

HOMOTOPY PERTURBATION METHOD FOR SOLVING MODELLING THE POLLUTION OF A SYSTEM OF LAKES

Mehmet MERDAN

Gümüşhane University, Department of Civil Engineering,
29000-Gümüşhane, TURKEY.
e-mail: merdan29@hotmail.com

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Abstract: In this article, homotopy perturbation method is implemented to give approximate and analytical solutions of nonlinear ordinary differential equation systems such as modelling the pollution of a system of lakes. The proposed scheme is based on homotopy perturbation method (HPM), Laplace transform and Padé approximants. The results to get the homotopy perturbation method (HPM) are applied Padé approximants. The accuracy of this method is examined by comparison with the Matlab ode23s. Our proposed approach showed results to analytical solutions of nonlinear ordinary differential equation systems. Some plots are presented to show the reliability and simplicity of the methods.

Key words: Padé approximants, homotopy perturbation method, modelling the pollution of a system of lakes

AMS Mathematics Subject Classifications (2000): 65L10, 65L06, 65K10, 65K05

BİR GÖLLER SİSTEMİNİN KİRLİLİK MODELİNİN HOMOTOPY PERTURBATION YÖNTEMİ İLE ÇÖZÜMÜ

Özet: Bu makale de bir göller sisteminin kirlilik modeli gibi lineer olmayan adi diferensiyel denklem sisteminin yaklaşık analitik çözümünü bulmak için homotopy perturbation yöntemi uygulandı. Önerilen yöntem homotopy perturbation yöntemi, Laplace dönüşümü ve Padé yaklaşımını baz alır. Homotopy perturbation yöntemin'den elde edilen sonuçlara Padé yaklaşımı önerildi. Bu yöntemin doğruluğu Matlab ode23s ile mukayese edildi. Önerdiğimiz yaklaşım, lineer olmayan adi diferensiyel denklem sistemlerinin analitik çözümlerini gösterdi. Yöntemin güvenilirliğini ve basitliğini göstermek için bazı grafikler sunuldu.

Anahtar kelimeler: Padé yaklaşımı, Homotopy perturbation yöntemi, Bir göller sisteminin kirlilik modeli

AMS Matematik Konu Sınıflandırması (2000): 65L10, 65L06, 65K10, 65K05

1. INTRODUCTION

Modelling the pollution of a system of lakes is examined in the study (BIAZAR & FARROKHI 2006). The system of three lakes that are modeled in this study (HOGGARD 2007). Each lake is considered to be a large compartment and the interconnecting channel as pipes between the compartments. The direction of flow in the channels or pipes is indicated by the arrows in (HOGGARD 2007). A pollutant is introduced into the first lake where $p(t)$ denotes the rate at which the pollutant enters the lake per unit time. The function $p(t)$ may be constant or may vary with time. We are interested in knowing the levels of pollution in each lake at any time.

The components of the basic three-component model are the amount of the pollutant in lake 1 at any time $t \geq 0$, the amount of the pollutant in lake 2 at any time $t \geq 0$ and the amount of the pollutant in lake 3 at any time $t \geq 0$, are denoted respectively by $x_1(t)$, $x_2(t)$ and $x_3(t)$. These quantities satisfy

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{F_{13}}{V_3} x_3(t) + p(t) - \frac{F_{31}}{V_1} x_1(t) - \frac{F_{21}}{V_1} x_1(t) \\ \frac{dx_2}{dt} &= \frac{F_{21}}{V_1} x_1(t) - \frac{F_{32}}{V_2} x_2(t) \\ \frac{dx_3}{dt} &= \frac{F_{31}}{V_1} x_1(t) + \frac{F_{32}}{V_2} x_2(t) - \frac{F_{13}}{V_3} x_3(t).\end{aligned}\tag{1}$$

with the initial conditions:

$x_1(0) = r_1$, $x_2(0) = r_2$, $x_3(0) = r_3$. Throughout this paper, we assume the following conditions:

$$\text{Lake 1: } F_{13} = F_{21} + F_{31},$$

$$\text{Lake 2: } F_{21} = F_{32},$$

$$\text{Lake 3: } F_{31} + F_{32} = F_{13}$$

A technique for calculating the analytical solutions of nonlinear ordinary differential equation systems is developed in this paper. The developed technique depends only on the fundamental operation properties of Laplace transform and Padé approximants. The calculated results are exactly the same as those obtained by other analytical or approximate methods and demonstrate the reliability and efficiency of the technique. We will use Laplace transform and Padé approximant to deal with the truncated series. Padé' approximant (BAKER 1975) approximates a function by the ratio of two polynomials. The coefficients of the powers occurring in the polynomials are determined by the coefficients in the Taylor series expansion of the function. Generally, the Padé' approximant can enlarge the convergence domain of the truncated Taylor series and can improve greatly the convergence rate of the truncated Maclaurin series. Many different new methods have recently presented such as the variational iteration (HE 1998, HE 1999, HE 2000, MOMANI & ABUASAD 2006, SOLIMAN 2006, ABULWafa et al. 2006, ODIBAT & MOMANI 2006), the homotopy analysis method (LIAO 1992, LIAO 1995, LIAO 1997, LIAO 1999, LIAO 2002, LIAO 2003, LIAO

2004, LIAO 2005, AYUB et al. 2003, HAYAT et al. 2004, HAYAT & KHAN 2005, TAN & ABBASBANDY 2008), the homotopy perturbation method (EL-SHAHED 2005, SIDDIQUI et al. 2006, HE 2003, HE 2004, HE 2005, LIU 2005, GOLBABAI & JAVIDI 2007, YILDIRIM & TURGUT 2007) , the exp function method (HE 2006, WANG & LI 2005), the F-expansion method (ADOMIAN 1994), the adomian decomposition method (HE 2001) and others (BİLDİK & KONURALP 2006).

In this paper, the homotopy-perturbation method (HPM) (EL-SHAHED 2005, SIDDIQUI et al. 2006, HE 2003, HE 2004, HE 2005, LIU 2005, GOLBABAI & JAVIDI 2007, YILDIRIM & TURGUT 2007) and Padé approximants (BAKER 1975) used to solve of modelling the pollution of a system of lakes (1).

2. PADÉ APPROXIMATION

A rational approximation to $f(x)$ on $[a, b]$ is the quotient of two polynomials $P_N(x)$ and $Q_M(x)$ of degrees N and M , respectively. We use the notation $R_{N,M}(x)$ to denote this quotient. The $R_{N,M}(x)$ Padé approximations to a function $f(x)$ are given by

$$R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)} \quad \text{for } a \leq x \leq b \quad (2)$$

(BIAZAR & FARROKHI 2006).

The method of Padé requires that $f(x)$ and its derivative be continuous at $x = 0$. The polynomials used in (1) are

$$P_N(x) = p_0 + p_1x + p_2x^2 + \dots + p_Nx^N \quad (3)$$

$$Q_M(x) = 1 + q_1x + q_2x^2 + \dots + q_Mx^M \quad (4)$$

The polynomials in (3) and (4) are constructed so that $f(x)$ and $R_{N,M}(x)$ agree at $x = 0$ and their derivatives up to $N + M$ agree at $x = 0$. In the case $Q_0(x) = 1$, the approximation is just the Maclaurin expansion for $f(x)$. For a fixed value of $N + M$ the error is smallest when $P_N(x)$ and $Q_M(x)$ have the same degree or when $P_N(x)$ has degree one higher than $Q_M(x)$.

Notice that the constant coefficient of Q_M is $q_0 = 1$. This is permissible, because it notice be 0 and $R_{N,M}(x)$ is not changed when both $P_N(x)$ and $Q_M(x)$ are divided by the same constant. Hence the rational function $R_{N,M}(x)$ has $N + M + 1$ unknown coefficients. Assume that $f(x)$ is analytic and has the Maclaurin expansion

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_kx^k + \dots, \quad (5)$$

and from the difference $f(x)Q_M(x) - P_N(x) = Z(x)$:

$$\left[\sum_{i=0}^{\infty} a_i x^i \right] \left[\sum_{i=0}^M q_i x^i \right] - \left[\sum_{i=0}^N p_i x^i \right] = \left[\sum_{i=N+M+1}^{\infty} c_i x^i \right], \quad (6)$$

The lower index $j = N + M + 1$ in the summation on the right side of (6) is chosen because the first $N + M$ derivatives of $f(x)$ and $R_{N,M}(x)$ are to agree at $x = 0$.

When the left side of (6) is multiplied out and the coefficients of the powers of x^i are set equal to zero for $k = 0, 1, 2, \dots, N + M$, the result is a system of $N + M + 1$ linear equations:

$$\begin{aligned}
 a_0 - p_0 &= 0 \\
 q_1 a_0 + a_1 - p_1 &= 0 \\
 q_2 a_0 + q_1 a_1 + a_2 - p_2 &= 0 \\
 q_3 a_0 + q_2 a_1 + q_1 a_2 + a_3 - p_3 &= 0 \\
 q_M a_{N-M} + q_{M-1} a_{N-M+1} + a_N - p_N &= 0
 \end{aligned} \tag{7}$$

and

$$\begin{aligned}
 q_M a_{N-M+1} + q_{M-1} a_{N-M+2} + \dots + q_1 a_N + a_{N+2} &= 0 \\
 q_M a_{N-M+2} + q_{M-1} a_{N-M+3} + \dots + q_1 a_{N+1} + a_{N+2} &= 0 \\
 \cdot &\cdot \\
 \cdot &\cdot \\
 \cdot &\cdot \\
 q_M a_N + q_{M-1} a_{N+1} + \dots + q_1 a_{N+M+1} + a_{N+M} &= 0
 \end{aligned} \tag{8}$$

Notice that in each equation the sum of the subscripts on the factors of each product is the same, and this sum increases consecutively from 0 to $N + M$. The M equations in (8) involve only the unknowns $q_1, q_2, q_3, \dots, q_M$ and must be solved first. Then the equations in (7) are used successively to find $p_1, p_2, p_3, \dots, p_N$ (BAKER 1975).

3. HOMOTOPY PERTURBATION METHOD

To illustrate the homotopy perturbation method (HPM) for solving non-linear differential equations, He considered the following non-linear differential equation:

$$A(u) = f(r), \quad r \in \Omega \tag{9}$$

subject to the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \tag{10}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial n}$ denotes differentiation along the normal vector drawn outwards from Ω (HE 2000, HE 1999). The operator A can generally be divided into two parts M and N . Therefore, (9) can be rewritten as follows:

$$M(u) + N(u) = f(r), \quad r \in \Omega \tag{11}$$

He constructed a homotopy $v(r, p) : \Omega \times [0, 1] \rightarrow \Re$ which satisfies

$$H(v, p) = (1 - p)[M(v) - M(u_0)] + p[A(v) - f(r)] = 0, \tag{12}$$

which is equivalent to

$$H(v, p) = M(v) - M(u_0) + pM(v_0) + p[N(v) - f(r)] = 0, \tag{13}$$

where $p \in [0, 1]$ is an embedding parameter, and u_0 is an initial approximation of (9) (HE 2003, HE 2004). Obviously, we have

$$H(v, 0) = M(v) - M(u_0) = 0, \quad H(v, 1) = A(v) - f(r) = 0. \quad (14)$$

The changing process of p from zero to unity is just that of $H(v, p)$ from $M(v) - M(u_0)$ to $A(v) - f(r)$. In topology, this is called deformation and $M(v) - M(u_0)$ and $A(v) - f(r)$ are called homotopic. According to the homotopy perturbation method, the parameter p is used as a small parameter, and the solution of Eq. (12) can be expressed as a series in p in the form

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots \quad (15)$$

When $p \rightarrow 1$, Eq. (12) corresponds to the original one, Eqs. (11) and (15) become the approximate solution of Eq. (11), i.e.,

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + v_3 + \dots \quad (16)$$

The convergence of the series in Eq. (16) is discussed by He (HE 2000, HE 1999).

4. APPLICATIONS

In this section, we will apply the homotopy perturbation method to nonlinear ordinary differential systems (1).

4.1 A Homotopy Perturbation Method To Modelling The Pollution Of A System Of Lakes

According to homotopy perturbation method, we derive a correct functional as follows:

$$\begin{aligned} (1-p)(\dot{u}_1 - \dot{x}_0) + p \left(\dot{u}_1 - p - \frac{F_{13}}{V_3}u_3 + \frac{F_{31}}{V_1}u_1 + \frac{F_{21}}{V_1}u_1 \right) &= 0, \\ (1-p)(\dot{u}_2 - \dot{y}_0) + p \left(\dot{u}_2 - \frac{F_{21}}{V_1}u_1 + \frac{F_{32}}{V_2}u_2 \right) &= 0, \\ (1-p)(\dot{u}_3 - \dot{z}_0) + p \left(\dot{u}_3 - \frac{F_{31}}{V_1}u_1 - \frac{F_{32}}{V_2}u_2 + \frac{F_{13}}{V_3}u_3 \right) &= 0, \end{aligned} \quad (17)$$

where “dot” denotes differentiation with respect to t , and the initial approximations are as follows:

$$\begin{aligned} u_{1,0}(t) &= x_0(t) = x_1(0) = r_1, \\ u_{2,0}(t) &= y_0(t) = x_2(0) = r_2, \\ u_{3,0}(t) &= z_0(t) = x_3(0) = r_3. \end{aligned} \quad (18)$$

and

$$\begin{aligned} u_1 &= u_{1,0} + pu_{1,1} + p^2u_{1,2} + p^3u_{1,3} + \dots, \\ u_2 &= u_{2,0} + pu_{2,1} + p^2u_{2,2} + p^3u_{2,3} + \dots, \\ u_3 &= u_{3,0} + pu_{3,1} + p^2u_{3,2} + p^3u_{3,3} + \dots, \end{aligned} \quad (19)$$

where $v_{i,j}, i, j = 1, 2, 3, \dots$ are functions yet to be determined. Substituting Eqs.(18) and (19) into Eq. (17) and arranging the coefficients of “p” powers, we have

$$\begin{aligned}
 & \left(\dot{u}_{1,1} - p + \frac{F_{13}}{V_3} r_3 + \frac{F_{13}}{V_1} r_1 \right) p + \left(\dot{u}_{1,2} + \frac{F_{13}}{V_3} u_{3,1} + \frac{F_{13}}{V_1} u_{1,1} \right) p^2 \\
 & \quad + \left(\dot{u}_{1,3} + \frac{F_{13}}{V_3} u_{3,2} + \frac{F_{13}}{V_1} u_{1,2} \right) p^3 + \dots = 0, \\
 & \left(\dot{u}_{2,1} - \frac{F_{21}}{V_1} r_1 + \frac{F_{32}}{V_2} r_2 \right) p + \left(\dot{u}_{2,2} - \frac{F_{21}}{V_1} u_{1,1} + \frac{F_{32}}{V_2} u_{2,1} \right) p^2 \\
 & \quad + \left(\dot{u}_{2,3} - \frac{F_{21}}{V_1} u_{1,2} + \frac{F_{32}}{V_2} u_{2,2} \right) p^3 + \dots = 0, \\
 & \left(\dot{u}_{3,1} - \frac{F_{31}}{V_1} r_1 - \frac{F_{32}}{V_2} r_2 + \frac{F_{13}}{V_3} r_3 \right) p + \left(\dot{u}_{3,2} - \frac{F_{31}}{V_1} u_{1,1} - \frac{F_{32}}{V_2} u_{2,1} + \frac{F_{13}}{V_3} u_{3,1} \right) p^2 \\
 & \quad + \left(\dot{u}_{3,3} - \frac{F_{31}}{V_1} u_{1,2} - \frac{F_{32}}{V_2} u_{2,2} + \frac{F_{13}}{V_3} u_{3,2} \right) p^3 + \dots = 0,
 \end{aligned} \tag{20}$$

In order to obtain the unknowns $v_{i,j}(t), i, j = 1, 2, 3$, we must construct and solve the following system which includes nine equations with nine unknowns, considering the initial conditions

$$\begin{aligned}
 & v_{i,j}(0) = 0, i, j = 1, 2, 3, \\
 & \dot{u}_{1,1} - p + \frac{F_{13}}{V_3} r_3 + \frac{F_{13}}{V_1} r_1 = 0, \\
 & \dot{u}_{1,2} + \frac{F_{13}}{V_3} u_{3,1} + \frac{F_{13}}{V_1} u_{1,1} = 0, \\
 & \dot{u}_{1,3} + \frac{F_{13}}{V_3} u_{3,2} + \frac{F_{13}}{V_1} u_{1,2} = 0, \\
 & \dot{u}_{2,1} - \frac{F_{21}}{V_1} r_1 + \frac{F_{32}}{V_2} r_2 = 0, \\
 & \dot{u}_{2,2} - \frac{F_{21}}{V_1} u_{1,1} + \frac{F_{32}}{V_2} u_{2,1} = 0, \\
 & \dot{u}_{2,3} - \frac{F_{21}}{V_1} u_{1,2} + \frac{F_{32}}{V_2} u_{2,2} = 0, \\
 & \dot{u}_{3,1} - \frac{F_{31}}{V_1} r_1 - \frac{F_{32}}{V_2} r_2 + \frac{F_{13}}{V_3} r_3 = 0, \\
 & \dot{u}_{3,2} - \frac{F_{31}}{V_1} u_{1,1} - \frac{F_{32}}{V_2} u_{2,1} + \frac{F_{13}}{V_3} u_{3,1} = 0, \\
 & \dot{u}_{3,3} - \frac{F_{31}}{V_1} u_{1,2} - \frac{F_{32}}{V_2} u_{2,2} + \frac{F_{13}}{V_3} u_{3,2} = 0,
 \end{aligned} \tag{21}$$

From Eq. (16), if the three terms approximations are sufficient, we will obtain:

$$\begin{aligned}
 x_1(t) &= \lim_{p \rightarrow 1} u_1(t) = \sum_{k=0}^2 u_{1,k}(t), \\
 x_2(t) &= \lim_{p \rightarrow 1} u_2(t) = \sum_{k=0}^2 u_{2,k}(t), \\
 x_3(t) &= \lim_{p \rightarrow 1} u_3(t) = \sum_{k=0}^2 u_{3,k}(t),
 \end{aligned}
 \tag{22}$$

therefore

$$\begin{aligned}
 x_1(t) &= r_1 + \left(p - \frac{F_{13}}{V_3} r_3 - \frac{F_{13}}{V_1} r_1 \right) t \\
 &\quad - \frac{F_{13}}{2V_1^2 V_2 V_3^2} \left[r_1 V_1 V_2 V_3 F_{31} + r_2 V_1^2 V_3 F_{32} - r_3 V_1^2 V_2 F_{31} \right] t^2 \\
 x_2(t) &= r_2 + \left(\frac{F_{21}}{V_1} r_1 - \frac{F_{32}}{V_2} r_2 \right) t \\
 &\quad - \frac{1}{2V_1^2 V_2^2 V_3} \left[-p V_1 V_2^2 V_3 F_{21} + r_3 V_1 V_2^2 F_{13} F_{21} + r_1 V_3 V_2^2 F_{13} F_{21} \right] t^2 \\
 x_3(t) &= r_3 + \left(\frac{F_{31}}{V_1} r_1 + \frac{F_{32}}{V_2} r_2 - \frac{F_{13}}{V_3} r_3 \right) t \\
 &\quad - \frac{1}{2V_1^2 V_2^2 V_3^2} \left[\begin{aligned} &-p V_1 V_2^2 V_3^2 F_{31} - r_3 V_1 V_2^2 V_3 F_{13} F_{31} - r_1 V_2^2 V_3^2 F_{13} F_{31} \\ &r_1 V_1 V_2 V_3^2 F_{32} F_{21} - r_2 V_1^2 V_3^2 F_{32}^2 - r_1 V_1 V_2^2 V_3 F_{31}^2 \\ &-r_2 V_1^2 V_2 V_3 F_{31} + r_3 V_1^2 V_2^2 F_{31}^2 \end{aligned} \right] t^2 \\
 &\quad \vdots \\
 &\quad \vdots \\
 &\quad \vdots
 \end{aligned}
 \tag{23}$$

Here $x_1(0) = 0$, $x_2(0) = 0$ and $x_3(0) = 0$ for the four-component model. Parameters,

$$V_1 = 2900 \text{ mi}^3, V_2 = 850 \text{ mi}^3, V_3 = 1180 \text{ mi}^3,$$

$$F_{21} = 18 \text{ mi}^3/\text{year}, F_{32} = 18 \text{ mi}^3/\text{year}, F_{31} = 20 \text{ mi}^3/\text{year}, F_{13} = 38 \text{ mi}^3/\text{year},$$

A few first approximations for $x_1(t)$, $x_2(t)$ and $x_3(t)$ are calculated and presented below.

Three terms approximations:

$$\begin{aligned}
 x_1(t) &= 100t - 6.5517t^2 + 2492t^3, \\
 x_2(t) &= .3103t^2 - .0157t^3, \\
 x_3(t) &= .3448t^2 - .0148t^3.
 \end{aligned}
 \tag{24}$$

Four terms approximations:

$$\begin{aligned}
 x_1(t) &= 100t - 6.5517t^2 + 2492t^3 - 0.007t^4, \\
 x_2(t) &= .3103t^2 - .0157t^3 + .00046998t^4, \\
 x_3(t) &= .3448t^2 - .0148t^3 + .000409t^4.
 \end{aligned}
 \tag{25}$$

Five terms approximations:

$$\begin{aligned} x_1(t) &= 100t - 6.5517t^2 + 2492t^3 - 0.007t^4 + .15629e-3t^5, \\ x_2(t) &= .3103t^2 - .0157t^3 + .00046998t^4 - .10641e-4t^5, \\ x_3(t) &= .3448t^2 - .0148t^3 + .000409t^4 - .90081e-5t^5. \end{aligned} \quad (26)$$

Six terms approximations:

$$\begin{aligned} x_1(t) &= 100t - 6.5517t^2 + 2492t^3 - 0.007t^4 + .15629e-3t^5 - .29297e-5t^6, \\ x_2(t) &= .3103t^2 - .0157t^3 + .00046998t^4 - .10641e-4t^5 + .19924e-6t^6, \\ x_3(t) &= .3448t^2 - .0148t^3 + .000409t^4 - .90081e-5t^5 + .16753e-6t^6. \end{aligned} \quad (27)$$

In this section, we apply Laplace transformation to (27), which yields

$$\begin{aligned} L(x_1(s)) &= \frac{100}{s^2} - \frac{13.1034}{s^3} + \frac{1.4952}{s^4} - \frac{.168}{s^5} + \frac{.0187548}{s^6} \\ &\quad - \frac{.002109384}{s^7} \\ L(x_2(s)) &= \frac{.6206}{s^3} - \frac{.0942}{s^4} + \frac{.01127952}{s^5} - \frac{.00127692}{s^6} \\ &\quad + \frac{.0001434528}{s^7} \\ L(x_3(s)) &= \frac{.6896}{s^3} - \frac{.0888}{s^4} + \frac{.009816}{s^5} - \frac{.001080972}{s^6} \\ &\quad + \frac{.0001206216}{s^7} \end{aligned} \quad (28)$$

For simplicity, let $s = \frac{1}{t}$; then

$$\begin{aligned} L(x_1(t)) &= 100t^2 - 13.1034t^3 + 1.4952t^4 - .168t^5 + .0187548t^6 - .002109384t^7 \\ L(x_2(t)) &= .6206t^3 - .0942t^4 + .01127952t^5 - .00127692t^6 + .0001434528t^7 \\ L(x_3(t)) &= .6896t^3 - .0888t^4 + .009816t^5 - .001080972t^6 + .0001206216t^7 \end{aligned} \quad (29)$$

Padé approximant $[4/4]$ of (29) and substituting $t = \frac{1}{s}$, we obtain $[4/4]$ in terms of s .

By using the inverse Laplace transformation, we obtain

$$\begin{aligned} x_1(t) &= -1078.819711e^{-.1117100809t} + 1081.956487e^{-.01872861322t} - 3.136776224e^{.08017549675t} \\ x_2(t) &= 76.39376929e^{-.1109811493t} - 423.6377791e^{-.02937987211t} + .1588638879e^{-5e^{16.12398814t}} \\ x_3(t) &= 65.38393811e^{-.1105664459t} + .7617257846e^{-7e^{-11.31331389t}} \\ &\quad + e^{-.009101928498t} [-65.38393811\cos(.01587086927t) + 418.0079626\sin(.01587086927t)] \end{aligned} \quad (30)$$

Table 1. Differences between the 6-term HPM and the the Padé approximations solutions for the modelling the pollution of a system of lakes when $\varepsilon = 0.1$.

t	$x_1(t)$	$x_2(t)$	$x_3(t)$
0	2.2400e-007	1.5786e-006	7.6173e-008
0.1	2.2060e-007	7.9572e-006	2.3633e-007
0.2	2.1729e-007	3.9943e-005	7.3233e-007
0.3	2.1405e-007	2.0034e-004	2.2694e-006
0.4	2.1080e-007	0.0010	7.0335e-006
0.5	2.0728e-007	0.0050	2.1801e-005
0.6	2.0288e-007	0.0253	6.7576e-005
0.7	1.9632e-007	0.1267	2.0947e-004
0.8	1.8528e-007	0.6354	6.4930e-004
0.9	1.6577e-007	3.1866	0.0020
1	1.3144e-007	15.9803	0.0062

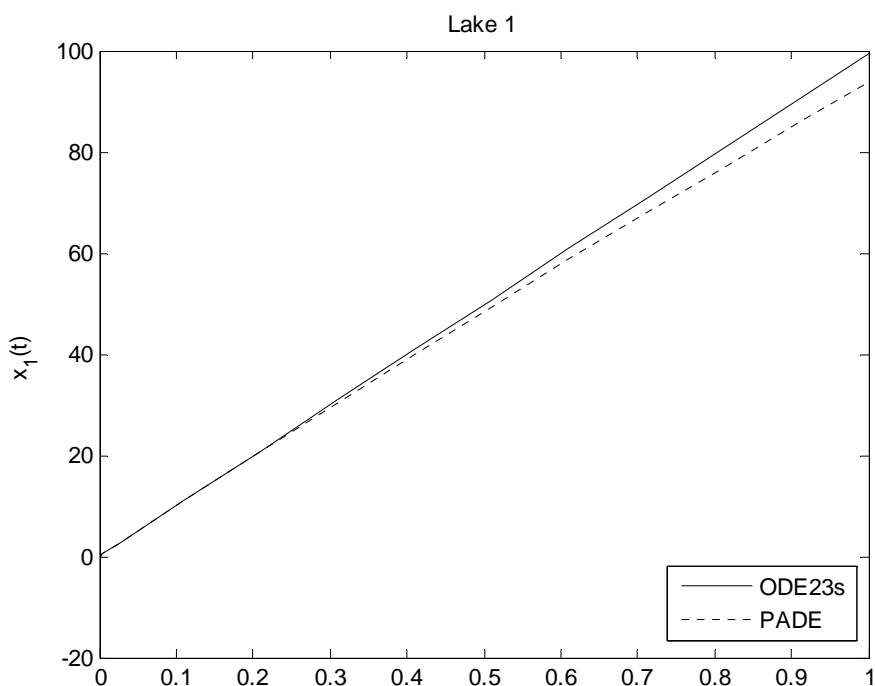


Figure 1. Local changes of $x_1(t)$

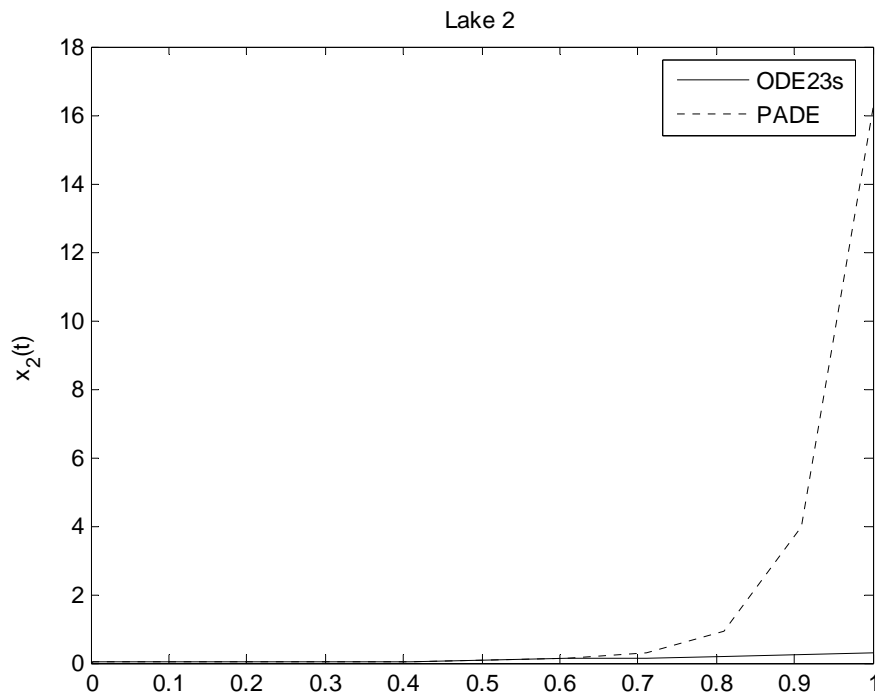


Figure 2. Local changes of $x_2(t)$

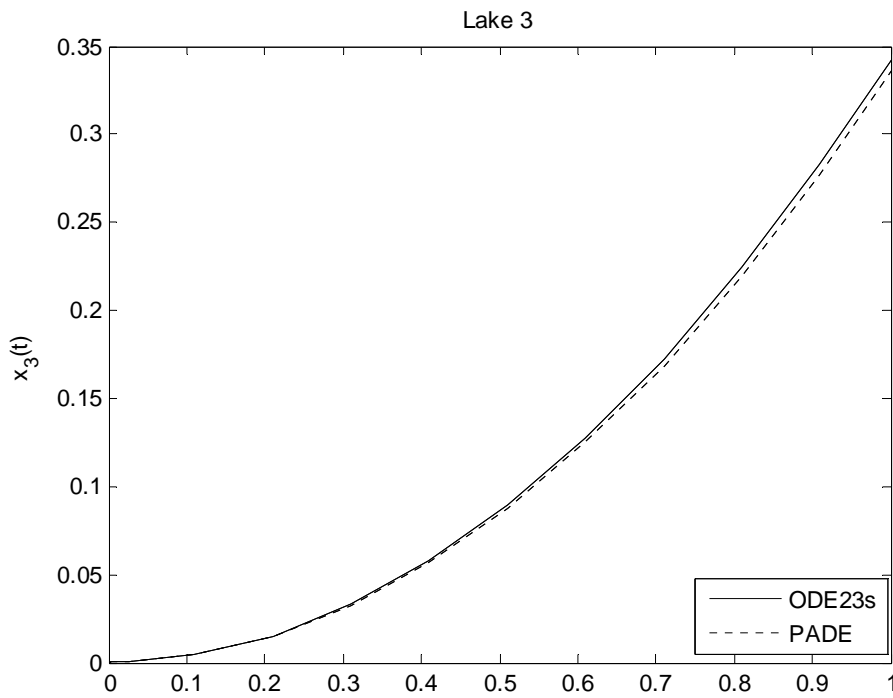


Figure 3. Local changes of $x_3(t)$

5. CONCLUSIONS

In this paper, we have presented an after treatment technique for the homotopy perturbation method. Because the Padé approximant usually improves greatly the Maclaurin series in the convergence region and the convergence rate, the at leads to a better analytic approximate solution from homotopy perturbation method truncated series The homotopy perturbation method was used for finding the solutions of nonlinear ordinary differential equation systems such as modelling the pollution of a system of lakes. We demonstrated the accuracy and efficiency of these methods by solving some ordinary differential equation systems. We use Laplace transformation and Padé approximant to obtain an analytic solution and to improve the accuracy of homotopy perturbation method. The reliability of the method and reduction in the size of computational domain give this method a wider applicability. It is observed that the results to get the homotopy perturbation method (HPM) applied Padé approximants is an effective and reliable tool for the solution of the nonlinear ordinary differential equation systems considered in the present paper. The homotopy perturbation method (HPM) can be successfully used to model lake systems. Results are compared with Matlab ODE23s.

The computations associated with the examples in this paper were performed using Maple 7 and Matlab 7.

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