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ON GENERALIZED ϕ – RECURRENT KENMOTSU MANIFOLDS

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Abstract: The purpose of this paper is to study generalized ϕ – recurrent Kenmotsu manifolds.

Key words: Kenmotsu manifold, generalized recurrent, ϕ – recurrent manifold, Einstein manifold.

GENELLEŞTİRİLMİŞ ϕ – RECURRENT KENMOTSU MANIFOLDLAR

Özet: Bu çalışmanın amacı genelleştirilmiş ϕ – recurrent Kenmotsu manifoldları çalışmaktır.

Anahtar kelimeler: Kenmotsu manifold, genelleştirilmiş ϕ – recurrent manifold, Einstein manifold.

1. INTRODUCTION

A Riemannian manifold (M^n, g) is called generalized recurrent (DE & GUHA 1991) if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = \alpha(X)R(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z],$$

where, α and β are two 1-forms, β is non-zero and these are defined by:

$$\alpha(X) = g(X,A), \ \beta(X) = g(X,B),$$

A and B are vector fields associated with 1-forms α and β, respectively.

ÖZGÜR (2007) studied generalized recurrent Kenmotsu manifolds. He showed that for a generalized recurrent Kenmotsu manifold $\alpha = \beta$.

In their study VENKATESHA & BAGEWADI (2006) studied pseudo-projective ϕ – recurrent Kenmotsu manifolds. It was shown that for a pseudo-projective ϕ – recurrent Kenmotsu manifold is an Einstein manifold and also a space of constant curvature.

Motivated by the above studies, in this paper, we define generalized ϕ – recurrent and generalized concircular ϕ – recurrent Kenmotsu manifolds and obtain some interesting results.

The paper is organized as follows. In Preliminaries, we give a brief account of Kenmotsu manifolds. In Section 3, we show that a generalized $\phi-$recurrent or a generalized concircular $\phi-$recurrent Kenmotsu manifold (M^{2n+1}, g) is an Einstein manifold. We also find some relations between the associated 1-froms α and β for a generalized $\phi-$recurrent and a generalized concircular $\phi-$recurrent Kenmotsu manifold.

2. PRELIMINARIES

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $2n+1$-dimensional almost contact Riemannian manifold, where ϕ is a (1, l)-tensor field, ξ is the structure vector field, η is a 1-form and g is the Riemannian metric. It is well known (ϕ, ξ, η, g)-structure satisfy the conditions (BLAIR 1976)

(2.1) $$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

(2.2) $$\phi^2 X = -X + \eta(X)\xi, \quad g(X,\xi) = \eta(X),$$

(2.3) $$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X and Y on M''. If moreover

(2.4) $$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X,$$

(2.5) $$\nabla_X \xi = X - \eta(X)\xi,$$

where ∇ denotes the Riemannian connection of g hold, then $(M^{2n+1}, \phi, \xi, \eta, g)$ is called a *Kenmotsu manifold.*

In this case, it is well known that KENMOTSU (1972)

(2.6) $$R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$$

(2.7) $$S(X,\xi) = -2n\eta(X),$$

where S denotes the Ricci tensor. From (2.6), it easily follows that

(2.8) $$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X,$$

(2.9) $$R(X,\xi)\xi = \eta(X)\xi - X,$$

(2.10) $$\eta(R(X,Y)V) = \eta(Y)g(X,V) - \eta(X)g(Y,V).$$

Since $S(X,Y) = g(QX,Y)$, we have $S(\phi X, \phi Y) = g(Q\phi X, \phi Y)$, where Q is the Ricci operator.

Using the properties $g(X, \phi Y) = -g(\phi X, Y)$, $Q\phi = \phi Q$, (2.2) and (2.7), we get

(2.11) $$S(\phi X, \phi Y) = S(X,Y) + 2n\eta(X)\eta(Y).$$

Also we have KENMOTSU (1972)

(2.12) $$(\nabla_X \eta)(Y) = g(X,Y) - \eta(X)\eta(Y)..$$

Kenmotsu manifold M^{2n+1} is said to be η-*Einstein* if its Ricci tensor *S* is of the form

(2.13) $$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

for any vector fields *X* and *Y,* where *a* and *b* are functions on M^n.

3. GENERALIZED $\phi -$RECURRENT KENMOTSU MANIFODS

Definition 3.1. Kenmotsu manifold (M^{2n+1}, g) is called generalized $\phi -$recurrent if its curvature tensor R satisfies the condition

(3.1) $$\phi^2((\nabla_W R)(X,Y)Z) = \alpha(W)R(X,Y)Z + \beta(W)[g(Y,Z)X - g(X,Z)Y]$$

where, α and β are two 1-forms, β is non-zero and these are defined by:

(3.2) $$\alpha(W) = g(W,A), \quad \beta(W) = g(W,B)$$

and A, B are vector fields associated with 1-forms α and β, respectively (TAKAHASHI 1977, DE & GUHA 1991).

From (3.1), using (2.2) we have

(3.3)
$$-(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = \alpha(W)R(X,Y)Z + \beta(W)[g(Y,Z)X - g(X,Z)Y]$$

from which it follows that

(3.4)
$$-g((\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)\eta(U) = \alpha(W)g(R(X,Y)Z,U) + \beta(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].$$

Let $\{e_i\}$, i = 1, 2, . . . , 2n+1, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.4) and taking summation over *i,* $1 \le i \le 2n+1$, we get

(3.5) $$-(\nabla_W S)(Y,Z) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) = \alpha(W)S(Y,Z) + 2n\beta(W)g(Y,Z).$$

The second term of (3.5) is reduced to

$$\sum_{i=1}^{2n+1}\eta((\nabla_W R)(e_i,Y)Z)\eta(e_i)=g((\nabla_W R)(\xi,Y)Z,\xi)$$

Using (2.5) and (2.6),we get

$$g((\nabla_W R)(\xi,Y)Z,\xi)=0.$$

So, the equation (3.5) has following form:

$$(\nabla_W S)(Y,Z)=-\alpha(W)S(Y,Z)-2n\beta(W)g(Y,Z).$$

Replacing Z by ξ in (3.5) and using (2.7) we have

$$-(\nabla_W S)(Y,\xi)=2n\alpha(W)\eta(Y)-2n\beta(W)\eta(Y). \tag{3.6}$$

Now we have $(\nabla_W S)(Y,\xi)=\nabla_W S(Y,\xi)-S(\nabla_W Y,\xi)-S(Y,\nabla_W\xi)$. Using (2.5) and (2.7) in the above relation, it follows that

$$(\nabla_W S)(Y,\xi)=-2ng(Y,W)-S(Y,W). \tag{3.7}$$

In view of (3.6) and (3.7) we obtain

$$-2ng(Y,W)-S(Y,W)=2n\eta(Y)(\alpha(W)-\beta(W)). \tag{3.8}$$

Replacing Y by ξ in (3.8) and then using (2.7), we get

$$\beta(W)=\alpha(W). \tag{3.9}$$

So using (3.9) in (3.8) we get

$$S(Y,W)=-2ng(Y,W) \tag{3.10}$$

This leads to the following results:

Theorem 3.1. A generalized $\phi-$recurrent Kenmotsu manifold (M^{2n+1},g) is an Einstein manifold.

Theorem 3.2. Let (M^{2n+1},g) be a generalized $\phi-$recurrent Kenmotsu manifold. Then $\beta=\alpha$.

Now from (3.1) we have

(3.11) $$\begin{aligned}(\nabla_W R)(X,Y)Z &= \eta((\nabla_W R)(X,Y)Z)\xi - \alpha(W)R(X,Y)Z \\ &\quad - \beta(W)[g(Y,Z)X - g(X,Z)Y]\end{aligned}$$

Then by the use of second Bianchi identity ,(3.11) and (3.9) we have

(3.12) $$\begin{aligned}&\alpha(W)R(X,Y)Z + \alpha(W)[g(Y,Z)X - g(X,Z)Y] \\ &+ \alpha(X)R(Y,W)Z + \alpha(X)[g(W,Z)Y - g(Y,Z)W] \\ &+ \alpha(Y)R(W,X)Z + \alpha(Y)[g(X,Z)W - g(W,Z)X] = 0.\end{aligned}$$

So by a suitable contraction from (3.12) we get

(3.13) $$\begin{aligned}&\alpha(W)S(X,U) + 2n\alpha(W)g(X,U) - \alpha(X)S(W,U) - 2n\alpha(X)g(W,U) \\ &- \alpha(R(W,X)U) + \alpha(X))g(W,U) - \alpha(W)g(X,U) = 0.\end{aligned}$$

Hence by the use of (3.9) , (3.10) in (3.13) it can be easily seen that:

(3.14) $$-\alpha(R(W,X)U) + \alpha(X))g(W,U) - \alpha(W)g(X,U) = 0.$$

Replacing X, U by ξ in (3.14), we have

$$\alpha(W) = \alpha(\xi)\eta(W)$$

or,

(3.15) $$\alpha(W) = \eta(A)\eta(W).$$

This leads to the following result:

Theorem 3.2. In a a generalized $\phi-$recurrent Kenmotsu manifold (M^{2n+1}, g), the characteristic vector field ξ and the vector field A associated to the 1-form α are co-directional and the 1-form α is given by (3.15).

Definition 3.2. A Kenmotsu manifold (M^{2n+1}, g) is called generalized concircular $\phi-$recurrent if its concircular curvature tensor $\overline{C}$ (YANO & KON 1984)

(3.17) $$\overline{C}(X,Y)W = R(X,Y)W - \frac{r}{(2n+1)2n}[g(Y,W)X - g(X,W)Y],$$

satisfies the condition

(3.18) $$\phi^2\left((\nabla_W \overline{C})(X,Y)Z\right) = \alpha(W)\overline{C}(X,Y)Z + \beta(W)[g(Y,Z)X - g(X,Z)Y],$$

[5] where α and β are defined as in (3.2) and r is the scalar curvature of (M^n, g).

Let us consider a generalized concircular $\phi-$recurrent Kenmotsu manifold. Then by virtue of (2.2) and (2.16) we have

(3.19)

$$-(\nabla_W \overline{C})(X,Y)Z + \eta((\nabla_W \overline{C})(X,Y)Z)\xi = \alpha(W)\overline{C}(X,Y)Z + \beta(W)[g(Y,Z)X - g(X,Z)Y],$$

from which it follows that

$$\begin{aligned} &-g((\nabla_W \overline{C})(X,Y)Z,U) + \eta((\nabla_W \overline{C})(X,Y)Z)\eta(U) \\ &= \alpha(W)g(\overline{C}(X,Y)Z,U) + \beta(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{3.20}$$

Let $\{e_i\}$, $i = 1, 2, \ldots, 2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (3.20) and taking summation over i, $1 \le i \le 2n+1$, we get

(3.21)

$$\begin{aligned} &-(\nabla_W S)(X,U) + \frac{W(r)}{(2n+1)2n}2ng(X,U) + (\nabla_W S)(X,\xi)\eta(U) - \frac{W(r)}{(2n+1)2n}2n\eta(X)\eta(U) \\ &= \alpha(W)\left[S(X,U) - \frac{r}{2n+1}g(X,U)\right] + 2n\beta(W)g(X,U). \end{aligned}$$

Replacing U by ξ in (3.3) and using (2.1) and (2.7), we have

$$0 = \alpha(W)\left[2n + \frac{r}{2n+1}\right]\eta(X) - 2n\beta(W)\eta(X). \tag{3.22}$$

Putting X=ξ in (3.22), we obtain

$$\alpha(W)\left[2n + \frac{r}{2n+1}\right] - 2n\beta(W) = 0. \tag{3.22}$$

This leads to the following results:

Theorem 3.3. Let (M^{2n+1}, g) be a generalized concircular $\phi-$recurrent Kenmotsu manifold. Then $\alpha(W)\left[2n + \frac{r}{2n+1}\right] - 2n\beta(W) = 0$.

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