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ON STATISTICALLY CONVERGENT IN FINITE DIMENSIONAL SPACES

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Abstract: In this paper, the notion of statistical convergence, which was introduced by Steinhaus (1951), was studied in $\mathbb{R}^m$; and some concepts and theorems, whose statistical correspondence for the real number sequences were given, were carried to $\mathbb{R}^m$. In addition, the concepts of the statistical limit point and the statistical cluster point were given and it was mentioned that these two concepts were'nt equal in Fridy's study in 1993. These concepts were given in $\mathbb{R}^m$ and the inclusion correlation between these concepts was studied.

Key words: Statistical convergence, statistical boundedness, statistically Cauchy sequence, statistical limit point, statistical cluster point.

Mathematics Subject Classifications (2000): 40A05

SONLU BOYUTLU UZAYLARDA İSTATİSTİKSEL YAKINSAKLIK ÜZERİNE

Özet: Bu makalede Steinhaus (1951) tarafından verilen istatistiksel yakınsaklık kavramı R^m de incelenip, reel sayı dizileri için istatistiksel benzerleri verilen bazı kavram ve teoremler R^m e taşınmıştır. 1993'de Fridy tarafından yapılan çalışmada istatistiksel limit noktası ve istatistiksel yığılma noktası kavramları verilip bu iki kavramın birbirine denk olmadığından bahsedilmiştir. Bu kavramlar R^m de verilip aralarındaki kapsama bağıntısı incelenmiştir.

Anahtar kelimeler: İstatistiksel Yakınsaklık, İstatistiksel Sınırlılık, İstatistiksel Cauchy Dizisi, İstatistiksel Limit Noktası, İstatistiksel Yığılma Noktası.

1. INTRODUCTION

The concept of statistically convergence was first given at a conference Wroclaw University by STEINHAUS (1951) in 1949. This concept was introduced by FAST (1951), BUCK (1953) and SCHOENBERG (1959) for real and complex sequences. MADDOX (1988) extended the concept for sequences in any locall convex topological vector spaces. A relation was constructed between statistical convergence and summability by FRIDY (1985), ŚALǍT (1980), CONNOR (1985), MADDOX (1988), RATH & TRIPATHY (1994). The concept of a statistical Cauchy sequence was studied by FRIDY (1951), RATH & TRIPATHY (1994). The notations of statistical limit and statistical cluster points were given by FRIDY (1993). FRIDY (1985) and CONNOR (1988) introduced different expressions of the Decomposition Theorem for statistical convergence. Statistical convergence in R^m was studied by PEHLIVAN & MAMEDOV (2000).

Statistical convergence is a type of convergence which is based on the natural density of positive integers. First of all let us define the concept of natural density and mention a few characteristics of it. If K is a subset of N, the set of K_n is defined as $\{k \in K : k \leq n\}$. $|K_n|$ shows the cardinality of K_n. According to this, the numbers

$$\underline{\delta}(K) = \liminf_{n\to\infty} \frac{|K_n|}{n}, \quad \overline{\delta}(K) = \limsup_{n\to\infty} \frac{|K_n|}{n}$$

are called the lower and upper asymptotic density of K, respectively. If there exists the limit $\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$, then $\delta(K)$ is said to be the asymptotic density of K and $\delta(K) = \underline{\delta}(K) = \overline{\delta}(K)$. Clearly, finite sets have zero density and $\delta(K^c) = 1 - \delta(K)$, whenever either side exists and $K^c = N / K$. If $\{x_{n(j)}\}$ is a subsequence of $x = (x_k)$ and $x = (x_k)$ and $K = \{n(j) : j \in N\}$ then we abbreviate $\{x_{n(j)}\}$ by $\{x\}_K$. If $\delta(K) = 0$, $\{x\}_K$ is called a subsequence of density zero, or a thin subsequence. On the other hand, $\{x\}_K$ is a nonthin subsequence of x if K does'nt have density zero. It should be noted that $\{x\}_K$ is a nonthin subsequence of x if either $\delta(K)$ is a positive number or K fails to have natural density. Now, we can define the concept of statistical convergence. The real number sequence $x = (x_k)$ is statistically convergent to the number L provided that for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left|\{k \in N : |x_k - L| \geq \varepsilon\}\right| = 0$$

i.e.

$$|x_k - L| < \varepsilon \ for\ a.a.k.$$

In this case we write $st - \lim x_k = L$. $x = (x_k)$ is called a statistical Cauchy sequence if, for each $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that

$$\lim_n \frac{1}{n} \left|\{k \leq n : |x_k - x_N| \geq \varepsilon\}\right| = 0.$$

The concept of statistical convergence in R^m is defined as follows.

Definition 1. Let $(x_k) = (\xi_j^k)$ be a sequence in R^m. When taken as

$$d(x_k, x) = \left[\sum_{j=1}^{m} \left(\xi_j^k - \xi_j\right)^2\right]^{\frac{1}{2}}, \text{ if}$$

$$\delta(\{k \in N : d(x_k, x) \geq \varepsilon\}) = 0$$

or

$$d(x_k, x) < \varepsilon, \text{ for a.a.k.}$$

the sequence of (x_k) is a statistically convergent to $x = (\xi_j)$ and denoted by $st - \lim d(x_k, x) = 0$.

Example 2. Let us define the sequence of $(x_k) = (\xi_j^k)$ in R^m for $1 \le j \le m$ as

$$(\xi_j^k) = \begin{cases} 0 & , \text{ if } j \text{ and } k \text{ square} \\ \dfrac{kj+1}{k} & , \text{ otherwise} \end{cases}$$

In case $\varepsilon = \dfrac{1}{2}$ and $(x) = (\xi_j) = (1,2,3,...,m)$, the set is

$$\{k \in N : d(x_k, x) \ge \varepsilon\} = \{1,4,9,16,...\}.$$

The density of this set is as follows:

$$\delta(\{k \in N : d(x_k, x) \ge \varepsilon\}) = \delta(\{1,4,9,16,...\}) = 0.$$

From this point we can say that the sequence of $(x_k) = (\xi_j^k)$ is statistically convergent to $(x) = (\xi_j)$. However, since the set of $\{k \in N : d(x_k, x) \ge \varepsilon\}$ includes an infinite number of elements, here doesn't exist ordinary convergence.

Theorem 3. $(x_k) = (\xi_j^k)$ and $(y_k) = (\eta_j^k)$, in R^m, are two sequences which are convergence to $(x) = (\xi_j)$, $(y) = (\eta_j)$, respectively. Then,

(i) $(x_k) + (y_k) \underset{\rightarrow}{st}\ x + y$

(ii) $(x_k) - (y_k) \underset{\rightarrow}{st}\ x - y$

Proof. **(i)** For $1 \le j \le m$ since $(x_k) = (\xi_j^k)$ is statistically convergent to $(x) = (\xi_j)$, as $A = \left\{k : d(x_k, x) \ge \dfrac{\varepsilon}{2}\right\}$ for each $\varepsilon > 0$ is $\delta(A) = 0$. Similarly, since $(y_k) = (\eta_j^k)$ is statistically convergent to $(y) = (\eta_j)$, as $B = \left\{k : d(y_k, y) \ge \dfrac{\varepsilon}{2}\right\}$ the density is $\delta(B) = 0$. Now, we prove that for each $\varepsilon > 0$, the density for the set $C = \{k : d(x_k + y_k, x + y)\} \ge \varepsilon$ is $\delta(C) = 0$. And to do this, it's sufficient to show $C \subseteq A \cup B$. We take $n \in C$, $d(x_n + y_n, x + y) \ge \varepsilon$ is obtained for each $\varepsilon > 0$. We suppose $n \notin A \cup B$. In this case $n \notin A$ and $n \notin B$. Since $n \notin A$ the metric is $d(x_n, x) < \dfrac{\varepsilon}{2}$ and since $n \notin B$ it is $d(y_n, y) < \dfrac{\varepsilon}{2}$. According to this

$$
\begin{aligned}
d(x_n + y_n, x+y) &\le d(x_n, x) + d(y_n, y) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
\end{aligned}
$$

is obtained. And this contradicts with the expression $n \in C$. In this case it's seen that $n \in A \cup B$. Here it is:

$$
\begin{aligned}
C &\subseteq A \cup B \\
\delta(C) &\le \delta(A \cup B) = 0 \\
\delta(C) &= 0.
\end{aligned}
$$

This shows that the sequence of $(x_k + y_k)$ is statistically convergent to $(\mathrm{x}+\mathrm{y})$. The proof of (ii) is carried out in a similar way to the proof of (i).

Example 4. We define $(x_k) = (\xi_j^k)$ sequence as

$$
(\xi_j^k) = \begin{cases} 0 & , \; \textit{if j and k are square} \\ \dfrac{kj+1}{k} & , \; \textit{otherwise} \end{cases}
$$

and $(y_k) = (\eta_j^k)$ as

$$
(\eta_j^k) = \begin{cases} 0 & , \; \textit{if } j+1 \textit{ and } k+1 \textit{ are square} \\ \dfrac{kj+1}{2k} & , \; \textit{otherwise.} \end{cases}
$$

Here $(x_k) = (\xi_j^k)$ is statistically convergent to $\mathrm{x} = (1, 2, 3, ..., \mathrm{m})$ and $(y_k) = (\eta_j^k)$ is to $y = \left(\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, ..., \frac{m}{2}\right)$. And the total sequence of $(x_k + y_k) = (\xi_j^k + \eta_j^k)$ is in the form of:

$$
(\xi_j^k + \eta_j^k) = \begin{cases} \dfrac{kj+1}{2k} & , \; \textit{if j and k are square} \\ \dfrac{kj+1}{k} & , \; \textit{if } j+1 \textit{ and } k+1 \textit{ are square} \\ \dfrac{3kj+3}{2k} & , \; \textit{otherwise.} \end{cases}
$$

This sequence is statistically convergent to $(x+y) = \left(\frac{3}{2}, 3, \frac{9}{2}, 6, ..., \frac{3m}{2}\right)$.

2. STATISTICAL BOUNDEDNESS, STATISTICAL CAUCHY SEQUENCE

In this section the concepts of statistical boundedness and statistical Cauchy sequence are defined the correspondences of some concepts and theorems given for ordinary convergence are presented.

Definition 5. Let $(x_k)=(\xi_j^k)$ be a sequence in R^m. If a finite positive number exists such that

$$\delta(\{k \in N : d(x_k,0) > A\}) = 0$$

or

$$d(x_k,0) < A \quad \text{a.a.k}$$

(x_k) sequence is called statistical boundedness.

Example 6. Let a $(x_k)=(\xi_j^k)$ sequence be defined for $1 \leq j \leq m$ as

$$(\xi_j^k) = \begin{cases} k & , \quad \text{if } j \text{ and } k \text{ are square} \\ j & , \quad \text{otherwise,} \end{cases}$$

When $A = \dfrac{m(m+1)(2m+1)}{6}$, the number is

$$\{k \in N : d(x_k,0) > \sqrt{A}\} \subseteq \{1,4,9,16,...\}.$$

When the upper densities of these sets are taken the results is as follows:

$$\overline{\delta}(\{k \in N : d(x_k,0) > \sqrt{A}\}) \leq \overline{\delta}(\{1,4,9,16,...\}) = 0 .$$

And this means that (x_k) sequence is statistically bounded. On the other hand this sequence isn't bounded.

Theorem 7. Let $(x_k)=(\xi_j^k)$ and $(y_k)=(\eta_j^k)$ be two sequences in R^m. If these sequences are statistically bounded, the sequence of $(x_k).(y_k)$ is also statistically bounded.

Proof. Let $(x_k)=(\xi_j^k)$ and $(y_k)=(\eta_j^k)$ be statistically bounded. There are, respectively, $A \geq 1$ and $B \geq 1$ finite numbers such that $\delta(\{k \in N : d(x_k,0) > A\}) = 0$ and $\delta(\{k \in N : d(y_k,0) > B\}) = 0$. We can write the inclusion

$$\{k \in N : d(x_k.y_k,0) > A.B\} \subseteq \{k \in N : d(x_k,0) > A\} \cup \{k \in N : d(y_k,0) > B\}$$

in the set of natural numbers. Here the density is

$$\delta(\{k \in N : d(x_k.y_k,0) > A.B\}) \leq \delta(\{k \in N : d(x_k,0) > A\}) + \delta(\{k \in N : d(y_k,0) > B\})$$

And this gives the statistical boundedness of $(x_k).(y_k)$ sequence.

Now we give the definition of statistical Cauchy sequence in R^m.

Definition 8. We take $(x_k)=(\xi_j^k)$ sequence in R^m. If, for each $\varepsilon > 0$, $M = M(\varepsilon)$ number exists such that the density of the set, $\{k : d(x_k, x_M) \geq \varepsilon\}$, is zero or

$d(x_k, x_M) < \varepsilon, a.a.k.$

(x_k) is called statistical Cauchy sequence in R^m.

Now we give the Cauchy convergence Criteria infinite dimensional space.

Theorem 9. We take $(x_k)=(\xi_j^k)$ and $(y_k)=(\eta_j^k)$. According to this the following expressions are equal.

(i) (x_k) is a statistically convergent sequence.

(ii) (x_k) is a statistically Cauchy sequence.

(iii) Let (x_k) be given. There is a $y=(y_k)$ sequence for almost all k such that $(x_k)=(y_k)$.

Proof. To prove that (i) implies (ii) we use an adaptation of familiar prof that a convergent sequence is a Cauchy sequence. We suppose that (x_k) is a statistically convergent sequence and $\varepsilon>0$, i.e. $\text{st-lim}\, d(x_k, x)=0$ and $\varepsilon>0$. Then if we choose M such that for almost all k of $d(x_k, x)<\frac{\varepsilon}{2}$ and $d(x_M, x)<\frac{\varepsilon}{2}$,

$$\begin{aligned} d(x_k, x_M) &< d(x_k, x)+d(x_M, x) \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \end{aligned}$$

is obtained for $a.a.k.$. So (x_k) is a statistical Cauchy sequence. Next, assume (ii) is true. That is $(x_k)=(\xi_j^k)$ statistically Cauchy so that the closed ball $B=\overline{B}(x_{N(1)},1)$ contains (x_k) $a.a.k.$ for some positive number $N(1)$. Also, apply hypothesis to choose M so that $B'=\overline{B}\left(x_M, \frac{1}{2}\right)$ contains $(x_k)=(\xi_j^k)$ $a.a.k.$. We will claim that $B_1=B\cap B'$ contains (x_k) $a.a.k.$.

Now since

$$\{k\le n : x_k\notin B\cap B'\}=\{k\le n : x_k\notin B\}=\{k\le n : x_k\notin B'\},$$

we have

$$\begin{aligned} \lim_n \frac{1}{n}\left|\{k\le n : x_k\notin B\cap B'\}\right| &\le \lim_n \frac{1}{n}\left|\{k\le n : x_k\notin B\}\right|+\lim_n \frac{1}{n}\left|\{k\le n : x_k\notin B'\}\right| \\ &= 0. \end{aligned}$$

Therefore B_1 is a closed ball of diameter less than or equal to 1 that contains (x_k) $a.a.k.$. Now we proceed by choosing $N(2)$ so that $B''=\overline{B}\left(x_{N(2)}, \frac{1}{4}\right)$, contains (x_k) $a.a.k.$ and by the preceding argument $B_2=B_1\cap B''$ contains (x_k) $a.a.k.$ and B_2 has a radius less than or equal to $\frac{1}{2}$. Continuing this process we construct a sequence $\{B_m\}_{m=1}^{\infty}$ of closed balls such that for each m, $B_m\supset B_{m+1}$, the diameter of B_m is'nt greater than $\frac{1}{2^{m-1}}$, and $(x_k)\in B_m$ $a.a.k.$

By the nested closed set theorem of a complete metric space we have $\bigcap_{m=1}^{\infty} B_m \neq \phi$. So there is a number λ such that $\lambda \in \bigcap_{m=1}^{\infty} B_m$. Using the fact that $(x_k) \in B_m$ *a.a.k.* we choose an increasing positive integer sequence $\{T_m\}_{m=1}^{\infty}$ such that

$$\frac{1}{n}\left|\{k \leq n : x_k \notin B_m\}\right| < \frac{1}{m},\ if\ \ n > T_m. \tag{1}$$

Now define a subsequence z of x consisting of all terms x_k such that $k > T_1$ and if $T_m < k \leq T_{m+1}$, then $(x_k) \notin B_m$. Next define the sequence $y = (y_k)$,

$$y_k = \begin{cases} \lambda & ;\ if\ \ x_k\ \ is\, a\, term\ of\ z \\ x_k & ;otherwise \end{cases}.$$

Then $\lim \mathrm{y_k} = \lambda : for,\ if\ \ \varepsilon > \frac{1}{m} > 0$ and $k > T_m$ then either (x_k) is a term of z, which means $y_k = \lambda$ or $y_k = x_k \in B_m$ and

$$d(y_k, \lambda) \leq B_m\ \ diameter\ of\ \leq \frac{1}{2^{m-1}}.$$

We also assert that $y_k = x_k$ *a.a.k.* To verify this we observe that if $T_m < n \leq T_{m+1}$. Then

$$\{k \leq n : y_k \neq x_k\} \subset \{k \leq n : x_k \notin B_m\}$$

and by (1)

$$\frac{1}{n}\left|\{k \leq n : y_k \neq x_k\}\right| \leq \frac{1}{n}\{k \leq n : x_k \notin B_m\} < \frac{1}{m}.$$

Hence, the limit, as $n \to \infty$, is 0 and $(x_k) = (y_k)$ *a.a.k.*. Therefore (ii) implies **(iii).** Finally, assume that (iii) holds, say $(x_k) = (y_k)$ *a.a.k.* and $d(y_k, x) < \varepsilon$ for every $\varepsilon > 0$. It's given $\varepsilon > 0$. Then for each k

$$\{k \leq n : d(x_k, x) \geq \varepsilon\} \subseteq \{k \leq n : x_k \neq y_k\} \cup \{k \leq n : d(y_k, x) > \varepsilon\}.$$

The last set here includes the finite number of set of integers since $d(y_k, x) < \varepsilon$, we call this $\ell = \ell(\varepsilon)$. So it's

$$\lim_n \frac{1}{n}\left|\{k \leq n : d(x_k, x) \geq \varepsilon\}\right| \leq \lim_n \frac{1}{n}\left|\{k \leq n : x_k \neq y_k\}\right| + \lim_n \frac{\ell}{n}$$
$$= 0$$

for *a.a.k.* since it's $(x_k) = (y_k)$. Hence $d(x_k, x) < \varepsilon$ *a.a.k.* and for every $\varepsilon > 0$, so (i) holds and the proof is complete.

As an immediate consequence of Theorem 9 we have the following result.

Corollary 10. If $x = (x_k)$ statistically convergent, i.e. $st - \lim d(x_k, x) = 0$, there exists a y subsequence of $x = (x_k)$ such that $d(y_k, x) < \varepsilon$.

3. STATISTICAL LIMIT POINT AND STATISTICAL CLUSTER POINT

In this chapter the concepts of statistical limit point and statistical cluster point are defined and the inclusion relation between them are studied.

Definition 11. If $(x_k)=(\xi_j^k)$ has a nonthin subsequence which is convergent to $\lambda \in R^m$, $\lambda \in R^m$ is called to be the statistical limit point of $(x_k)=(\xi_j^k)$.
The set of all the statistical limit points of $(x_k)=(\xi_j^k)$ is expressed with Λ_x.
Let us give an example regarding this definition.

Example 12. Let $(x_k)=(\xi_j^k)$ be a sequence in R^m. For $1 \le j \le m$ it's defined as follows:

$$(\xi_j^k)=\begin{cases} j+2 & , \quad \text{if } k \text{ is square} \\ j & , \quad \text{if } k \text{ is odd and not square} \\ j+1 & , \quad \text{if } k \text{ is even and not square.} \end{cases}$$

This sequence has two statistical limit points such that as $\lambda_1=(1,2,...,m)$ and $\lambda_2=(2,3,...,m+1)$. Thus $\Lambda x=\{\lambda_1,\lambda_2\}$. Also the ordinary limit pointsof this sequence are $L_x=\{\lambda_1,\lambda_2,\ell\}$ where $\ell=(3,4,5...,m+2)$.

Theorem 13. In R^m the statistical limit point of a statistically convergentsequence is unique.

Proof. Let $(x_k)=(\xi_j^k)$ be given as a statistical convergence in R^m. We suppose that this sequence is statistically convergent to two different values such as λ_1 and λ_2 and $d(\lambda_1,\lambda_2)>2\varepsilon>0$. Since $(x_k)=(\xi_j^k)$ is statistically convergent to λ_1 the density of the following set is zero,

$$\{k : d(x_k,\lambda_1)\ge \varepsilon\}.$$

In addition since (x_k) is statistically convergent to λ_2,

$$\delta(\{k : d(x_k,\lambda_2)\ge \varepsilon\})=0.$$

Here

$$\{k : d(x_k,\lambda_2)<\varepsilon\}\subset\{k : d(x_k,\lambda_1)\ge \varepsilon\}$$
$$\delta(\{k : d(x_k,\lambda_2)<\varepsilon\})\le\delta(\{k : d(x_k,\lambda_1)\ge \varepsilon\})=0$$

is obtained. This contradicts with the fact that (x_k) is statistically convergent to λ_2. Hence it's obligatory $d(\lambda_1,\lambda_2)=0$.

Now we give the definition of statistical cluster point.

Definition 14. Let $(x_k)=(\xi_j^k)$ be a number sequence in R^m. If, for each $\varepsilon>0$, the set of $\{k\in N: d(x_k,\gamma)<\varepsilon\}$g doesn't have zero density, γ number is called a tatistical Cluster point of (x_k). The set of all statistical cluster points of (x_k) is expressed with Γ_x.

Proposition 15. For a number sequence $(x_k)=(\xi_j^k)$ in R^m, $\Lambda_x\subseteq\Gamma_x$.

Proof. We suppose $\lambda\in\Lambda_x$. According to this (x_k) has a nonthin subsequence which is convergent to $\lambda\in R^m$, i.e. $\lim_n x_{k(n)}=\lambda$ and $\delta(\{k(n): n 2\mathrm{N}\})=d>0$. Then for each $\varepsilon>0$ the $A=\{n: d(x_{k(n)},\lambda)\geq\varepsilon\}$ set is finite. Hence for $K\subset\mathrm{N}$

$$\{k\in \mathrm{N}: d(x_k,\lambda)<\varepsilon\}\supseteq\{k\in K: d(x_k,\lambda)<\varepsilon\}\supseteq\{k(n): n\in\mathrm{N}\}/A$$

$$\delta(\{k\in \mathrm{N}: d(x_k,\lambda)<\varepsilon\}\geq\delta(\{k\in K: d(x_k,\lambda)<\varepsilon\})\geq\delta(\{k(n): n\in\mathrm{N}\})-\delta(A)$$

$$\delta(\{k\in \mathrm{N}: d(x_k,\lambda)<\varepsilon\}\geq\delta(\{k(n): n\in\mathrm{N}\})=d.$$

Thus

$$\delta(\{k\in \mathrm{N}: d(x_k,\lambda)<\varepsilon\}\neq 0$$

is obtained. And this means $\lambda\in\Gamma_x$. Here it's seen that $\Lambda_x\subseteq\Gamma_x$.

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REFERENCES

BUCK RC, 1953. Generalized asymptotic density, *American Journal of Mathematics,* 75, 335-346.

CONNOR JS, 1985. Some applications of functional analysis to summability theory, Ph.D thesis, Kent State University.

CONNOR JS, 1988. The statistical and strong P-Cesáro convergence of sequences, *Analysis*, 8, 47-63.

FAST H, 1951. Sur la convergence statistique, *Collog. Math*, 2, 241-244.

FRIDY JA, 1985. On statistical convergence, *Analysis*, 5, 301-313.

FRIDY JA, 1993. Statistical limit point, *Proceedings of the American Mathematical Society*, 4, 1187-1192.

MADDOX IJ 1988. Statistical convergence in a locally convex sequence space, *Math. Proc. Camp. Phil. Soc.*, 104, 141-145.

PEHLIVAN S, MAMEDOV M, 2000. Statistical cluster point and turnpike, *Optimization*, 48, 93-106.

RATH D, TRIPATHY BC, 1994. On statistically convergent and statistically Cauchy sequences, *Indian Journal of Pure and Applied Mathematics*, 25, 381-386.

SALAT T, 1980. On Statistically convergent sequences of real numbers, *Mathematica. Slovaca*, 30, 139-150.

SCHOENBERG IJ, 1959. The integrability of certain functions and related summability methods, *American. Mathematical. Monthly*, 66, 361-375.
STEINHAUS H, 1951. Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math*, 2, 73-74.