Universal Journal of Mathematics and Applications, 3 (3) (2020) 121-128
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.756304

Existence and Uniqueness of Generalised Fractional Cauchy-Type Problem

Ahmad Y. A. Salamooni[1*] and D. D. Pawar[1]

[1] *School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University, Nanded-431606, India*
[*] *Corresponding author*

Article Info

Keywords: *Generalized Fractional Cauchy-type problem, Existence and uniqueness, Hilfer-Hadamard-type fractional derivative, Equivalence with Volterra integral equation.*
2010 AMS: *34A08, 34A34, 34A12.*
Received: *22 June 2020*
Accepted: *22 September 2020*
Available online: *29 September 2020*

Abstract

In this paper, we study the existence and uniqueness of Generalized Fractional Cauchy-type problem involving Hilfer-Hadamard-type fractional derivative for a nonlinear fractional differential equation. Also, we prove an equivalence between the Cauchy-type problem and Volterra integral equation(VIE).

1. Introduction

We consider the Cauchy-type problem

$$\begin{cases} {}_H D_{a+}^{\alpha,\beta} x(t) = \varphi(t, x(t)), \quad n-1 < \alpha < n, 0 \le \beta \le 1, \\ {}_H D_{a+}^{\gamma-j} x(t)\big|_{t=a} = x_{a_j}, \quad (j = 1, 2, ..., n), \quad \gamma = \alpha + \beta(n-\alpha). \end{cases} \tag{1.1}$$

From the above initial condition and by definition 2.3(in this paper), it is clear that

$${}_H D_{a+}^{\gamma-j} x(t) = \delta^{n-j}\ {}_H I_{a+}^{n-\gamma} x(t),$$

where ${}_H D_{a+}^{\alpha,\beta}$ is the Hilfer-Hadamard-type fractional derivative of order α and type β [1,2] Fractional differential equations have numerous applications in science, physics, chemistry, and engineering [3,6].
Recently, the theory and applications of fractional derivatives have received considerable attention by researchers. They have studied some results of the existence and uniqueness of solutions for fractional differential equations on the different finite intervals such as the examples in [1,21] and references therein.
In this paper, we find a variety of results for the initial values problem (1.1), which are equivalent with (VIE), existence and uniqueness. In section 2, we present some preliminaries. In section 3, we establish the equivalence of the Cauchy-type problem (1.1) and (VIE). In section 4, we prove the existence and uniqueness results for a solution of the Cauchy-type problem (1.1) in the weighted space.

2. Preliminaries

In this section, we introduce some notations, Lemmas, definitions and weighted spaces, which are important for developing some theories in this paper. For further explanations, see [5].
Let $0 < a < b < +\infty$. Assume that $C[a,b], AC[a,b], C^n[a,b]$ and $C^n_\mu[a,b]$ be the spaces of continuous, absolutely continuous, n-times continuous and continuously differentiable functions on $[a,b]$ respectively. And let $L^p(a,b)$ with $p \ge 1$ be the space of Lebesgue integrable

Email addresses: ayousss83@gmail.com (A.Y.A. Salamooni), dypawar@yahoo.com (D.D. Pawar)

functions on (a,b). Moreover, we recall some of weighted spaces [5] in definition 2.1.
Definition 2.1 [5] Let $\Omega=[a,b]$ $(0<a<b<+\infty)$ is a finite interval and $0\leq\mu<1$. We introduce the weighted space $C_{\mu,\log}[a,b]$ of continuous functions φ on $(a,b]$

$$C_{\mu,\log}[a,b]=\{\varphi:(a,b]\to\mathbb{R}:[\log(t/a)]^{\mu}\varphi(t)\in C[a,b]\}$$

with the norm

$$\|\varphi\|_{C_{\mu,\log}}=\left\|[\log(t/a)]^{\mu}\varphi(t)\right\|_{C},\quad C_{0,\log}[a,b]=C[a,b].$$

And for $n\in\mathbb{N}$ and $\delta=t\frac{d}{dt}$, we have

$$C^{n}_{\delta,\mu}[a,b]=\left\{\varphi:\|\varphi\|_{C^{n}_{\delta,\mu}}=\sum_{k=0}^{n-1}\|\delta^{k}\varphi\|_{C}+\|\delta^{n}\varphi\|_{C_{\mu,\log}}\right\},C^{0}_{\delta,\mu}[a,b]=C_{\mu,\log}[a,b].$$

The space $C_{\mu,\log}[a,b]$ is the complete metric space defined with the distance as

$$d(x_1,x_2)=\|x_1-x_2\|_{C_{\mu,\log}}[a,b]:=\max_{t\in[a,b]}\left|[\log(t/a)]^{\mu}\left[x_1(t)-x_2(t)\right]\right|,$$

where $\log(.)=\log_{e}(.)$.
Definition 2.2 [4, 5] Let $0<a<b<+\infty$. The Hadamard fractional integral of order $\alpha\in\mathbb{R}^{+}$ for a function $\varphi:(a,+\infty)\to\mathbb{R}$ is defined as

$${}_{H}I^{\alpha}_{a+}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_{a}^{t}(\log\frac{t}{\tau})^{\alpha-1}\quad\frac{\varphi(\tau)}{\tau}d\tau,\qquad(t>a).$$

Definition 2.3 [4, 5] Let $0<a<b<+\infty$. The Hadamard fractional derivative of order α applied to the function $\varphi:(a,+\infty)\to\mathbb{R}$ is defined as

$${}_{H}D^{\alpha}_{a+}\varphi(t)=\delta^{n}({}_{H}I^{n-\alpha}_{a+}\varphi(t)),\quad n-1<\alpha<n,\quad n=[\alpha]+1,$$

where $\delta^{n}=(t\frac{d}{dt})^{n}$, and $[\alpha]$ denotes the integer part of the real number α.
Lemma 2.4 [5] Let $n\in\mathbb{N}_0=\{0,1,2,...\}$ and let $\mu_1,\mu_2\in\mathbb{R}$ such that $0\leq\mu_1\leq\mu_2<1$. The following embeddings hold:

$$C^{n}_{\delta}[a,b]\longrightarrow C^{n}_{\delta,\mu_1}[a,b]\longrightarrow C^{n}_{\delta,\mu_2}[a,b],$$

with

$$\|\varphi\|_{C^{n}_{\delta,\mu_2}}\leq K_{\delta}\|\varphi\|_{C^{n}_{\delta,\mu_1}},\quad K_{\delta}=\min\left[1,\left(\log(b/a)\right)^{\mu_2-\mu_1}\right],\ a\neq0.$$

In particular,

$$C[a,b]\longrightarrow C_{\mu_1,\log}[a,b]\longrightarrow C_{\mu_2,\log}[a,b]$$

with

$$\|\varphi\|_{C_{\mu_2,\log}}\leq\left(\log(b/a)\right)^{\mu_2-\mu_1}\|\varphi\|_{C_{\mu_1,\log}},\ a\neq0.$$

Lemma 2.5 [5]
(a_1) If $\Re(\alpha)\geq0,\Re(\beta)\geq0$ and $0<a<b<+\infty$, then

$$\left[{}_{H}I^{\alpha}_{a+}\left(\log(\tau/a)\right)^{\beta-1}\right](x)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(\log(t/a))^{\alpha+\beta-1},\quad x>a,$$

$$\left[{}_{H}D^{\alpha}_{a+}\left(\log(\tau/a)\right)^{\beta-1}\right](x)=\frac{\Gamma(\beta)}{\Gamma(\alpha-\beta)}(\log(t/a))^{\alpha-\beta-1},\quad x>a.$$

(a_2) Let $\Re(\alpha)\geq0,n=[\Re(\alpha)]+1$ and $0<a<b<+\infty$. The equality $({}_{H}D^{\alpha}_{a+}x)(t)=0$ is valid if and only if

$$x(t)=\sum_{k=1}^{n}c_k(\log(t/a))^{\alpha-k},$$

where $c_k\in\mathbb{R}(k=1,2,...,n)$ are arbitrary constants.
(a_3) Let $\Re(\alpha)\geq0,\Re(\beta)\geq0$ and $0\leq\mu<1$. If $0<a<b<+\infty$, then for $\varphi\in C_{\mu,\log}[a,b]$

$${}_{H}I^{\alpha}_{a+}\,{}_{H}I^{\beta}_{a+}\varphi={}_{H}I^{\alpha+\beta}_{a+}\varphi$$

holds at any point $t\in(a,b]$. When $\varphi\in C[a,b]$, then this relation will be valid at any point $t \in (a,b]$.

Theorem 2.6 [5] Let $\Re(\alpha) \geq 0, n = [\Re(\alpha)] + 1$, and $0 < a < b < +\infty$. Also, let ${}_H I_{a+}^{n-\alpha}\varphi$ be the Hadamard-type fractional integral of order $n-\alpha$ of the function φ. If $\varphi \in C_{\mu,\log}[a,b]$ $(0 \leq \mu < 1)$ and ${}_H I_{a+}^{n-\alpha}\varphi \in C_{\delta,\mu}^{n}[a,b]$, then

$$({}_H I_{a+}^{\alpha}\, {}_H D_{a+}^{\alpha}\varphi)(t) = \varphi(t) - \sum_{k=1}^{n} \frac{(\delta^{n-k}({}_H I_{a+}^{n-\alpha}\varphi))(a)}{\Gamma(\alpha-k+1)} (\log\frac{t}{a})^{\alpha-k}.$$

Lemma 2.7 [5] Let $0 < a < b < +\infty, \Re(\alpha) \geq 0, n = [\Re(\alpha)] + 1$ and $0 \leq \Re(\mu) < 1$.

(a) If $\Re(\mu) > \Re(\alpha) > 0$, then the fractional integration operator ${}_H I_{a+}^{\alpha}$ is bounded from $C_{\mu,\log}[a,b]$ into $C_{\mu-\alpha,\log}[a,b]$:

$$\| {}_H I_{a+}^{\alpha}\varphi\|_{C_{\mu-\alpha,\log}} \leq k_1 \|\varphi\|_{C_{\mu,\log}},$$

where

$$k_1 = \left(\log(b/a)\right)^{\Re(\alpha)} \frac{\Gamma[\Re(\alpha)]|\Gamma(1-\Re(\mu))|}{|\Gamma(\alpha)|\Gamma(1+\Re(\alpha-\mu))}.$$

In particular, ${}_H I_{a+}^{\alpha}$ is bounded in $C_{\mu,\log}[a,b]$.

(b) If $\Re(\mu) \leq \Re(\alpha)$, then the fractional integration operator ${}_H I_{a+}^{\alpha}$ is bounded from $C_{\mu,\log}[a,b]$ into $C[a,b]$:

$$\| {}_H I_{a+}^{\alpha}\varphi\|_{C} \leq k_2 \|\varphi\|_{C_{\mu,\log}},$$

where

$$k_2 = \left(\log(b/a)\right)^{\Re(\alpha-\mu)} \frac{\Gamma[\Re(\alpha)]|\Gamma(1-\Re(\mu))|}{|\Gamma(\alpha)|\Gamma(1+\Re(\alpha-\mu))}.$$

In particular, ${}_H I_{a+}^{\alpha}$ is bounded in $C_{\mu,\log}[a,b]$.

Definition 2.8 [2] Let $n-1 < \alpha < n, 0 \leq \beta \leq 1$, and $\varphi \in L^1(a,b)$. The Hilfer-Hadamard fractional derivative ${}_H D^{\alpha,\beta}$ of order α and type β of φ is defined as

$$\begin{aligned}({}_H D^{\alpha,\beta}\varphi)(t) &= \left({}_H I^{\beta(n-\alpha)}(\delta)^n\, {}_H I^{(n-\alpha)(1-\beta)}\varphi\right)(t)\\ &= \left({}_H I^{\beta(n-\alpha)}(\delta)^n\, {}_H I^{n-\gamma}\varphi\right)(t); \quad \gamma = \alpha + n\beta - \alpha\beta\\ &= \left({}_H I^{\beta(n-\alpha)}\, {}_H D^{\gamma}\varphi\right)(t),\end{aligned}$$

where ${}_H I^{(\cdot)}$ and ${}_H D^{(\cdot)}$ is the Hadamard fractional integral and derivative defined by definitions 2.2 and 2.3 respectively.

Definition 2.9 [5, 13] Assume that $\varphi(x,y)$ is defined on set $(a,b] \times G, G \subset \mathbb{R}$. The function $\varphi(x,y)$ satisfies Lipschitz condition with respect to y, if for all $x \in (a,b]$ and for all $y_1, y_2 \in G$,

$$|\varphi(x,y_1) - \varphi(x,y_2)| \leq L|y_1 - y_2|,$$

where $L > 0$ is Lipschitz constant.

Definition 2.10 [1, 12] Let $0 < \alpha < 1, 0 \leq \beta \leq 1$. The weighted space $C_{1-\gamma}^{\alpha,\beta}[a,b]$ is defined by

$$C_{1-\gamma}^{\alpha,\beta}[a,b] = \{\varphi \in C_{1-\gamma}[a,b] : D_{a+}^{\alpha,\beta}\varphi \in C_{1-\gamma}[a,b]\}, \gamma = \alpha + \beta - \alpha\beta.$$

Lemma 2.11 [9] Let $0 < a < b < +\infty, \alpha > 0, 0 \leq \mu < 1$ and $\varphi \in C_{\mu,\log}[a,b]$. If $\alpha > \mu$, then ${}_H I_{a+}^{\alpha}\varphi$ is continuous on $[a,b]$ and

$${}_H I_{a+}^{\alpha}\varphi(a) = \lim_{t \to a^+} {}_H I_{a+}^{\alpha}\varphi(t) = 0.$$

Lemma 2.12 [2] Let $\Re(\alpha) > 0, 0 \leq \beta \leq 1, \gamma = \alpha + n\beta - \alpha\beta, n-1 < \gamma \leq n,\ n = [\Re(\alpha)] + 1$ and $0 < a < b < \infty$. If $\varphi \in L^1(a,b)$ and $({}_H I_{a+}^{n-\gamma}\varphi)(t) \in AC_{\delta}^{n}[a,b]$, then

$${}_H I_{a+}^{\alpha}\, ({}_H D_{a+}^{\alpha,\beta}\varphi)(t) = {}_H I_{a+}^{\gamma}\, ({}_H D_{a+}^{\gamma}\varphi)(t) = \varphi(t) - \sum_{j=1}^{n} \frac{(\delta^{(n-j)}({}_H I_{a+}^{n-\gamma}\varphi))(a)}{\Gamma(\gamma-j+1)} (\log\frac{t}{a})^{\gamma-j}.$$

Lemma 2.13 [13] Let $0 < a < b < +\infty, 0 \leq \mu < 1, \varphi \in C_{\mu,\log}[a,c]$ and $\varphi \in C_{\mu,\log}[c,b]$. Then, $\varphi \in C_{\mu,\log}[a,b]$ and

$$\|\varphi\|_{C_{\mu,\log}[a,b]} \leq \max\left\{\|\varphi\|_{C_{\mu,\log}[a,c]}, \left(\log(b/a)\right)^{\mu} \|\varphi\|_{C[c,b]}\right\}.$$

Theorem 2.14 [5] Let (U, d) be a non-empty complete metric space. Let $0 \leq \omega < 1$ and let $\mathrm{T} : \mathrm{U} \to \mathrm{U}$ be the map such that, for every $u, v \in \mathrm{U}$, the relation

$$d(\mathrm{T}u, \mathrm{T}v) \leq \omega\, d(u,v), \qquad 0 \leq \omega < 1$$

holds. Then, the operator T has a unique fixed point $u^* \in \mathrm{U}$.
Furthermore, if $\mathrm{T}^k (k \in \mathbb{N})$ is the sequence of operators defined by

$$\mathrm{T}^1 = \mathrm{T}, \qquad \mathrm{T}^k = \mathrm{T}\mathrm{T}^{k-1} \in \mathbb{N} \setminus \{1\},$$

then, for any $u_0 \in \mathrm{U}$, the sequence $\{\mathrm{T}^k u_0\}_{k=1}^{+\infty}$ converges to the above fixed point u^*.

3. Equivalence of the Cauchy-Type Problem (1.1) and (VIE)

In this section, we are going to prove the equivalence of the Cauchy-type problem (1.1) and (VIE). So, we need the following definition:
Definition 3.1 Let $n-1<\alpha<n,\ 0\leq\beta\leq 1, \gamma=\alpha+n\beta-\alpha\beta$ and $0\leq\mu<1$. We consider the underlying spaces defined by

$$C^{\alpha,\beta}_{\delta;n-\gamma,\mu}[a,b]=\{\varphi\in C_{n-\gamma,\log}[a,b]:\ {}_HD^{\alpha,\beta}_{a+}\varphi\in C_{\mu,\log}[a,b]\}$$

and

$$C^{\gamma}_{n-\gamma,\log}[a,b]=\{\varphi\in C_{n-\gamma,\log}[a,b]:\ {}_HD^{\gamma}_{a+}\varphi\in C_{n-\gamma,\log}[a,b]\},$$

where $C_{n-\gamma,\log}[a,b]$ and $C_{\mu,\log}[a,b]$ are weighted spaces of continuous functions on $(a,b]$ defined by

$$C_{\gamma,\log}[a,b]=\left\{\varphi:(a,b]\to\mathbb{R}:\left(\log t/a\right)^{\gamma}\varphi(t)\in C[a,b]\right\}.$$

In the next theorem, we studied the equivalence between the Cauchy-type problem (1.1) and (VIE) of the second kind

$$x(t)=\sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log(t/a))^{\gamma-k}+\frac{1}{\Gamma(\alpha)}\int_a^t(\log(t/\tau))^{\alpha-1}\varphi(\tau,x(\tau))\frac{d\tau}{\tau},\quad t>a. \tag{3.1}$$

Theorem 3.2 Let $n-1<\alpha<n, 0\leq\beta\leq 1, \gamma=\alpha+\beta(n-\alpha)$, and assume that $\varphi(.,x(.))\in C_{\mu,\log}[a,b]$, where $\varphi:(a,b]\times\mathbb{R}\to\mathbb{R}$ be a function for any $x\in C_{\mu,\log}[a,b](n-\gamma\leq\mu<n-\beta(n-\alpha))$. If $x\in C^{\gamma}_{n-\gamma,\log}[a,b]$, then x satisfies (1.1) if and only if x satisfies the integral equation (3.1).

Proof. First part, we will prove the necessity.
Assume that $x\in C^{\gamma}_{n-\gamma,\log}[a,b]$ is a solution of (1.1). We prove that x is a solution of (3.1) as follows:
By the definition 3.1 of $C^{\gamma}_{n-\gamma,\log}[a,b]$, Lemma 2.7 (b) and definition 2.3, we have

$${}_HI^{n-\gamma}_{a+}x\in C[a,b],\qquad {}_HD^{\gamma}_{a+}x=\delta^n\ {}_HI^{n-\gamma}_{a+}x\in C_{n-\gamma,\log}[a,b].$$

Thus, by definition 2.1, we get

$${}_HI^{n-\gamma}_{a+}x\in C^n_{\delta,n-\gamma}[a,b].$$

Now, by applying Theorem 2.6, we obtain

$${}_HI^{\gamma}_{a+}\ {}_HD^{\gamma}_{a+}x(t)=x(t)-\sum_{k=1}^{n}\frac{(\delta^{n-k}({}_HI^{n-\gamma}_{a+}\varphi))(a)}{\Gamma(\gamma-k+1)}(\log\frac{t}{a})^{\gamma-k},\quad t\in(a,b],$$

or

$${}_HI^{\gamma}_{a+}\ {}_HD^{\gamma}_{a+}x(t)=x(t)-\sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log\frac{t}{a})^{\gamma-k},\quad t\in(a,b], \tag{3.2}$$

where x_{a_k} comes from the initial condition of (1.1). By our hypothesis $\varphi(.,x(.))\in C_{\mu,\log}[a,b]$ and since $x\in C_{n-\gamma,\log}[a,b]\subset C_{\mu,\log}[a,b]$, Lemma 2.7, we can see that the integral ${}_HI^{\alpha}_{a+}\varphi(.,x(.))\in C_{\mu-\alpha,\log}[a,b]$ for $\mu>\alpha$ and ${}_HI^{\alpha}_{a+}\varphi(.,x(.))\in C[a,b]$ for $\mu\leq\alpha$. By applying the operator ${}_HI^{\alpha}_{a+}$ to both sides of the problem of Cauchy-type (1.1) and Lemma 2.12 we obtain

$${}_HI^{\gamma}_{a+}\ {}_HD^{\gamma}_{a+}x=\ {}_HI^{\alpha}_{a+}\ {}_HD^{\alpha,\beta}_{a+}x=\ {}_HI^{\alpha}_{a+}(\ {}_HD^{\alpha,\beta}_{a+}x)=\ {}_HI^{\alpha}_{a+}\varphi. \tag{3.3}$$

From (3.2) and (3.3) we get

$$x(t)=\sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log\frac{t}{a})^{\gamma-k}+\ {}_HI^{\alpha}_{a+}[\varphi(\tau,x(\tau))](t),\quad t\in(a,b], \tag{3.4}$$

which is the (VIE)(3.1).

Second part, we will prove the sufficiency.
Assume that $x\in C^{\gamma}_{n-\gamma,\log}[a,b]$ satisfies (3.1) which is written as (3.4). Then, ${}_HD^{\gamma}_{a+}x$ exists and ${}_HD^{\gamma}_{a+}x\in C_{n-\gamma,\log}[a,b]$. Now, by applying the operator ${}_HD^{\gamma}_{a+}$ to both sides of (3.4), we get

$${}_HD^{\gamma}_{a+}x(t)=\ {}_HD^{\gamma}_{a+}\left[\sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log\frac{t}{a})^{\gamma-k}+\ {}_HI^{\alpha}_{a+}[\varphi(\tau,x(\tau))](t)\right].$$

By using Lemma 2.5 (a_2) and (a_3), and definition 2.3, we obtain

$$\begin{aligned}{}_HD^{\gamma}_{a+}x&=\ {}_HD^{\gamma}_{a+}\left[{}_HI^{\alpha}_{a+}\varphi\right]\\&=\delta^n({}_HI^{n-\gamma}_{a+}\ {}_HI^{\alpha}_{a+}\varphi)\\&=\delta^n({}_HI^{n-\beta(n-\alpha)}\varphi)\\&=\ {}_HD^{\beta(n-\alpha)}_{a+}\varphi\end{aligned} \tag{3.5}$$

From (3.5) and the hypothesis ${}_HD^{\gamma}_{a+}x \in C_{n-\gamma,\log}[a,b]$, we have

$$ {}_HD^{\beta(n-\alpha)}_{a+}\varphi \in C_{n-\gamma,\log}[a,b]. $$

Now, by applying ${}_HI^{\beta(n-\alpha)}_{a+}$ to both sides of (3.5) we obtain

$$ ({}_HI^{\beta(n-\alpha)}_{a+} \, {}_HD^{\gamma}_{a+}x)(t) = ({}_HI^{\beta(n-\alpha)}_{a+} \, {}_HD^{\beta(n-\alpha)}_{a+} \varphi(\tau,x(\tau)))(t); $$

that is,

$$ {}_HI^{\beta(n-\alpha)}_{a+} \delta^n ({}_HI^{n-\gamma}_{a+}x)(t) = ({}_HI^{\beta(n-\alpha)}_{a+} \, {}_HD^{\beta(n-\alpha)}_{a+} \varphi(\tau,x(\tau)))(t). $$

Since

$$ \delta^n ({}_HI^{n-\beta(n-\alpha)}_{a+} \varphi(t,x(t))) = {}_HD^{\beta(n-\alpha)}_{a+} \varphi(.,x(.)) \in C_{n-\gamma,\log}[a,b], $$

and $\gamma > \beta(n-\alpha)$ and by definition 2.1, we have ${}_HI^{n-\beta(n-\alpha)}_{a+}\varphi \in C^n_{\delta;n-\gamma}[a,b]$ (also that which is found in the first part of this proof, or by Lemma 2.7 (b) with $\mu < n-\beta(n-\alpha)$, for a continuity of ${}_HI^{n-\beta(n-\alpha)}_{a+}\varphi$). Then, Theorem 2.6 with definition 2.8 allow us to write

$$ {}_HD^{\alpha,\beta}_{a+}x(t) = \varphi(t,x(t)) - \sum_{k=1}^{n} \frac{(\delta^{n-k}({}_HI^{n-\beta(n-\alpha)}_{a+}\varphi))(a)}{\Gamma(\beta(k-\alpha))} (\log \frac{t}{a})^{\beta(n-\alpha)-k}, \tag{3.6} $$

since $\mu < n-\beta(n-\alpha)$. Then, it follows by Lemma 2.11 that

$$ \left[{}_HI^{n-\beta(n-\alpha)}_{a+}\varphi) \right](a) = 0. $$

Therefore, we can write the relation (3.6) as

$$ {}_HD^{\alpha,\beta}_{a+}x(t) = \varphi(t,x(t)), \qquad t \in (a,b]. $$

Finally, we will show that the initial condition of (1.1) also holds. For that, we apply ${}_HD^{\gamma-j}_{a+} = \delta^{n-j} \, {}_HI^{n-\gamma}_{a+} (j = 1,2,...,n)$ to both sides of (3.4) and by using Lemma 2.5 (a_1) and (a_3), we obtain

$$ {}_HD^{\gamma-j}_{a+}x(t) = x_{a_j} + \left[\delta^{n-j}(\, {}_HI^{n-\beta(n-\alpha)}_{a+}\varphi(\tau,x(\tau))) \right](t) \tag{3.7} $$

Now, taking the limit as $t \to a$, in (3.7), we get

$$ {}_HD^{\gamma-j}_{a+}x(t)\big|_{t=a} = x_{a_j}, \qquad (j = 1,2,...,n). $$

The proof of this theorem is complete.

Remark 3.3 For $0 < \alpha < 1$, Theorem 3.2 is reduced to Theorem 21 (see[9]).

4. Existence and Uniqueness

In this section, we will prove the existence and uniqueness results for a solution of the Cauchy-type problem (1.1) in the weighted space $C^{\alpha,\beta}_{n-\gamma,\log}[a,b]$ by using the Banach fixed point theorem. For that, we need the following Lemma.

Lemma 4.1 If $\mu \in \mathbb{R}(0 \le \mu < 1)$, then the Hadamard-type fractional integral operator ${}_HI^{\alpha}_{a+}$ *with* $\alpha \in \mathbb{C}(\Re(\alpha) > 0)$ is bounded from $C_{\mu,\log}[a,b]$ *into* $C_{\mu,\log}[a,b]$ such that,

$$ \| \, {}_HI^{\alpha}_{a+}\varphi\|_{C_{\mu,\log}[a,b]} \le \frac{\Gamma(1-\mu)}{\Gamma(1+\alpha-\mu)} (\log(t/a))^{\alpha} \|\varphi\|_{C_{\mu,\log}[a,b]}. \tag{4.1} $$

Proof. By Lemma 2.7, the result of this Lemma follows. Now, we will prove the inequality (4.1). By definition 2.1 of the weighted space $C_{\mu,\log}[a,b]$, we have

$$ \begin{aligned} \| \, {}_HI^{\alpha}_{a+}\varphi\|_{C_{\mu,\log}[a,b]} &= \left\| (\log(t/a))^{\mu} \, {}_HI^{\alpha}_{a+}\varphi \right\|_{C[a,b]} \\ &\le \left\| \varphi \right\|_{C_{\mu,\log}[a,b]} \left\| {}_HI^{\alpha}_{a+} (\log(t/a))^{-\mu} \right\|_{C_{\mu,\log}[a,b]}. \end{aligned} $$

Now, by using Lemma 2.5 (a_1) (with β replaced by $1-\mu$) we obtain

$$ \| \, {}_HI^{\alpha}_{a+}\varphi\|_{C_{\mu,\log}[a,b]} \le \frac{\Gamma(1-\mu)}{\Gamma(1+\alpha-\mu)} (\log(t/a))^{\alpha} \|\varphi\|_{C_{\mu,\log}[a,b]}. $$

Hence, the proof of this Lemma is complete.

Theorem 4.2 Let $n-1 < \alpha < n, 0 \le \beta \le 1, \gamma = \alpha + \beta(n-\alpha)$, and assume that $\varphi(.,x(.)) \in C_{\mu,\log}[a,b]$, where $\varphi : (a,b] \times \mathbb{R} \to \mathbb{R}$ be a function for any $x \in C_{\mu,\log}[a,b](n-\gamma \leq \mu < n-\beta(n-\alpha))$ and satisfies the Lipschitz condition given in definition 2.9 with respect to x. Then, there exists a unique solution $x(t)$ for the Cauchy-type problem (1.1) in the weighted space $C^{\alpha,\beta}_{\delta;n-\gamma,\mu}[a,b]$.

Proof. First, we will prove the existence of the unique solution $x(t) \in C_{n-\gamma,\log}[a,b]$. According to Theorem 3.2, it is sufficient to prove the existence of the unique solution $x(t) \in C_{n-\gamma,\log}[a,b]$ to the nonlinear (VIE)(3.1) and that is based on Theorem 2.14 (Banach fixed point theorem). Since the equation (3.1) makes sense in any interval $[a,t_1] \subset [a,b]$, then we choose $t_1 \in (a,b]$ such that the following estimate holds

$$\omega_1 := L\,\frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)}(\log(t_1/a))^{\alpha} \;<\; 1, \tag{4.2}$$

where $L > 0$ is a Lipschitz constant. So, we will prove the existence of the unique solution $x(t) \in C_{n-\gamma,\log}[a,t_1]$ to the equation (3.1) on the interval $(a,t_1]$. For this we know that the space $C_{n-\gamma,\log}[a,t_1]$ is a complete metric space defined with the distance as

$$d(x_1,x_2) = \|x_1 - x_2\|_{C_{n-\gamma,\log}}[a,t_1] := \max_{t\in[a,t_1]}\left|[\log(t/a)]^{n-\gamma}\big[x_1(t)-x_2(t)\big]\right|.$$

The equation (3.1) we be rewritten as the following:

$$x(t) = (\mathrm{T}x)(t),$$

where T is the operator defined by

$$(\mathrm{T}x)(t) = x_0(t) + \left[\,{}_H I^{\alpha}_{a+}\varphi(\tau,x(\tau))\right](t). \tag{4.3}$$

with

$$x_0(t) = \sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log(t/a))^{\gamma-k}. \tag{4.4}$$

Now, we claim that T maps from $C_{n-\gamma,\log}[a,t_1]$ into $C_{n-\gamma,\log}[a,t_1]$. In fact, it is clear from (4.4) that $x_0(t) \in C_{n-\gamma,\log}[a,t_1]$. And since $\varphi(t,x(t)) \in C_{n-\gamma,\log}[a,t_1]$, then, by Lemma 2.7 and Lemma 4.1 [with $\mu = n-\gamma, b = t_1$ and $\varphi(.) = \varphi(.,x(.))$], the integral in the right-hand side of (4.1) is relevant to $C_{n-\gamma,\log}[a,t_1]$. Thus, $(\mathrm{T}x)(t) \in C_{n-\gamma,\log}[a,t_1]$.
Next, we will prove that T is the contraction. That is, we will prove that the following estimate holds:

$$\left\|\mathrm{T}x_1 - \mathrm{T}x_2\right\|_{C_{n-\gamma,\log}[a,t_1]} \leq \omega_1 \left\|x_1 - x_2\right\|_{C_{n-\gamma,\log}[a,t_1]},\;\; 0 < \omega_1 < 1. \tag{4.5}$$

By equations (4.1) and (4.4), and using the Lipschitz condition given in definition 2.9 and applying the estimate (4.1) [with $\mu = n-\gamma, b = t_1$ and $\varphi(t) = \varphi(t,x_1(t)) - \varphi(t,x_2(t))$], we get

$$\begin{aligned}\left\|\mathrm{T}x_1 - \mathrm{T}x_2\right\|_{C_{n-\gamma,\log}[a,t_1]} &= \left\|\,{}_H I^{\alpha}_{a+}\varphi(t,x_1(t)) - \,{}_H I^{\alpha}_{a+}\varphi(t,x_2(t))\right\|_{C_{n-\gamma,\log}[a,t_1]}\\ &\leq \left\|\,{}_H I^{\alpha}_{a+}\left[|\varphi(t,x_1(t)) - \varphi(t,x_2(t))|\right]\right\|_{C_{n-\gamma,\log}[a,t_1]}\\ &\leq L\left\|\,{}_H I^{\alpha}_{a+}\left[|x_1(t)) - x_2(t))|\right]\right\|_{C_{n-\gamma,\log}[a,t_1]}\\ &\leq L\,\frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)}(\log(t_1/a))^{\alpha}\|x_1 - x_2\|_{C_{n-\gamma,\log}[a,t_1]}\\ &= \omega_1\|x_1 - x_2\|_{C_{n-\gamma,\log}[a,t_1]},\end{aligned}$$

which yields (4.5), $0 < \omega_1 < 1$. According to (4.2) and by applying the Theorem 2.14 (Banach fixed point theorem), we obtain a unique solution $x^* \in C_{n-\gamma,\log}[a,t_1]$ to (VIE)(3.1) on the interval $(a,t_1]$.
This solution x^* is given from a limit of the convergent sequence $(\mathrm{T}^m x_0^*)(t)$:

$$\lim_{m\to\infty}\left\|\mathrm{T}^m x_0^* - x^*\right\|_{C_{n-\gamma,\log}[a,t_1]} = 0,$$

where x_0^* is any function in $C_{n-\gamma,\log}[a,t_1]$ and

$$\begin{aligned}(\mathrm{T}^m x_0^*)(t) &= (\mathrm{T}\mathrm{T}^{m-1}x_0^*)(t)\\ &= x_0(t) + \left[\,{}_H I^{\alpha}_{a+}\varphi(\tau,(\mathrm{T}^{m-1}x_0^*)(\tau))\right](t),\end{aligned}$$

Let us put $x_0^*(t) = x_0(t)$ with $x_0(t)$, which is defined by (4.4).
If we indicate $x_m(t) := (\mathrm{T}^m x_0^*)(t)$, then it is clear that

$$\lim_{m\to+\infty}\left\|x_m(t) - x^*\right\|_{C_{n-\gamma,\log}[a,t_1]} = 0. \tag{4.6}$$

Next, we consider the interval $[t_1,b]$. From the (VIE)(3.1) we have

$$\begin{aligned}x(t) &= \sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log(t/a))^{\gamma-k} + \frac{1}{\Gamma(\alpha)}\int_a^{t_1}(\log(t/\tau))^{\alpha-1}\varphi(\tau,x(\tau))\frac{d\tau}{\tau}\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(\log(t/\tau))^{\alpha-1}\varphi(\tau,x(\tau))\frac{d\tau}{\tau}\\ &= x_{01} + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(\log(t/\tau))^{\alpha-1}\varphi(\tau,x(\tau))\frac{d\tau}{\tau},\end{aligned} \tag{4.7}$$

where x_{01} is defined by

$$x_{01}=\sum_{k=1}^{n}\frac{x_{a_k}}{\Gamma(\gamma-k+1)}(\log(t/a))^{\gamma-k}+\frac{1}{\Gamma(\alpha)}\int_a^{t_1}(\log(t/\tau))^{\alpha-1}\varphi(\tau,x(\tau))\frac{d\tau}{\tau}, \tag{4.8}$$

and is the known function. We note that $x_{01}\in C_{n-\gamma,\log}[t_1,b]$. Now, we will prove the existence of the unique solution $x(t)\in C_{n-\gamma,\log}[t_1,b]$ to the equation (3.1) on the interval $(t_1,b]$. Also, we use Theorem 2.14 (Banach fixed point theorem) for the space $C_{n-\gamma,\log}[t_1,t_2]$, where $t_2\in(t_1,b]$ (with $t_2=t_1+h_1,\ h_1>0,\ t_2\leq b$) satisfies

$$\omega_2:=L\,\frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)}(\log(t_2/t_1))^{\alpha}<1.$$

The space $C_{n-\gamma,\log}[t_1,t_2]$ is a complete metric space defined with the distance as

$$d(x_1,x_2)=\|x_1-x_2\|_{C_{n-\gamma,\log}}[t_1,t_2]=\max_{t\in[t_1,t_2]}\left|[\log(t/a)]^{n-\gamma}\big[x_1(t)-x_2(t)\big]\right|.$$

Also, we can rewrite equation (4.6) as the following:

$$x(t)=(\mathrm{T}x)(t), \tag{4.9}$$

where T is the operator given by

$$(\mathrm{T}x)(t)=x_{01}(t)+\left[\,{}_{H}I^{\alpha}_{t_1+}\varphi(\tau,x(\tau))\right](t).$$

As in the beginning part of this proof, since $x_{01}(t)\in C_{n-\gamma,\log}[t_1,t_2]$ and $\varphi(t,x(t))\in C_{n-\gamma,\log}[t_1,t_2]$, then, by Lemma 2.7 and Lemma 4.1 [with $\mu=n-\gamma, b=t_2$ and $\varphi(.)=\varphi(.,x(.))$], the integral in the right-hand side of (4.9) also belongs to $C_{n-\gamma,\log}[t_1,t_2]$. Thus, $(\mathrm{T}x)(t)\in C_{n-\gamma,\log}[t_1,t_2]$.
Furthermore, using the Lipschitz condition given in definition 2.9 and applying the estimate (4.1) [with $\mu=n-\gamma, b=t_2$ and $\varphi(t)=\varphi(t,x_1(t))-\varphi(t,x_2(t))$], we get

$$\begin{aligned}\left\|\mathrm{T}x_1-\mathrm{T}x_2\right\|_{C_{n-\gamma,\log}[t_1,t_2]}&=\left\|\,{}_{H}I^{\alpha}_{t_1+}\varphi(t,x_1(t))-\,{}_{H}I^{\alpha}_{t_1+}\varphi(t,x_2(t))\right\|_{C_{n-\gamma,\log}[t_1,t_2]}\\&\leq\left\|\,{}_{H}I^{\alpha}_{t_1+}\big[|\varphi(t,x_1(t))-\varphi(t,x_2(t))|\big]\right\|_{C_{n-\gamma,\log}[t_1,t_2]}\\&\leq L\left\|\,{}_{H}I^{\alpha}_{t_1+}\big[|x_1(t))-x_2(t))|\big]\right\|_{C_{n-\gamma,\log}[t_1,t_2]}\\&\leq\ \omega_2\|x_1-x_2\|_{C_{n-\gamma,\log}[t_1,t_2]}.\end{aligned}$$

This, together with (4.8), $0<\omega_2<1$, indicates that T is a contraction. And by applying the Theorem 2.14 (Banach fixed point theorem), we obtain a unique solution $x_1^*\in C_{n-\gamma,\log}[t_1,t_2]$ to (VIE)(3.1) on the interval $(t_1,t_2]$. Moreover, this solution x_1^* is given from a limit of the convergent sequence $(\mathrm{T}^m x_{01}^*)(t)$:

$$\lim_{m\to+\infty}\left\|\mathrm{T}^m x_{01}^*-x_1^*\right\|_{C_{n-\gamma,\log}[t_1,t_2]}=0,$$

where x_{01}^* is any function in $C_{n-\gamma,\log}[t_1,t_2]$. Again, we can put $x_{01}^*(t)=x_{01}(t)$ defined by (4.7). Hence,

$$\lim_{m\to+\infty}\left\|x_m(t)-x_1^*\right\|_{C_{n-\gamma,\log}[t_1,t_2]}=0,$$

where

$$\begin{aligned}x_m(t)&=(\mathrm{T}^m x_{01}^*)(t)\\&=x_{01}(t)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(\log(t/\tau))^{\alpha-1}\varphi(\tau,x(\tau))\frac{d\tau}{\tau}.\end{aligned}$$

Next, if $t_2\neq b$, we consider the interval $[t_2,t_3]$ such that $t_3=t_2+h_2$ with $h_2>0,\ t_3\leq b$ and

$$\omega_3:=L\,\frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)}(\log(t_3/t_2))^{\alpha}<1.$$

By using the same argument as above, we conclude that there exists a unique solution $x_2^*\in C_{n-\gamma,\log}[t_2,t_3]$ to (VIE)(3.1) on $[t_2,t_3]$. If $t_3\neq b$, then we continue the previous process until we get a unique solution $x(t)$ to the (VIE)(3.1) and $x(t)=x_i^*$ such that $x_i^*\in C_{n-\gamma,\log}[t_{i-1},t_i]$ for $i=1,2,...,L$, where $a=t_0<t_1<t_2<...<t_L=b$ and

$$\omega_i:=L\,\frac{\Gamma(\gamma-n+1)}{\Gamma(\alpha+\gamma-n+1)}(\log(t_i/t_{i-1}))^{\alpha}<1.$$

Thus, by using Lemma 2.13, it yields that there exists a unique solution $x(t)\in C_{n-\gamma,\log}[a,b]$ to the (VIE)(3.1) on the whole interval $(a,b]$. Therefore, $x(t)\in C_{n-\gamma,\log}[a,b]$ is a unique solution to the Cauchy-type problem (1.1).

Finally, we will show that such unique solution $x(t)\in C_{n-\gamma,\log}[a,b]$ is in the weighted space $C^{\alpha,\beta}_{n-\gamma,\mu}[a,b]$. By definition 3.1, it is sufficient to prove that ${}_{H}D^{\alpha,\beta}_{a+}x\in C_{\mu,\log}[a,b]$. From the above proof, a solution $x(t)\in C_{n-\gamma,\log}[a,b]$ is a limit of the sequence $x_m(t)\in C_{n-\gamma,\log}[a,b]$ such that

$$\lim_{m\to+\infty}\left\|x_m-x\right\|_{C_{n-\gamma,\log}[a,b]}=0. \tag{4.10}$$

Hence, by using equation (1.1), Lipschitz condition given in definition 2.9 and Lemma 2.4, we have

$$\begin{aligned}\left\| {}_HD_{a+}^{\alpha,\beta} x_m(t) - {}_HD_{a+}^{\alpha,\beta} x(t)\right\|_{C_{\mu,\log}[a,b]} &= \left\|\varphi(t,x_m(t)) - \varphi(t,x(t))\right\|_{C_{\mu,\log}[a,b]} \\ &\leq L\,(\log(b/a))^{\mu-n+\gamma}\left\|x_m(t)-x(t)\right\|_{C_{n-\gamma,\log}[a,b]}.\end{aligned} \tag{4.11}$$

Clearly, the equations (4.10) and (4.10) yield that

$$\lim_{m\to+\infty}\left\| {}_HD_{a+}^{\alpha,\beta} x_m(t) - {}_HD_{a+}^{\alpha,\beta} x(t)\right\|_{C_{\mu,\log}[a,b]} = 0,$$

and, hence, $({}_HD_{a+}^{\alpha,\beta}x) \in C_{\mu,\log}[a,b]$. Thus, the proof of this theorem is complete.

Remark 4.3 For $0<\alpha<1$, Theorem 4.2 is reduced to Theorem 22 (see[9]).

Acknowledgement

The authors are grateful to the referee for his/her important remarks and suggestions.

References

[1] M. D. Qassim, K. M. Furati, N-e. Tatar, *On a differential equation involving Hilfer-Hadamard fractional derivative.* Abstract Appl. Anal. **2012**, 2012:17, Article ID 391062.
[2] Ahmad Y. A. Salamooni, D. D. Pawar, *Existence and uniqueness of boundary value problems for Hilfer-Hadamard-type fractional differential equations.* arXiv:1801.10400v1[math.AP] 31 Jan (2018).
[3] I. Podlubny, *Fractional Differential Equations.* Academic Press, San Diego, (1999).
[4] S. G. Samko, A. A. Kilbas; O. I. Marichev; *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach, New York (1993). [Translation from the Russian edition, Nauka i Tekhnika, Minsk (1987)]
[5] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies 204, Editor: Jan Van Mill, Elsevier, Amsterdam, The Netherlands, (2006).
[6] R. Hilfer, *Applications of Fractional Calculus in Physics.* World Scientific Publ. Co., Singapore, (2000).
[7] R. Hilfer, *Threefold introduction to fractional derivatives.* In: Anomalous transport: foundations and applications, **2008**, 17-73.
[8] J. A. Nanware, D. B. Dhaigude, *Existence and uniqueness of solution of Riemann-Liouville fractional differential equations with integral boundary conditions.* Int. J. Nonlinear Sci., **14**, (2012), 410-415.
[9] M. D. Kassim and N. E. Tatar, *Well-Posedness and Stability for a Differential Problem with Hilfer-Hadamard Fractional Derivative.* Abst. Appl. Anal., **2013**, (2013), 1-12, Article ID 605029.
[10] R. Hilfer, Y. Luchko, Ž. Tomovski, *Operational method for solution of the fractional differential equations with the generalized Riemann-Liouville fractional derivatives,* Fract. Cal. Appl. Anal., **12**, (2009), 299-318.
[11] K. Diethelm, N. J. Ford, *Analysis of fractional differential equations.* J. Math. Anal. Appl., **265**, (2002), 229-248.
[12] K. M. Furati, M. D. Kassim and N.e.Tatar, *Existence and uniqueness for a problem involving Hilfer fractional derivative.* Computers Math. Appl., (2012), 1616-1626.
[13] C. Kou, J. Liu, and Y. Ye, *Existence and uniqueness of solutions for the Cachy-type problems of fractional differential equaitions.* Discrete Dyn. Nat. Soc., Article ID 142175, (2010), 1-15.
[14] H.M. Srivastava, Ž. Tomovski, *Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel.* Appl. Math. Comput., **211**, (2009), 198-210.
[15] Živorad Tomovski, R. Hilfer, H. Srivastava, *Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions.* Integral Transforms Spec. Funct., **21**, (11), (2010), 797-814.
[16] J. A. Nanware, D. B. Dhaigude, *Existence and uniqueness of solutions of differential equations of fractional order with integral boundary conditions.* J. Nonlinear Sci. Appl., **7**, (2014), 246-254.
[17] Ahmad Y. A. Salamooni, D. D. Pawar, *Existence and uniqueness of nonlocal boundary conditions for Hilfer-Hadamard-type fractional differential equations.* arXiv:1802.04262v1[math.AP] 12 Feb (2018).
[18] D. Vivek, K. Kanagarajan and E. M. Elsayed, *Some Existence and Stability Results for Hilfer-fractional Implicit Differential Equations with Nonlocal Conditions.* Mediterr. J. Math. **2018** 15:15, https://doi.org/10.1007/s00009-017-1061-0, ©Springer International Publishing AG, part of Springer Nature 2018.
[19] S. Abbas, M. Benchohra, , M. Bohner, *Weak solutions for implicit differential equations with Hilfer–Hadamard fractional derivative.* Adv. Dyn. Syst. Appl., **12**, (2017), 1-16.
[20] S. Abbas, M. Benchohra, J.-E. Lazreg, Y. Zhou, *A survey on Hadamard and Hilfer fractional differential equations: analysis and stability.* Chaos Solitons Fractals, **102**, (2017), 47-71.
[21] A. Y. A. Salamooni, D. D. Pawar, *Unique positive solution for nonlinear Caputo-type fractional q−difference equations with nonlocal and Stieltjes integral boundary conditions.* Fractional Differential Calculus, **9**(2), (2019), 295-307.