SDÜ FEN EDEBİYAT FAKÜLTESİ
FEN DERGİSİ (E-DERGİ). 2008, 3(1), 72-79

ON THE DETERMINATION OF A DEVELOPABLE TIMELIKE RULED SURFACE

Mustafa KAZAZ, Ali ÖZDEMİR, Tuba GÜROĞLU

Department of Mathematics, Faculty of Science, University of Celal Bayar
Muradiye Campus, 45047, Manisa, TURKEY, e-mail: m_kazaz@hotmail.com.
Received: 04 June 2007, Accepted: 11 January 2008

Abstract: This paper gives a method for determining a developable timelike ruled surface by using dual vector calculus. A developable timelike ruled surface can be parameterized in the form $m(t, u) = p(t) + u\,x(t)$ ($p(t)$ is called the base curve of $m(t, u)$). The dual vectorial expression of a developable timelike ruled surface is obtained from the coordinates and the first derivatives of the base curve.

Key words: Minkowski space**,** Developable surface, Timelike ruled surface.

Mathematics Subject Classifications (2000): 53C50, 53A04, 53A05.

BİR AÇILABİLİR TİMELİKE REGLE YÜZEYİN BELİRLENMESİ ÜZERİNE

Özet: Bu makalede dual vektör analizi kullanılarak bir açılabilir timelike regle yüzeyin belirlenmesi için bir metot verilmiştir. Bir açılabilir timelike regle yüzey $m(t, u) = p(t) + u\,x(t)$ formunda parametreleştirilebilir ($p(t)$ ye $m(t, u)$ nın dayanak eğrisi denir). Bir açılabilir timelike regle yüzeyin dual vektörel ifadesi koordinatlar ve dayanak eğrisinin ilk türevlerinden elde edildi.

Anahtar kelimeler: Minkowski Uzayı, Açılabilir Yüzey, Timelike Regle Yüzey.

1. INTRODUCTION

Ruled surfaces, particularly developable surfaces, have been widely studied and applied in mathematics and engineering. The generation and machining of ruled surfaces play an important role in design and manufacturing of products and many other areas (RAVANI & KU 1991, ZHA 1997).

Dual numbers, dual vectors, dual inner products, dual cross product, dual angle, E. Study Mapping, etc. are the most important notions on the geometry of lines and kinematics. E. Study mapping plays a fundamental role between the real and dual Lorentzian spaces. By this mapping, there exist one-to-one correspondence between the vectors of dual unit sphere S^2 and the directed lines of the space of lines E^3 (STUDY 1903). Therefore, the motion locus of a straight line in E^3 can be described by that of a point on the surface of dual unit sphere S^2 in dual space D^3. Then a ruled surface in E^3

corresponds to a unique dual curve on the surface of S^2 (GUGENHEIMER 1977). Therefore, the generation of a ruled surface can be converted into the determination of a unique corresponding spherical curve.

The correspondence of E. Study's mapping states that there is a one-to-one correspondence between an oriented straight timelike line in 3-dimensional Minkowski space IR_1^3 and a dual point on the surface of a dual hyperbolic unit sphere $\tilde{H}_0^2$ in 3-dimentional dual Lorentzian space D_1^3 (UĞURLU & ÇALIŞKAN 1996). Thus, a timelike ruled surface in IR_1^3 corresponds to a unique dual curve on the surface of $\tilde{H}_0^2$.

In this study, using methods given in (KÖSE 1999), we determine a method of determination of a developable timelike ruled surface and obtain a linear differential equation of first order.

2. DUAL NUMBERS AND DUAL LORENTZIAN VECTORS

In this section we give a brief summary of the theory of dual numbers and dual Lorentzian vectors.

Let IR_1^3 be a 3-dimensional Minkowski space over the field of real numbers IR with the Lorentzian inner product $<\,,\,>$ given by

$$<a,b> = a_1b_1 + a_2b_2 - a_3b_3,$$

where $a=(a_1,a_2,a_3)$ and $b = (b_1, b_2, b_3) \in IR^3$.

A vector $a=(a_1,a_2,a_3)$ of IR_1^3 is said to be timelike if $<a,a><0$, spacelike if $<a,a>>0$ or $a=0$, and lightlike (or null) if $<a,a>=0$ and $a \neq 0$ (O' NEILL 1983). The norm of a vector a is defined by $|a|=\sqrt{|<a,a>|}$. Now let $a=(a_1, a_2, a_3)$ and $b=(b_1, b_2, b_3)$ be two vectors in IR_1^3. Then the Lorentzian cross product is given by

$$a\times b = (\ a_3b_2 - a_2b_3\ ,\ a_1b_3 - a_3b_1,\ a_1b_2 - a_2b_1)$$

(AKUTAGAWA & NISHIKAWA 1990).

A dual number has the form $\hat{\lambda} := \lambda + \varepsilon\lambda^*$, where λ and λ^* are real numbers, and ε stands for the dual unit which is subject to the rules:

$$\varepsilon \neq 0\ ,\ \ \varepsilon^2 = 0,\ \ 0\varepsilon = \varepsilon 0 = 0,\ \ 1\varepsilon = \varepsilon 1 = \varepsilon.$$

We denote the set of dual numbers by D:

$$D = \left\{ \hat{\lambda} = \lambda + \varepsilon\lambda^* \middle| \lambda, \lambda^* \in IR,\ \ \varepsilon^2 = 0 \right\}.$$

Addition and multiplication are defined in D by

$$(\lambda+\varepsilon\lambda^*)+(\beta+\varepsilon\beta^*)=(\lambda+\beta)+\varepsilon(\lambda^*+\beta^*),$$

and

$$(\lambda+\varepsilon\lambda^*)\cdot(\beta+\varepsilon\beta^*)=\lambda\beta+\varepsilon(\lambda\beta^*+\lambda^*\beta),$$

respectively. Then it is easy to show that $(D,+,.)$ is a commutative ring with unity.

Now let f be a differentiable function. Then the Maclaurin series generated by f is

$$f(\hat{x})= f(x+\varepsilon x^*)= f(x)+\varepsilon x^* f'(x),$$

where $f'(x)$ is the derivative of f.

Let D^3 be the set of all triples of dual numbers, i.e.

$$D^3 = \left\{ \tilde{a} = (A_1,\ A_2,\ A_3) \middle| A_i \in D,\ \ i = 1,\ 2,\ 3 \right\}.$$

The elements of D^3 are called as dual vectors. A dual vector $\tilde{a}$ may be expressed in the form $\tilde{a} = a + \varepsilon a^*$, where a and a^* are the vectors of IR^3.

Now let $\tilde{a} = a + \varepsilon a^*$, $\tilde{b} = b + \varepsilon b^* \in D^3$ and $\hat{\lambda} = \lambda_1 + \varepsilon \lambda_1^* \in D$. Let us define

$$\tilde{a} + \tilde{b} = a + b + \varepsilon (a^* + b^*),$$

$$\hat{\lambda}\, \tilde{a} = \lambda_1 a + \varepsilon (\lambda_1 a^* + \lambda_1^* a).$$

Then D^3 is a module together with these operations. It is called as D-module or dual space.

The Lorentzian inner product of dual vectors $\tilde{a} = a + \varepsilon a^*$ and $\tilde{b} = b + \varepsilon b^*$ is defined by

$$< \tilde{a}, \tilde{b} > = < a, b > + \varepsilon (< a, b^* > + < a^*, b >),$$

where $< a, b >$ is the Lorentzian inner product of the vectors a and b in the Minkowski 3-space IR_1^3 (UĞURLU&ÇALIŞKAN 1996).

A dual vector $\tilde{a} = a + \varepsilon a^*$ is said to be timelike if $< a, a > < 0$, spacelike if $< a, a > > 0$ or $a = 0$ and lightlike (or null) if $< a, a > = 0$ and $a \neq 0$.

The set of dual timelike, spacelike and lightlike vectors is called dual Lorentzian space and it is denoted by D_1^3, i.e.

$$D_1^3 = \left\{ \tilde{a} = a + \varepsilon a^* \middle| a,\ a^* \in IR_1^3 \right\}.$$

The Lorentzian cross product of dual vectors $\tilde{a}$ and $\tilde{b} \in D_1^3$ is defined by

$$\tilde{a} \times \tilde{b} = a \times b + \varepsilon (a^* \times b + a \times b^*),$$

where $a \times b$ is the Lorentzian cross product in IR_1^3 (UĞURLU & ÇALIŞKAN 1996).

Lemma 2.1. Let $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in D_1^3$. Then we have

$$< \tilde{a} \times \tilde{b}, \tilde{c} > = -\det(\tilde{a}, \tilde{b}, \tilde{c})$$

$$\tilde{a} \times \tilde{b} = -\tilde{b} \times \tilde{a},$$

$$(\tilde{a} \times \tilde{b}) \times \tilde{c} = -< \tilde{a}, \tilde{c} > \tilde{b} + < \tilde{b}, \tilde{c} > \tilde{a},$$

$$< \tilde{a} \times \tilde{b}, \tilde{c} \times \tilde{d} > = -< \tilde{a}, \tilde{c} > < \tilde{b}, \tilde{d} > + < \tilde{a}, \tilde{d} > < \tilde{b}, \tilde{c} >,$$

$$< \tilde{a} \times \tilde{b}, \tilde{a} > = 0 \text{ ; and } < \tilde{a} \times \tilde{b}, \tilde{b} > = 0$$

(UĞURLU & ÇALIŞKAN 1996).

Let $\tilde{a} = a + \varepsilon a^* \in D_1^3$. Then $\tilde{a}$ is said to be dual timelike (resp. spacelike) unit vector if the vectors a and a^* satisfy the following equations

$$< a, a > = -1,\ \ < a, a^* > = 0\ \ (\text{resp.} < a, a > = 1,\ \ < a,\ a^* > = 0).$$

The set of all dual timelike unit vectors is called the dual hyperbolic unit sphere and is denoted by $\tilde{H}_0^2$:

$$\tilde{H}_0^2 = \left\{ \tilde{a} = a + \varepsilon a^* \in IR_1^3 \middle| < a,\ a > = -1 \right\}.$$

Theorem 2.2 (E. Study's Mapping) The dual timelike unit vectors of the dual hyperbolic unit sphere $\tilde{H}_0^2$ are in one to one correspondence with the directed timelike lines of the Minkowski 3-space IR_1^3 (UĞURLU & ÇALIŞKAN 1996).

3. THE DUAL VECTOR FORMULATION

By using the dual vector representation, the Plücker vectors x and $p\times x$ of a timelike line L can be collected into a single dual timelike vector $\tilde{x} = x + \varepsilon p \times x = x + \varepsilon x^*$, where x is the direction vector of L and p is the position vector of any point on L.
A timelike ruled surface $m(t,u) = p(t) + u\,x(t)$ is written as the dual vector function $\tilde{x}(t)$ given by

$$\tilde{x}(t) = x(t) + \varepsilon p(t) \times x(t) = x(t) + \varepsilon x^*(t). \tag{1}$$

Since the spherical image of $x(t)$ is a timelike unit vector, the dual vector $\tilde{x}(t)$ also has unit magnitude:

$$\begin{aligned} <\tilde{x}, \tilde{x}> &= < x + \varepsilon p \times x,\ x + \varepsilon p \times x > \\ &= < x, x > + < 2\varepsilon x,\ p \times x > + \varepsilon^2 < p \times x,\ p \times x > \\ &= < x, x > = -1. \end{aligned}$$

Thus, a timelike ruled surface can be represented by a dual curve on the surface of a dual hyperbolic unit sphere.
The dual arc-length of a timelike ruled surface $\tilde{x}(t)$ is defined by

$$\tilde{s}(t) = \int_0^t \left\| \frac{d\tilde{x}}{dt} \right\| dt. \tag{2}$$

The integrant of Equation (2) is the dual speed , $\tilde{\delta}$, of $\tilde{x}(t)$ and is

$$\tilde{\delta} = \left\| \frac{d\tilde{x}}{dt} \right\| = \left\| \frac{dx}{dt} \right\| \left(1 + \varepsilon \frac{< \frac{dx}{dt}, \frac{dp}{dt} \times x >}{\left\| \frac{dx}{dt} \right\|^2} \right),$$

or

$$\tilde{\delta} = \left\| \frac{dx}{dt} \right\| \left(1 + \varepsilon \frac{< \frac{dx}{dt}, \frac{dx^*}{dt} >}{\left\| \frac{dx}{dt} \right\|^2} \right) = \delta(1 + \varepsilon \Delta). \tag{3}$$

The curvature function

$$\Delta = \frac{< \frac{dx}{dt}, \frac{dp}{dt} \times x >}{\left\| \frac{dx}{dt} \right\|^2} = \frac{< \frac{dx}{dt}, \frac{dx^*}{dt} >}{\left\| \frac{dx}{dt} \right\|^2} \tag{4}$$

is the well-known distribution parameter of the timelike ruled surface.

The Gaussian curvature of a surface is given by $K = \dfrac{eg - f^2}{EG - F^2}$, where $E =< m_t, m_t >$, $F =< m_t, m_u >, G =< m_u, m_u >$ are the first fundamental coefficients and $e =< m_{tt}, \dfrac{m_t \times m_u}{\|m_t \times m_u\|} >$, $f =< m_{tu}, \dfrac{m_t \times m_u}{\|m_t \times m_u\|} >$, $g =< m_{uu}, \dfrac{m_t \times m_u}{\|m_t \times m_u\|} >$ are the second fundamental coefficients. Then the relation between the Gaussian curvature K and the distribution parameter Δ of a timelike ruled surface $m(t, u)$ is given by (KASAP et al. 2005)

$$K = \frac{\Delta^2}{\left(\Delta^2 + u^2\right)^2} \quad . \tag{5}$$

If K is zero everywhere, that is, Δ is zero everywhere then the timelike ruled surface is said to be developable.

4. THE DETERMINATION OF A DEVELOPABLE TIMELIKE RULED SURFACE

The dual coordinates $\tilde{x}_i = x_i + \varepsilon x_i^*$ $(i = 1,\ 2,\ 3)$ of an arbitrary point $\tilde{x}$ of the dual hyperbolic unit sphere $\tilde{H}_0^2$, centered at the origin, may be expressed as

$$\begin{aligned} \tilde{x}_1 &= x_1 + \varepsilon x_1^* = \sinh\tilde{\theta}\cos\tilde{\varphi} \ , \\ \tilde{x}_2 &= x_2 + \varepsilon x_2^* = \sinh\tilde{\theta}\sin\tilde{\varphi}, \\ \tilde{x}_3 &= x_3 + \varepsilon x_3^* = \cosh\tilde{\theta} \end{aligned} \tag{6}$$

where $\tilde{\theta} = \theta + \varepsilon\theta^*$ and $\tilde{\varphi} = \varphi + \varepsilon\varphi^*$ are dual hyperbolic and spacelike angles with $\theta \in IR$ and $0 \le \varphi \le 2\pi$, respectively.

Since $\varepsilon^2 = \varepsilon^3 = \cdots = 0$, according to the Taylor series expansion from Equation (6), we obtain

$$\begin{aligned} \tilde{x}_1 = x_1 + \varepsilon x_1^* &= \sinh(\theta + \varepsilon\theta^*)\cos(\varphi + \varepsilon\varphi^*) \\ &= (\sinh\theta + \varepsilon\theta^*\cosh\theta)(\cos\varphi - \varepsilon\varphi^*\sin\varphi) \\ &= \sinh\theta\cos\varphi + \varepsilon(\theta^*\cosh\theta\cos\varphi - \varphi^*\sinh\theta\sin\varphi), \\ \tilde{x}_2 = x_2 + \varepsilon x_2^* &= \sinh(\theta + \varepsilon\theta^*)\sin(\varphi + \varepsilon\varphi^*) \\ &= (\sinh\theta + \varepsilon\theta^*\cosh\theta)(\sin\varphi + \varepsilon\varphi^*\cos\varphi) \\ &= \sinh\theta\sin\varphi + \varepsilon(\theta^*\cosh\theta\sin\varphi + \varphi^*\sinh\theta\cos\varphi), \end{aligned}$$

and

$$\begin{aligned} \tilde{x}_3 = x_3 + \varepsilon x_3^* &= \cosh(\theta + \varepsilon\theta^*) \\ &= \cosh\theta + \varepsilon\theta^*\sinh\theta. \end{aligned}$$

Then we obtain the real parts as

$$\begin{aligned} x_1 &= \sinh\theta\cos\varphi, \\ x_2 &= \sinh\theta\sin\varphi, \\ x_3 &= \cosh\theta, \end{aligned} \tag{7}$$

and the dual parts as

$$x_1^* = \theta^* \cosh\theta \ \ \cos\varphi - \varphi^* \sinh\theta \sin\varphi,$$
$$x_2^* = \theta^* \cosh\theta \sin\varphi + \varphi^* \sinh\theta \cos\varphi, \tag{8}$$
$$x_3^* = \theta^* \sinh\theta.$$

Now let us consider a dual curve $\tilde{x}(t) = x(t) + \varepsilon x^*(\mathrm{t})$ on the dual hyperbolic unit sphere corresponding to a timelike ruled surface $m(t, u) = p(t) + u\, x(t)$ in IR_1^3. Then we write

$$\tilde{x}(t) = x(t) + \varepsilon x^*(\mathrm{t}) = \left(\sinh\theta(t)\cos\varphi(t)\ ,\ \sinh\theta(t)\sin\varphi(t)\ ,\ \cosh\theta(t)\right)$$
$$+\ \varepsilon\Big(\theta^*(t)\ \cosh\theta(t)\cos\varphi(t) - \varphi^*(t)\ \sinh\theta(t)\ \sin\varphi(t),\ \theta^*(t)\ \cosh\theta(t)\ \sin\varphi(t) \tag{9}$$
$$+\varphi^*(t)\sinh\theta(t)\ \cos\varphi(t),\ \theta^*(t)\ \sinh\theta(t)\Big).$$

Since $x^*(t) = p(t) \times x(t)$, we have the following system of linear equations in $p_1,\ p_2$ and p_3 (where $p_1,\ p_2$ and p_3 are the coordinates of $p(t)$):

$$-p_2\cosh\theta + p_3 \sinh\theta\ \sin\varphi = \ \theta^*\cosh\theta\ \cos\varphi - \varphi^* \sinh\theta\ \sin\varphi$$
$$p_1\cosh\theta - p_3\ \sinh\theta\ \cos\varphi\ = \ \theta^* \cosh\theta\ \sin\varphi + \varphi^* \sinh\theta\ \cos\varphi$$
$$p_1 \sinh\theta \sin\varphi - p_2\sinh\theta\ \cos\varphi = \theta^* \sinh\theta$$

The matrix of coefficients of unknowns p_1, p_2 and p_3 is the Lorentzian skew-symmetric matrix

$$\begin{pmatrix} 0 & -\cosh\theta & \sinh\theta\sin\varphi \\ \cosh\theta & 0 & -\sinh\theta\cos\varphi \\ \sinh\theta\sin\varphi & -\sinh\theta\cos\varphi & 0 \end{pmatrix}$$

and therefore its rank is 2 with $\theta(t) \in IR$. Also the rank of the augmented matrix

$$\begin{pmatrix} 0 & -\cosh\theta & \sinh\theta\cos\varphi & \theta^*\cosh\theta\cos\varphi - \varphi^*\sinh\theta\sin\varphi \\ \cosh\theta & 0 & -\sinh\theta\cos\varphi & \theta^*\cosh\theta\sin\varphi + \varphi^*\sinh\theta\cos\varphi \\ \sinh\theta\sin\varphi & -\sinh\theta\cos\varphi & 0 & \theta^*\sinh\theta \end{pmatrix}$$

is 2. Hence this system has infinite solutions given by

$$p_1 = (p_3 + \varphi^*)\tanh\theta\ \cos\varphi + \theta^*\sin\varphi,$$
$$p_2 = (p_3 + \varphi^*)\ \ \sinh\theta\ \sin\varphi - \theta^* \cos\varphi\ , \tag{10}$$
$$p_3 = p_3.$$

Since $p_3(t)$ can be chosen arbitrarily, then we may take $p_3(t) = -\varphi^*(t)$. In this case, Equation (10) reduces to

$$p_1 = \theta^* \sin\varphi,$$
$$p_2 = -\theta^* \cos\varphi, \tag{11}$$
$$p_3 = -\varphi^*.$$

The distribution parameter of the timelike ruled surface given by Equation (9) is

$$\Delta = \frac{<\dfrac{dx}{dt}\ ,\ \dfrac{dx^*}{dt}>}{\left\|\dfrac{dx}{dt}\right\|^2} = \frac{\dfrac{d\theta^*}{dt}\dfrac{d\theta}{dt} + \dfrac{d\varphi}{dt}\ \dfrac{d\varphi^*}{dt}\sinh^2\theta + \theta^*\left(\dfrac{d\varphi}{dt}\right)^2 \sinh\theta\cosh\theta}{\left(\dfrac{d\theta}{dt}\right)^2 + \sinh^2\theta\ \left(\dfrac{d\varphi}{dt}\right)^2} \tag{12}$$

If this ruled surface is a developable one, then $\Delta = 0$ and then Equation (12) becomes

$$\frac{d\theta^*}{dt}\frac{d\theta}{dt} + \frac{d\varphi}{dt}\frac{d\varphi^*}{dt}\sinh^2\theta + \theta^*\left(\frac{d\varphi}{dt}\right)^2 \sinh\theta\cosh\theta = 0. \qquad (13)$$

Dividing the Equation (13) by $\sinh^2\theta$ gives

$$\frac{d\theta^*}{dt}\frac{d}{dt}(\coth\theta) - \theta^*\left(\frac{d\varphi}{dt}\right)^2 \coth\theta - \frac{d\varphi}{dt}\frac{d\varphi^*}{dt} = 0.$$

Setting

$$y(t) = \coth\theta,\ A(t) = \frac{\theta^*\left(\frac{d\varphi}{dt}\right)^2}{\frac{d\theta^*}{dt}}, \text{ and } B(t) = \frac{\frac{d\varphi}{dt}\frac{d\varphi^*}{dt}}{\frac{d\theta^*}{dt}},$$

we are lead to a linear differential equation of first order

$$\frac{dy}{dt} - A(t)y - B(t) = 0. \qquad (14)$$

In the case that the hyperbolic angle $\theta(t)$ and spacelike angle $\varphi(t)$ are both constant, this equation is identically zero, that is, the ruled surface $\tilde{x}(t)$ is a timelike cylinder .

Let $p(t)$ be a curve. Then we can find a developable timelike ruled surface such that its base curve is the curve $p(t)$. In fact from Equation (11), we have

$$\tan\varphi = -\frac{p_1}{p_2},\ \theta^* = \pm\sqrt{p_1^2 + p_2^2}, \text{ and } \varphi^* = -p_3$$

Now only $\varphi(t)$ remains to be determined. The solution of the linear differential Equation (14) gives $\coth\theta$. This solution includes an integral constant. Therefore we have infinitely many developable timelike ruled surfaces such that its base curve is $p(t)$.

It is to be noted that $\theta^*(t)$ has two values; when we use the minus sign we obtain the reciprocal of the timelike ruled surface $\tilde{x}(t)$ obtained by using the plus sign for a given integral constant.

REFERENCES

AKUTAGAWA K, NISHIKAWA S, 1990. The Gauss map and space-like surfaces with prescribed mean curvature in Minkowski 3-space, *Tohoku Mathematical Journal*, 42, 67–82.

GUEGENHEIMER H, 1977. *Differential Geometry*, Dover, New York, pp.378.

KASAP E, AYDEMİR İ, KURUOĞLU N, 2005. Erratum to: ‘‘Ruled surfaces with timelike rulings’’ [Appl. Math. Comput. 147(2004) 241–248], *Applied Mathematics and Computation*. 168, 1461-1468.

KÖSE Ö, 1999. A Method of the Determination of a Developable Ruled Surface, *Mechanism and Machine Theory,* 34, 1187-1193.

O'NEILL B, 1983. *Semi-Riemannian Geometry, With Applications to Relativity*, Academic Press Inc., London, pp.468.

RAVANI B, KU TS, 1991. Bertrand offsets of ruled and developable surfaces, *Computer Aided Geometric Design. Elsevier* , 23(2), 145-152.

STUDY E, 1903. *Geometrie der Dynamen,* Leipzig.

UĞURLU HH, ÇALIŞKAN A, 1996. The Study Mapping for Directed Spacelike and Timelike Lines in Minkowski 3-Space IR_1^3 ,*Mathematical and Computational Applications,* 1(2), 142-148.

ZHA XF, 1977. A New Approach to Generation of Ruled Surface and its Applications in Engineering, *International Journal of Advanced Manufacturing Technology,* 13, 155-163.