}\\ &-\Big\{\frac{t_{\sigma(k+1)}-t}{t_{\sigma(k+1)}-t_{\sigma k}}\varphi(t_{\sigma k})+\frac{t-t_{\sigma k}}{t_{\sigma(k+1)}-t_{\sigma k}}\varphi(t_{\sigma(k+1)})\\ &-\frac{t_{\nu(k+1)}-t_0}{t_{\nu(k+1)}-t_{\nu k}}\varphi(t_{\nu k})-\frac{t_0-t_{\nu k}}{t_{\nu(k+1)}-t_{\nu k}}\varphi(t_{\nu(k+1)})\Big\}\frac{1}{t-t_0}dt.\end{aligned} \qquad (9)$$

Passing now to the estimation of the expression (9), we have for $t_0\in \overset{\frown}{t_\nu t_{\nu+1}}$ and $\sigma\neq\nu$, the relation

$$\left| \begin{array}{l} \sum_{\substack{\sigma=0\\ \sigma\neq\nu}}^{N-1} \sum_{k=0}^{m-1} \int_{t_{\sigma k} t_{\sigma k+1}} \frac{\varphi(t)-\varphi(t_0)}{t-t_0} \\ -\{\frac{t_{\sigma(k+1)}-t}{t_{\sigma(k+1)}-t_{\sigma k}}\varphi(t_{\sigma k})+\frac{t-t_{\sigma k}}{t_{\sigma(k+1)}-t_{\sigma k}}\varphi(t_{\sigma(k+1)}) \\ -\frac{t_{\nu(k+1)}-t_0}{t_{\nu(k+1)}-t_{\nu k}}\varphi(t_{\nu k})-\frac{t_0-t_{\nu k}}{t_{\nu(k+1)}-t_{\nu k}}\varphi(t_{\nu(k+1)})\}\frac{1}{t-t_0}dt. \end{array} \right| = O(\frac{\ln mN}{m^{\mu}N^{\mu}}).$$

Naturally, the estimation given above is obtained with the help of which using the density φ as an element of the Hölder space $H(\mu)$ (MUSKHELISHVILI 1968).

$$|\varphi(x)-\varphi(y)| \leq A\,|x-y|^{\mu}$$

Besides, it is easy to see that

$$\max_{t_0\in t_\nu t_{\nu+1}} |O(\frac{1}{m^{\mu}N^{\mu}}) \sum_{\substack{\sigma=0\\ \sigma\neq\nu}}^{N-1} \sum_{k=0}^{m-1} \int_{t_{\sigma k} t_{\sigma(k+1)}} \frac{dt}{t-t_0}| = O(\frac{\ln mN}{m^{\mu}N^{\mu}}).$$

Further, for the case where $\sigma=\nu$, using again the condition $\varphi\in H(\mu)$ and the condition of smoothness of Γ, we obtain

$$\left|\int_{t_\nu t_{\nu+1}} \frac{\varphi(t)-\varphi(t_0)}{t-t_0}dt\right| \leq A\int_{s_\nu}^{s_{\nu+1}} |s-s_0|^{\mu-1}\,ds = O(N^{-\mu})$$

Numerical experiments. Using our approximation, we apply the algorithms to singular integrals and we present results concerning the accuracy of the calculations, in this numerical experiments each table I represents the exact principal value of the singular integral and $\tilde{I}$ corresponds to the approximate calculation produced by our approximation at points values interpolation.

Example 1. Consider the singular integral,

$$I = F(t_0) = \frac{1}{\pi i}\int_{\Gamma} \frac{\varphi(t)}{t-t_0}dt = \frac{1}{\pi i}\int_{\Gamma} \frac{1}{t(t-t_0)}dt,$$

where the curve Γ designate the unit circle and the function density φ is,

$$\varphi(t) = \frac{1}{t}.$$

N	m	$\Vert I-\tilde{I}\Vert_1$	$\Vert I-\tilde{I}\Vert_2$	$\Vert I-\tilde{I}\Vert_\infty$
10	3	4.7835599E - 03	4.7812131E - 03	4.3059266E - 03
10	4	2.8606000E - 03	2.4213300E - 03	2.2799596E - 03
10	5	1.8950893E - 03	1.5464505E - 03	1.4939693E - 03

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Example 2. We take the singular integral,

$$I = F(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t - t_0}dt = \frac{1}{\pi i}\int_\Gamma \frac{\sin t + \cos t}{t(t - t_0)}dt,$$

also, the curve Γ designate the unit circle and the function density φ is,

$$\varphi(t) = \frac{\sin t + \cos t}{t}.$$

N	m	$\|I - \tilde{I}\|_1$	$\|I - \tilde{I}\|_2$	$\|I - \tilde{I}\|_\infty$
10	3	4.6366839E - 03	4.0219417E - 03	3.5962313E - 03
10	4	3.8213600E - 03	3.2955576E - 03	3.0960667E - 03
10	5	1.8740431E - 03	1.4832552E - 03	1.3956153E - 03

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