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BISHOP FRAME OF THE TIMELIKE CURVE IN MINKOWSKI 3-SPACE

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Abstract: In this study, we study the result which are obtained by BISHOP (1975) for a timelike curve in Minkowski 3-Space.In addition , the Darboux vector(matrix) for the timelike curve is found. Furthermore, using the derivative of the tangent vector T of the timelike curve, the relations between the curvature funtions κ, τ and k_1, k_2 are found.

Key words: Bishop Frame, Natural Frenet Frame, Fermi-Walker Frame (RPAF),Timelike Curve, Minkowski 3-Space.

MINKOWSKI 3-UZAYINDA TİMELİKE EĞRİNİN BİSHOP ÇATISI

Özet: Biz bu çalışmada, BISHOP (1975) tarafından elde edilen sonucu, Minkowski 3-uzayında timelike eğriler için çalıştık. İlaveten, timelike eğriler için Darboux vectörü(matrisi) bulundu. Ayrıca, timelike eğrinin T teğet vektörünün türevi kullanılarak κ, τ ve k_1, k_2 eğrilik fonksiyonları arasındaki ilişkiler bulundu.

Anahtar kelimeler: Bishop Çatısı, Doğal Frenet Çatısı, Fermi-Walker Çatısı, Timelike Eğri, Minkowski 3-uzayı.

1. PRELIMINARIES

Let $R^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in R\}$ be a 3-dimensional vector space, and let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in R^3. The Lorentz scalar product of and y is defined by

$$\langle x, y \rangle_L = -x_1 y_1 + x_2 y_2 + x_3 y_3 ,$$

$R_1^3 = (R^3, \langle x, y \rangle_L)$ is called 3-dimensional Lorentzian space or Minkowski 3-Space.The vector x in R_1^3 is called a spacelike vector, a null vector or a timelike vector if $\langle x, y \rangle_L > 0$ or $x = 0$, $\langle x, y \rangle_L = 0$ and $x \neq 0$ or $\langle x, y \rangle_L < 0$, respectively. For $x \in R_1^3$, the norm of the vector x defined by $\|x\|_L = \sqrt{|\langle x, x\rangle_L|}$, and x is called a unit vector if $\|x\|_L = 1$ (PETROVIC-TORGASEV & SUCUROVIC 2001). For any $x, y \in R_1^3$, Lorentzian vectoral product of x and y is defined by

$$x \wedge_L y = \left(x_3 y_2 - x_2 y_3, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1\right).$$

The equations $C.P = \vec{c} \wedge_L \vec{p}$ and $\vec{c} \wedge_L (\vec{c} \wedge_L \vec{p}) = -\langle \vec{c}, \vec{p}\rangle_L \vec{c} + \langle \vec{c}, \vec{c}\rangle_L \vec{p}$ are valid (KARACAN 2004) . Let P be a point in Lorentzian plane and $r > 0$. The curve $\left\{X \in R_1^2 \,\middle|\, \|PX\|_L = r\right\}$ has two branches and each of them is called Lorentzian circle with center P and radius r (YUCE & KURUOGLU 2006).The Lorentzian sphere of center $m = (m_1, m_2, m_3)$ and radius $r \in R^+$ in the Minkowski 3-space is defined by

$$S_1^2 = \left\{a = (a_1, a_2, a_3) \in R_1^3 \,\middle|\, \langle a - m, a - m\rangle_L = r^2\right\}.$$

Denote by $\{T, N, B\}$ the moving Frenet frame along the curve α. Then T, N and B are the tangent, the principal normal and the binormal vector of the curve α, respectively. If α is a timelike curve, then this set of orthogonal unit vectors, known as the Frenet-Serret frame, has the following properties (PETROVIC-TORGASEV & SUCUROVIC 2001):

$$T' = \kappa N \;, N' = \kappa T + \tau B, \; B' = -\tau N,$$
$$\langle T, T\rangle = -1, \langle N, N\rangle = 1, \; \langle B, B\rangle = 1.$$

2. INTRODUCTION

The Frenet frame of a 3-times continuously differentiable non-degenerate timelike curve invariant under semi-Euclidean space has long been the standart vehicle for analyzing properties of the timelike curve invariant under semi-Euclidean motions. For arbitrary moving frames that is, orthonormal basis fields, we can express the derivatives of the frame with respect to the timelike curve parameter in term of the frame its self, and due to semi-orthonormality the coefficient matrix is always semi- skew symmetric. Thus it generally has three nonzero entries. The Frenet frame gains part of its special significance from the fact that one of the three derivatives is zero. Another feature of the Frenet frame is that it is adapted to the timelike curve: the members are either tangent to or perpendicular to the timelike curve. It is the purpose of this paper to show that there are other frames which have these same advantages and to compare them with the Frenet frame.

3. PARALLEL FIELDS

3.1. Relatively Parallel Fields: We say that a normal vector field N along a curve α is relatively parallel if its derivative is tangential. Such a vector field turns only whatever amount is necessary for it to remain normal, so it is as close to being parallel as possible without losing normality. Since its derivative is perpendicular to it, a relatively parallel normal vector field has constant length. Such fields occur classically in the discussion of curves which are said to be parallel to given curve. Indeed, if α is a curve, considered as a displacement vector function of a parameter t, then if N is relatively parallel, the curve with displacement vector $\alpha+N$ has velocity $(\alpha+N)'=(v+f)T$, where T is the unit tangent vector field of α, v is the speed of α and $N'=fT$. Thus the segment between the two curves are perpendicular to both. Whether or not this segment is locally a segment of minimum length between the two curves depends on the curvature and the length of N. It is easily verified that the segment local minimizes length if N is short enough. Conversely, a curve which runs at constant distance from α must be given by $\alpha+N$, where N is relatively parallel.

A single normal vector field N_0 at a point $\alpha(t_0)$ generates a unique relatively parallel field N such that $N(t_0)=N_0$. The uniqueness is trivial: the difference of two relatively parallel fields is obviously relatively parallel, so if two such coincide at one point, their difference has constant length 0. To show existence one takes auxiliary adapted frames; the Frenet frame would do if it exists, but we want existence even for degenerate curves, that is, those which have curvature vanishing at some points. Such frames can be constructed locally by appling the Gram-Schmidt process to T and two parallel fields.

Theorem 3.1. Let α be a timelike curve. If $\{T,N_1,N_2\}$ is an adapted frame, then the derivatives of the frame with respect to the curve parameter are as the following:

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix}=\begin{bmatrix} 0 & \rho_{01} & \rho_{02} \\ \rho_{01} & 0 & \rho_{12} \\ \rho_{02} & \rho_{12} & 0 \end{bmatrix}\begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}.$$

Proof. We can write

$$T'=\rho_{00}T+\rho_{01}N_1+\rho_{02}N_2 \tag{3.1}$$

for some the functions ρ_{00},ρ_{01} and ρ_{02}. From equation (3.1), we find

$$-\rho_{00}=\langle T',T\rangle_L=0,$$
$$\rho_{01}=\langle T',N_1\rangle_L,$$
$$\rho_{02}=\langle T',N_2\rangle_L.$$

So we get

$$T'=0.T+\rho_{01}N_1+\rho_{02}N_2.$$

Similarly, we can write

$$N_1'=\rho_{10}T+\rho_{11}N_1+\rho_{12}N_2$$

and

$$N_2'=\rho_{20}T+\rho_{21}N_1+\rho_{22}N_2,$$

where

$$\rho_{10} = -\langle N_1', T \rangle_L = \langle T', N_1 \rangle_L = \rho_{01},$$
$$\rho_{11} = \langle N_1', N_1 \rangle_L = 0,$$
$$\rho_{12} = \langle N_1', N_2 \rangle_L$$

and

$$\rho_{20} = -\langle N_2', T \rangle_L = \langle T', N_2 \rangle_L = \rho_{02},$$
$$\rho_{21} = \langle N_2', N_1 \rangle_L = \langle N_1', N_2 \rangle_L = -\rho_{12},$$
$$\rho_{22} = \langle N_2', N_2 \rangle_L = 0.$$

Thus we get

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & \rho_{01} & \rho_{02} \\ \rho_{01} & 0 & \rho_{12} \\ \rho_{02} & \rho_{12} & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}$$

or shortly,

$$X' = K X .$$

Moreover K is semi skew-matrix for satisyfing $K^T = -\varepsilon K \varepsilon$, where ε is a diagonal matrix whose components are 1,1 and -1.Now we find the condition for a normal field of constant length L to be relatively parallel. There is a smooth function θ such that

$$N = L[N_1 \cos\theta + N_2 \sin\theta].$$

Differentiating, we have

$$N' = L[(\theta' + \rho_{12})(-N_1 \sin\theta + N_2 \cos\theta) + (\rho_{01}\cos\theta + \rho_{02}\sin\theta)T].$$

From this we see that N is relatively parallel iff $\theta' = -\rho_{12}$.

Since there is a solution for θ satisfying any initial condition, this shows that local relatively parallel normal fields exist. To get global existence we can patch together local ones, which exist on a covering by intervals. Smoothness at the points where they link together is a consequence of the uniqueness part. We define a tangential field to be relatively parallel if it is a constant multiple of the unit tangent field T. An arbitrary field is relatively parallel if its tangential and normal components are reletively parallel. We spell out the complete hypotheses for the existence and uniqueness of this fields as follows.

Theorem 3.2. Let α be a C^k timelike curve in Minkowski 3-space which is regular, that is, the velocity never vanishes $(k \geq 2)$. Then for any vector X_0 at $\alpha(t_0)$ there is a unique C^{k-1} relatively parallel field X along α such that $X(t_0) = X_0$ and the scalar product of two relatively parallel fields are constant.

Proof. To prove that the scalar product $\langle X, Y \rangle_L$ of two relatively parallel fields X, Y is constant, we observe that it is trivial for tangential ones and may be verified for the tangential and normal parts separately. Thus we assume X and Y are normal, with derivatives fT and gT. Then the derivative of $\langle X,Y\rangle_L$ is

$$\begin{aligned}\frac{d}{dt}\langle X,Y\rangle_L &= \langle X',Y\rangle_L + \langle X,Y'\rangle_L\\ &= \langle fT,Y\rangle_L + \langle X,gT\rangle_L\\ &= f\langle T,Y\rangle_L + g\langle X,T\rangle_L\\ &= f.0+g.0\\ &= 0,\end{aligned}$$

as desired. Thus, $\langle X,Y\rangle_L$ is constant.

3.2. Special Adapted Frames. It should be clear that the relatively parallel fields on a C^2 regular curve form a 3-dimensional vector space over R with distinguished subspaces consisting of an oriented 1-dimensional tangential part and a 2-dimensional normal part, and there is a Lorentzian scalar product inherited from the pointwise scalar product on the ambient semi-Euclidean space.We call an semi-orthonormal basis of this vector space which fits the two subspaces a relatively parallel adapted frame or RPAF. If we assume that the ambient semi-Euclidean space has a preferred orientation, then so does the normal space of the timelike curve, and we may refer to a properly oriented RPAF. The totality of RPAF'S are in the form of two lorentzian circles,one in each orientation class, since they can be parametrized by the 2-dimensional semi-orthogonal group, according to the following obvious result.

Theorem 3.3. If $\{T,N_1,N_2\}$ is a relatively paralel adapted frame, then the totality RPAF's consists of frames the form $\{T,aN_1+bN_2,cN_1+dN_2\}$, where

$$\begin{bmatrix} a & b\\ c & d\end{bmatrix}$$

runs through semi-orthogonal matrices having constant entries.

Now if $\{T,N_1,N_2\}$ is a RPAF, denoting derivatives with respect to arc length by a dot, we have

$$\begin{bmatrix} T^{\bullet}\\ N_1^{\bullet}\\ N_2^{\bullet}\end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2\\ k_1 & 0 & 0\\ k_2 & 0 & 0\end{bmatrix}\begin{bmatrix} T\\ N_1\\ N_2\end{bmatrix}. \tag{3.2}$$

This shows that we accomplished our original goal of showing that there are other adapted frames which have only two nonzero entires in their Cartan matrices. In fact, given one such RPAF, Theorem 3.3 tells us that possible Cartan matrices for RPAF's are

$$K=\begin{bmatrix}0 & ak_1+bk_2 & ck_1+dk_2\\ * & 0 & 0\\ * & 0 & 0\end{bmatrix}$$

where $*$ denotes an entry which can be determined by using semi skew-symmetry. The Frenet frame has semi-Cartan matrix

$$\begin{bmatrix}0 & \kappa & 0\\ \kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix},$$

and is unique once the orientation of the ambient space and a convention on the sign of the torsion τ have been chosen. The only other possibility for a semi-Cartan matrix with one entry vanishing would be

$$\begin{bmatrix}0 & 0 & f\\ 0 & 0 & g\\ f & -g & 0\end{bmatrix},\begin{bmatrix}0 & f & g\\ f & 0 & 0\\ g & 0 & 0\end{bmatrix}.$$

It is simple to relate the entries of the various semi-Cartan matrices. Indeed,

$$\kappa=\left\|T^{\bullet}\right\|_L=\left\|k_1N_1+k_2N_2\right\|_L=\sqrt{k_1^2+k_2^2}.$$

Writing the principal normal as

$$N=N_1\cos\theta+N_2\sin\theta=\left(\frac{k_1}{\kappa}\right)N_1+\left(\frac{k_2}{\kappa}\right)N_2, \tag{3.3}$$

and differentiating we obtain

$$N^{\bullet}=\kappa T+\tau B=\kappa T+\theta^{\bullet}\left(-N_1\sin\theta+N_2\cos\theta\right).$$

If $\{T,N_1,N_2\}$ is properly oriented, we conclude that $B=-N_1\sin\theta+N_2\cos\theta$ and hence $\theta^{\bullet}=\tau$ .Thus κ and an indefinite integral $\int\tau(s)ds$ are polar coordinates for the curve (k_1,k_2).

4. THE NORMAL DEVELOPMENT OF A TIMELIKE CURVE

We want to view (k_1,k_2) as a sort of invariant of the timelike curve α . This slightly more difficult to conceive than in a the case of (κ,τ), since the RPAF is not unique. However, we have spelled out what degree of freedom there is Theorem 3.3: (k_1,k_2) is determined up to a semi-orthogonal transformation in the non oriented case and up to a semi-rotation about the origin in the oriented case. Thus we must think of (k_1,k_2) as a parametrized (by an arc-length parameter for α) continuous timelike curve in a centro semi-Euclidean plane, that is, a semi-Euclidean plane having a distinguished point. When conceived of in this way we call α the normal development of timelike curve α . This situation is not really so different from the case of the Frenet invariants (κ,τ), because in the non oriented case (κ,τ) and the Frenet frame are determined only up to an action by the two-element group, with the non identity changing the sign of τ and κ. That is, (κ,τ) cannot be distinguished from $(\kappa,-\tau)$. The standart facts about the relation between (κ,τ) and the timelike the curve α as an object of semi-Euclidean geometry correspond to similar facts about (k_1,k_2) and α. The proofs are identical with the Frenet case, and in fact are partly given in unified form in (O'NEILL 1996) .

Theorem 4.1. Two C^2 regular timelike curves in semi-Euclidean space are congurent if and only if they have the same normal development. For any parametrised continous curve in a centro semi-Euclidean plane there is a C^2 regular curve in semi - Euclidean space having the given curve as its normal development.i.e. Two curves are congruent iff they have the same arc-length parametrisation of their curvature and torsion.
The modifications for the oriented case are clear: make both the semi-Euclidean space and the centro semi-Euclidean plane be oriented and congurences be proper.

Theorem 4.2. Let α be a C^2 regular timelike curve. Then the curve α lies on a Lorentzian sphere of radius r and center p iff its normal development lies on a line not through the origin. The distance of this line from the origin and the radius of the Lorentzian sphere are reciprocals.

Proof. $\Rightarrow$: If α lies on a Lorentzian sphere with center p and radius r, then

$$\langle \alpha - p, \alpha - p\rangle_L = r^2. \tag{4.1}$$

Differentiating with respect to arc length gives

$$\langle T, \alpha - p\rangle_L = 0. \tag{4.2}$$

so

$$\alpha - p = fN_1 + gN_2 \tag{4.3}$$

for some functions f,g. From equation (4.3), we get

$$f = \langle \alpha - p, N_1\rangle_L \ , \ g = \langle \alpha - p, N_2\rangle_L \tag{4.4}$$

Derivating equation (4.4), we have

$$\begin{aligned} f^{\bullet} &= \left(\langle \alpha - p, N_1\rangle_L\right)^{\bullet} \\ &= \langle T, N_1\rangle_L + \langle \alpha - p, N_1^{\bullet}\rangle_L \\ &= 0 + \langle \alpha - p, k_1 T\rangle_L \\ &= 0. \end{aligned}$$

This means that $f = constant$.
Similarly, g is constant. Then differentiating equation (4.2) and by using theorem 3.2, we get

$$\langle T, \alpha - p\rangle_L = 0$$
$$\langle T^{\bullet}, \alpha - p\rangle_L + \langle T, T\rangle_L = 0$$
$$\langle k_1 N_1 + k_2 N_2, \alpha - p\rangle_L - 1 = 0$$
$$k_1\langle N_1, \alpha - p\rangle_L + k_2\langle N_2, \alpha - p\rangle_L - 1 = 0$$
$$fk_1 + gk_2 - 1 = 0 \ .$$

That is, (k_1, k_2) is on the line
$$fx + gy - 1 = 0. \tag{4.5}$$

Moreover, distance of line l from the origin is
$$\frac{1}{f^2 + g^2} = \frac{1}{r^2} = d.$$
$\Leftarrow$:Conversely, suppose that $fx + gy - 1 = 0$, where f and g are constant. Let $\overrightarrow{p\alpha} = fN_1 + gN_2$, then
$$\begin{aligned} -p^{\bullet} &= -\alpha^{\bullet} + fN_1^{\bullet} + gN_2^{\bullet} \\ &= -T + fk_1 T + gk_2 T \\ &= [-1 + fk_1 + gk_2]T \\ &= 0.T \\ &= 0, \end{aligned}$$
so p is constant. Moreover, let
$$\left\|\overrightarrow{p\alpha}\right\|_L^2 = \langle \alpha - p, \alpha - p\rangle_L = r^2,$$
then
$$\left(\langle \alpha - p, \alpha - p\rangle_L\right)^{\bullet} = 2\langle T, \alpha - p\rangle_L = 0$$
is obtained. Therefore
$$\|p\alpha\|_L^2 = \langle \alpha - p, \alpha - p\rangle_L = r^2 = \text{constant}.$$
So timelike curve α lies on a Lorentzian sphere of radius r and center p.

Definition 4.1. If a rigid body moves along a timelike curve α which we suppose that the curve is unit speed, then the motion of body consists of translation along timelike curve α and about the rotation timelike curve α .The rotation is determined by an angular velocity vector ϖ which satifies
$$T^{\bullet} = \varpi \wedge_L T,\ N_1^{\bullet} = \varpi \wedge_L N_1,\ N_2^{\bullet} = \varpi \wedge_L N_2.$$
The vector ϖ is called the Darboux vector.

Theorem 4.3. Let ϖ be the Darboux vector of the curve α , $\alpha : I \to R_1^3$ with curvatures $\{k_1, k_2\}$ and W be the matrix which corresponds to ϖ .Then the following hold:

$$\varpi = -k_2 N_1 + k_1 N_2$$

and

$$W = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix}.$$

Proof. First of all let us find the Darboux vector ϖ . Then we write

$$\varpi = aT + bN_1 + cN_2 \tag{4.6}$$

and take cross products with T, N_1 and N_2 to determine a, b and c . So we get

$$\begin{aligned} T^\bullet &= \varpi \wedge_L T = -T \wedge_L (aT + bN_1 + cN_2) \\ &= -b(T \wedge_L N_1) - c(T \wedge_L N_2) \\ k_1 N_1 + k_2 N_2 &= cN_1 - bN_2 \end{aligned}$$

then $c = k_1$ and $b = -k_2$ .Similarly,

$$\begin{aligned} N_1^\bullet &= \varpi \wedge_L N_1 = -N_1 \wedge_L (aT + bN_1 + cN_2) \\ &= -a(N_1 \wedge_L T) - c(N_1 \wedge_L N_2) \\ k_1 T &= cT + aN_2, \; a = 0, \end{aligned}$$

and

$$\begin{aligned} N_2^\bullet &= \varpi \wedge_L N_2 = -N_2 \wedge_L (aT + bN_1 + cN_2) \\ &= -a(N_2 \wedge_L T) - b(N_2 \wedge_L N_1) \\ k_2 T &= -bT + aN_1, \; b = -k_2. \end{aligned}$$

Thus we can write the Darboux vector as follows,

$$\varpi = -k_2 N_1 + k_1 N_2 = (0, -k_2, k_1)_{\{T, N_1, N_2\}}.$$

Moreover we get

$$W = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix}.$$

Theorem 4.4. If T is tangent vector of a regular timelike curve α , then the following formulas hold:

a) $T^\bullet \wedge_L T^{\bullet\bullet} = -\kappa^2 \varpi - (k_2^\bullet k_1 - k_2 k_1^\bullet) T, \; k_1 \neq 0$

b) $\det(T, T^\bullet, T^{\bullet\bullet}) = k_2^\bullet k_1 - k_2 k_1^\bullet$

c) $\dfrac{\det(T, T^\bullet, T^{\bullet\bullet})}{\left\| T^\bullet \wedge_L T^{\bullet\bullet} \right\|_L^2} = \theta^\bullet \; or \; \tau,$

where ϖ is the Darboux vector of the timelike curve α .

Proof. (a) We can write

$$\left\langle T^{\bullet\bullet}, T\right\rangle_L = -\kappa^2,$$
$$\left\langle T^{\bullet\bullet}, N_1\right\rangle_L = k_1^{\bullet},$$
$$\left\langle T^{\bullet\bullet}, N_2\right\rangle_L = k_2^{\bullet},$$

since

$$\begin{aligned} T^{\bullet\bullet} &= k_1^{\bullet} N_1 + k_2^{\bullet} N_2 + k_1(k_1 T) + k_2(k_2 T) \\ &= \left(k_1^2 + k_2^2\right)T + k_1^{\bullet} N_1 + k_2^{\bullet} N_2 \\ &= \kappa^2 + k_1^{\bullet} N_1 + k_2^{\bullet} N_2. \end{aligned}$$

From definition of the Darboux vector ϖ, we write

$$T^{\bullet} = \varpi \wedge_L T.$$

So we get

$$\begin{aligned} T^{\bullet} \wedge_L T^{\bullet\bullet} &= -T^{\bullet\bullet} \wedge_L (\varpi \wedge_L T) \\ &= \left\langle T^{\bullet\bullet}, T\right\rangle_L \varpi - \left\langle T^{\bullet\bullet}, \varpi\right\rangle_L T \\ &= -\kappa^2 \varpi - \left\langle T^{\bullet\bullet}, -k_2 N_1 + k_1 N_2\right\rangle_L T \\ &= -\kappa^2 \varpi + k_2 \left\langle T^{\bullet\bullet}, N_1\right\rangle_L T - k_1 \left\langle T^{\bullet\bullet}, N_2\right\rangle_L T \\ &= -\kappa^2 \varpi - k_2 k_1^{\bullet} T - k_1 k_2^{\bullet} T \\ &= -\kappa^2 \varpi - \left(k_2 k_1^{\bullet} - k_1 k_2^{\bullet}\right)T. \end{aligned}$$

(b) From the last equality we can write

$$\left\langle T^{\bullet} \wedge_L T^{\bullet\bullet}, T\right\rangle_L = \left\langle -\kappa^2 \varpi - \left(k_2 k_1^{\bullet} - k_1 k_2^{\bullet}\right)T, T\right\rangle_L$$

and then we get

$$\begin{aligned} \det\left(T, T^{\bullet}, T^{\bullet\bullet}\right) &= -\kappa^2 \left\langle \varpi, T\right\rangle_L + \left(k_2 k_1^{\bullet} - k_1 k_2^{\bullet}\right)\left\langle T, T\right\rangle_L \\ &= -\kappa^2 \left\langle -k_2 N_1 + k_1 N_2, T\right\rangle_L + \left(k_2 k_1^{\bullet} - k_1 k_2^{\bullet}\right)(-1) \\ &= k_1 k_2^{\bullet} - k_2 k_1^{\bullet}. \end{aligned}$$

(c) From the equality $T^{\bullet} = \varpi \wedge_L T$, we can write the vector $\left(T \wedge_L T^{\bullet}\right)$ as the following :

$$\begin{aligned} T \wedge_L T^{\bullet} &= T \wedge_L (\varpi \wedge_L T) \\ &= -\left\langle T, T\right\rangle_L \varpi + \left\langle T, \varpi\right\rangle_L T \\ &= \varpi + 0.T \\ &= \varpi. \end{aligned}$$

And then if we take the norm of the equality $T \wedge_L T^{\bullet} = \varpi$, we find

$$\left\| T \wedge_L T^{\bullet} \right\|_L^2 = k_1^2 + k_2^2.$$

From equality (3.3), we immediately see that

$$\tan\theta = \frac{k_2}{k_1} \tag{4.7}$$

or

$$\theta = \arctan\left(\frac{k_2}{k_1}\right).$$

Differentiating from equality (4.7), we find

$$\theta^{\bullet} = \frac{\left(\frac{k_2}{k_1}\right)^{\bullet}}{1+\left(\frac{k_2}{k_1}\right)^2}.$$

Thus we have

$$\theta^{\bullet} = \frac{\det\left(T, T^{\bullet}, T^{\bullet\bullet}\right)}{\left\|T^{\bullet} \wedge_L T^{\bullet\bullet}\right\|_L^2} = or\ \tau.$$

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