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On the Existence of the Solutions of A Nonlinear Fredholm Integral Equation in Hölder Spaces

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Abstract
In this article, we prove the theorem concerning the existence of the solutions for some nonlinear integral equations. As an application, we investigate the problem of existence of solutions of Fredholm integral equations using the technique of relative compactness in conjunction with fixed point theorem. Our solutions are placed in the space of functions satisfying the Hölder condition. Our work is more general than the previous works in [1–3]. In the last section, we show the efficiency of this approach on one numerical example.

Keywords: *Hölder condition; Fredholm integral equation; Schauder fixed point theorem.*

AMS Subject Classification (2020): *Primary: 45B05 ; Secondary: 45G10; 47H10.*

1. Introduction and Preliminaries

Integral equations appear in most applied areas and are as important as differential equations. Nonlinear integral equations are frequently studied in research articles [1–32].

The symbol $\mathbb{R}$ will stand for the set of real numbers and put $\mathbb{R}^+ = [0, \infty)$. Let's give some inequalities that we use in some sections of the article.

Lemma 1.1. *Let u, v be arbitrary real numbers such that $1 \leq v < u$. Moreover, let a be an arbitrarily fixed nonnegative number. Then, the following inequality is satisfied*

$$\left| (x^v + a)^{\frac{1}{u}} - (y^v + a)^{\frac{1}{u}} \right| \leq |x - y|^{\frac{v}{u}} \tag{1.1}$$

for all $x, y \in \mathbb{R}$,[4].

Lemma 1.2. *[4] Observe that using the notation of the generalized root of an arbitrary degree $u (u > 0)$, i.e. putting $\sqrt[u]{x} = x^{\frac{1}{u}}$ for $x \in \mathbb{R}^+$, we can represent inequality (1.1) in a more transparent form*

$$\left| \sqrt[u]{(x^v + a)} - \sqrt[u]{(y^v + a)} \right| \leq \sqrt[u]{|x - y|^v}.$$

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Observe that in the case when v is a natural even number inequality (1.1) can be extended to the whole real axis $\mathbb{R}$ i.e., if $v = 2n$, where $n \in \mathbb{N}$, then for an arbitrary number $u > 2n$ the following inequality is satisfied

$$\left|(x^{2n}+a)^{\frac{1}{u}} - (y^{2n}+a)^{\frac{1}{u}}\right| \leq |x-y|^{\frac{2n}{u}}$$

that is

$$\left|\sqrt[u]{(x^{2n}+a)} - \sqrt[u]{(y^{2n}+a)}\right| \leq \sqrt[u]{(x-y)^{2n}}$$

for all $x, y \in \mathbb{R}$ and $a \geq 0$.

In the case when $a = 0$ we have that $f(x) = x^{\frac{v}{u}}$. Applying the standard methods of mathematical analysis (second derivative, the concavity and the subadditivity of the function f) we can easily show that

$$\left|x^{\frac{v}{u}} - y^{\frac{v}{u}}\right| \leq |x-y|^{\frac{v}{u}}$$

for all $x, y \in \mathbb{R}^+$. The following known definitions are available in [1, 2, 31, 32].

Let $[\lambda, \mu]$ be a closed interval in $\mathbb{R}$, by $C[\lambda, \mu]$ we indicate the space of continuous functions defined on $[\lambda, \mu]$ equipped with the supremum norm, i.e.,

$$\|x\|_\infty = \sup\{|x(u)| : u \in [\lambda, \mu]\}$$

for $x \in C[\lambda, \mu]$. For a fixed α with $0 < \alpha \leq 1$, by $H_\alpha[\lambda, \mu]$ we will state the spaces of the real functions x defined on $[\lambda, \mu]$ and satisfying the Hölder condition, that is, those functions x for which there exists a constant H_x^α such that

$$|x(u) - x(v)| \leq H_x^\alpha |u-v|^\alpha \tag{1.2}$$

for all $u, v \in [\lambda, \mu]$. It is well proved that $H^\alpha[\lambda, \mu]$ is a linear subspaces of $C[\lambda, \mu]$. Also, for $x \in H^\alpha[\lambda, \mu]$, by H_x^α we will state the least possible stable for which inequality (1.2) is satisfied. Rather, we put

$$H_x^\alpha = \sup\left\{\frac{|x(u) - x(v)|}{|u-v|^\alpha} : u, v \in [\lambda, \mu] \text{ and } u \neq v\right\}. \tag{1.3}$$

The space $H_\alpha[\lambda, \mu]$ with $0 < \alpha \leq 1$ may be equipped with the norm

$$\|x\|_\alpha = |x(\lambda)| + H_x^\alpha$$

for $x \in H_\alpha[\lambda, \mu]$. Here, H_x^α is defined by (1.3). In [1], the authors show that $(H_\alpha[\lambda, \mu], \|\cdot\|_\alpha)$ with $0 < \alpha \leq 1$ is a Banach space.

Theorem 1.1 (Schauder's Fixed Point Theorem)**.** *Let E be a nonempty, compact subset of a Banach space $(X, \|\cdot\|)$, convex and let $T : E \to E$ be a continuity mapping. Then T has at least one fixed point in E,[9].*

Lemma 1.3. *For $0 < \alpha < \beta \leq 1$, we have*

$$H_\beta[\lambda, \mu] \subset H_\alpha[\lambda, \mu] \subset C[\lambda, \mu].$$

Furthermore, for $x \in H_\beta[\lambda, \mu]$, we have:

$$\|x\|_\alpha \leq \max\left(1, (\mu-\lambda)^{\beta-\alpha}\right)\|x\|_\beta.$$

Particularly, the inequality $\|x\|_\infty \leq \|x\|_\alpha \leq \|x\|_\beta$ is satisfied for $\lambda = 0$ and $\mu = 1$, [1].

Lemma 1.4. *Let's assume that $0 < \alpha < \beta \leq 1$ and E is a bounded subset in $H_\beta[\lambda, \mu]$, then E is a relatively compact subset in $H_\alpha[\lambda, \mu]$, [2].*

Lemma 1.5. *Assume that $0 < \alpha < \beta \leq 1$ and by B_r^β we state the ball centered at θ and radius r in the space $H_\beta[\lambda, \mu]$, i.e., $B_r^\beta = \{x \in H_\beta[\lambda, \mu] : \|x\|_\beta \leq r\}$. B_r^β is a closed subset of $H_\alpha[\lambda, \mu]$, [2].*

Corollary 1.1. *Assume that $0 < \alpha < \beta \leq 1$ and B_r^β is a relatively compact subset in $H_\alpha[\lambda, \mu]$ and is a closed subset of $H_\alpha[\lambda, \mu]$, then B_r^β is a compact subset in the space $H_\alpha[\lambda, \mu]$, [2].*

2. Main Result

J. Banaś and R. Nalepa et al. [1] study the following equation;

$$x(u) = p(u) + x(u)\int_{\lambda}^{\mu} k(u,\tau)x(\tau)d\tau. \tag{2.1}$$

Also, J. Caballero, M. Darwish and K. Sadarangani et al. [2] study the following equation;

$$x(u) = p(u) + x(u)\int_{0}^{1} k(u,\tau)x(r(\tau))d\tau. \tag{2.2}$$

Further, S. Peng, J. Wang and J. Chen et al. [3] study the following equation;

$$x(u) = f(u,x(u)) + x(u)\int_{\lambda}^{\mu} k(u,\tau)x(\tau)d\tau. \tag{2.3}$$

The purpose of this paper is to examine the existence of solutions of the following integral equation of Fredholm type with a changed argument,

$$x(u) = (Gx)(u) + x(u)\int_{0}^{1} k(u,\tau)x(q(\tau))d\tau, \ \ u \in I = [0,1]. \tag{2.4}$$

The equation (2.4) is more general than many equations considered up to now and includes (2.1), (2.2) and (2.3) as special cases. Notice that the equation (2.1) in [1] for $\lambda = 0$ and $\mu = 1$ is a particular case of (2.4) with $q(\tau) = \tau$ and $(Gx)(u) = p(u)$. Also, it should be noted that the equation (2.4) is the more general than the equation (2.2) considered in [2]. If we take $(Gx)(u) = p(u)$, then the equation

$$x(u) = p(u) + x(u)\int_{0}^{1} k(u,\tau)x(r(\tau))d\tau$$

is obtained from the equation (2.4). Further, notice that equation (2.3) in study [3] for $\lambda = 0$ and $\mu = 1$ is a particular case of (2.4), for $(Gx)(u) = f(u,x(u)), q(\tau) = \tau$.

Theorem 2.1. *Assume that the following conditions $(i) - (iv)$ are satisfied:*

(i) *The operator $G : H_\beta[0,1] \to H_\beta[0,1]$ is continuous on the space $H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$, where $0 < \alpha < \beta \leq 1$ and there exists function $w : \mathbb{R}_+ \to \mathbb{R}_+$ which is non-decreasing such that it holds the inequality*

$$\|Gx\|_\beta \leq w\left(\|x\|_\beta\right),$$

for any $x \in H_\beta[0,1]$.

(ii) *$k : [0,1]\times[0,1] \to \mathbb{R}$ is a continuous function such that there exists a constant k_β such that*

$$|k(u,\tau) - k(v,\tau)| \leq k_\beta |u-v|^\beta,$$

for any $u, v, \tau \in [0,1]$.

(iii) *$q : [0,1] \to [0,1]$ is a measurable function.*

(iv) *There exists a positive solution r_0 of the inequality*

$$w(r) + (2K + k_\beta)r^2 \leq r,$$

where the constant K is defined by

$$\sup\left\{\int_0^1 |k(u,\tau)|d\tau : u \in [0,1]\right\} \leq K.$$

Then the equation (2.4) has at least one solution $x = x(u)$ belonging to space $H_\alpha[0,1]$.

Proof. Note that we suppose unless stated otherwise that α and β are arbitrarily fixed numbers such that $0 < \alpha < \beta \leq 1$. Now, let us consider $x \in H_\beta[0,1]$ and the operator N defined on the space $H_\beta[0,1]$ by the formula:

$$(Nx)(u) = (Gx)(u) + x(u)\int_0^1 k(u,\tau)x(q(\tau))d\tau,$$

for $u \in [0,1]$. Then for arbitrarily fixed $u, v \in [0,1], (u \neq v)$, in view of our assumptions we get

$$\begin{aligned}
(Nx)(u) - (Nx)(v) &= (Gx)(u) + x(u)\int_0^1 k(u,\tau)x(q(\tau))d\tau - (Gx)(v) - x(v)\int_0^1 k(v,\tau)x(q(\tau))d\tau \\
&= (Gx)(u) - (Gx)(v) + x(u)\int_0^1 k(u,\tau)x(q(\tau))d\tau - x(v)\int_0^1 k(v,\tau)x(q(\tau))d\tau \\
&\quad + x(v)\int_0^1 k(u,\tau)x(q(\tau))d\tau - x(v)\int_0^1 k(u,\tau)x(q(\tau))d\tau \\
&= (Gx)(u) - (Gx)(v) + (x(u) - x(v))\int_0^1 k(u,\tau)x(q(\tau))d\tau \\
&\quad + x(v)\int_0^1 (k(u,\tau) - k(v,\tau))x(q(\tau))d\tau
\end{aligned}$$

and

$$\begin{aligned}
\frac{|(Nx)(u) - (Nx)(v)|}{|u-v|^\beta} &\leq \frac{|(Gx)(u) - (Gx)(v)|}{|u-v|^\beta} + \frac{|x(u) - x(v)|}{|u-v|^\beta}\int_0^1 |k(u,\tau)||x(q(\tau))|d\tau \\
&\quad + \frac{|x(v)|}{|u-v|^\beta}\int_0^1 |k(u,\tau) - k(v,\tau)||x(q(\tau))|d\tau \\
&\leq \frac{|(Gx)(u) - (Gx)(v)|}{|u-v|^\beta} + \|x\|_\infty \|x\|_\beta \int_0^1 |k(u,\tau)|d\tau \\
&\quad + \|x\|_\infty \|x\|_\infty \int_0^1 \frac{|k(u,\tau) - k(v,\tau)|}{|u-v|^\beta}d\tau \\
&\leq \frac{|(Gx)(u) - (Gx)(v)|}{|u-v|^\beta} + \|x\|_\beta^2 K + \|x\|_\beta^2 \int_0^1 k_\beta \frac{|u-v|^\beta}{|u-v|^\beta}d\tau \\
&\leq \frac{|(Gx)(u) - (Gx)(v)|}{|u-v|^\beta} + (K + k_\beta)\|x\|_\beta^2. \qquad (2.5)
\end{aligned}$$

Considering the (i) hypothesis, this demonstrates that the operator N maps $H_\beta[0,1]$ into itself. Besides, for any $x \in H_\beta[0,1]$, we get

$$\begin{aligned}
|(Nx)(0)| &\leq |(Gx)(0)| + |x(0)|\int_0^1 |k(0,\tau)||x(q(\tau))|d\tau \\
&\leq |(Gx)(0)| + \|x\|_\infty \|x\|_\infty \int_0^1 |k(0,\tau)|d\tau \\
&\leq |(Gx)(0)| + \|x\|_\beta^2 K. \qquad (2.6)
\end{aligned}$$

By the inequalities by (2.5) and (2.6), we derive that

$$\|Nx\|_\beta \leq \|Gx\|_\beta + (2K + k_\beta)\|x\|_\beta^2. \qquad (2.7)$$

Since positive number r_0 is the solution of the inequality given in hypothesis (iv), from (2.7) and function $w : \mathbb{R}_+ \to \mathbb{R}_+$ which is non-decreasing, we conclude that the inequality

$$\|Nx\|_\beta \leq w(r_0) + (2K + k_\beta)r_0^2 \leq r_0. \qquad (2.8)$$

As a results, it follows that N transform the ball

$$B_{r_0}^\beta = \{x \in H_\beta[0,1] : \|x\|_\beta \leq r_0\}$$

into itself. That is, $N : B^\beta_{r_0} \to B^\beta_{r_0}$. Thus, we have that the set $B^\beta_{r_0}$ is relatively compact in $H_\alpha[0,1]$ for any $0 < \alpha < \beta \le 1$. Furthermore, $B^\beta_{r_0}$ is a compact subset in $H_\alpha[0,1]$.

In the sequel, we will demonstrate that the operator N is continuous on $B^\beta_{r_0}$ with respect to the norm $\|\cdot\|_\alpha$, where $0 < \alpha < \beta \le 1$.

Let $y \in B^\beta_{r_0}$ be an arbitrary point in $B^\beta_{r_0}$. Then, we get

$$\begin{aligned}(Nx)(u) - (Ny)(u) - ((Nx)(v) - (Ny)(v)) &= (Gx)(u) + x(u)\int_0^1 k(u,\tau)x(q(\tau))d\tau \\ &\quad -(Gy)(u) - y(u)\int_0^1 k(u,\tau)y(q(\tau))d\tau \\ &\quad -(Gx)(v) - x(v)\int_0^1 k(v,\tau)x(q(\tau))d\tau \\ &\quad +(Gy)(v) + y(v)\int_0^1 k(v,\tau)y(q(\tau))d\tau \qquad (2.9)\end{aligned}$$

for any $x \in B^\beta_{r_0}$ and $u, v \in [0,1]$. The equality (2.9) can be rewritten as

$$\begin{aligned}&(Nx)(u) - (Ny)(u) - ((Nx)(v) - (Ny)(v)) \\ &= (Gx)(u) - (Gy)(u) - ((Gx)(v) - (Gy)(v)) + (x(u) - y(u))\int_0^1 k(u,\tau)x(q(\tau))d\tau \\ &+y(u)\left[\int_0^1 k(u,\tau)x(q(\tau))d\tau - \int_0^1 k(u,\tau)y(q(\tau))d\tau\right] \\ &-(x(v) - y(v))\int_0^1 k(v,\tau)x(q(\tau))d\tau \\ &-y(v)\left[\int_0^1 k(v,\tau)x(q(\tau))d\tau - \int_0^1 k(v,\tau)y(q(\tau))d\tau\right]. \qquad (2.10)\end{aligned}$$

By (2.10), we have

$$\begin{aligned}(Nx)(u) - (Ny)(u) - ((Nx)(v) - (Ny)(v)) &= (Gx)(u) - (Gy)(u) - ((Gx)(v) - (Gy)(v)) \\ &\quad +[x(u) - y(u) - (x(v) - y(v))]\int_0^1 k(u,\tau)x(q(\tau))d\tau \\ &\quad +(x(v) - y(v))\left[\int_0^1 k(u,\tau)x(q(\tau))d\tau - \int_0^1 k(v,\tau)x(q(\tau))d\tau\right] \\ &\quad +y(u)\int_0^1 k(u,\tau)(x(q(\tau)) - y(q(\tau))d\tau \\ &\quad -y(v)\int_0^1 k(v,\tau)(x(q(\tau)) - y(q(\tau))d\tau. \qquad (2.11)\end{aligned}$$

(2.11) yields the following inequality:

$$\begin{aligned}|(Nx)(u) - (Ny)(u)) - ((Nx)(v) - (Ny)(v)| &\le |(Gx)(u) - (Gy)(u) - ((Gx)(v) - (Gy)(v))| \\ &\quad +|x(u) - y(u) - (x(v) - y(v))|\int_0^1 |k(u,\tau)||x(q(\tau))|d\tau \\ &\quad +|x(v) - y(v)|\int_0^1 |k(u,\tau) - k(v,\tau)||x(q(\tau))|d\tau \\ &\quad +|y(u) - y(v)|\int_0^1 |k(u,\tau)||(x(q(\tau)) - y(q(\tau))|d\tau \\ &\quad +|y(v)|\int_0^1 |k(u,\tau) - k(v,\tau)||(x(q(\tau)) - y(q(\tau))|d\tau. \qquad (2.12)\end{aligned}$$

Hence, taking into account (2.12), we can write:

$$\begin{aligned}\frac{|(Nx)(u)-(Ny)(u)-((Nx)(v)-(Ny)(v))|}{|u-v|^\alpha} \quad\leq\quad & \frac{|(Gx)(u)-(Gy)(u)-((Gx)(v)-(Gy)(v))|}{|u-v|^\alpha} \\ & + \frac{|(x(u)-y(u))-(x(v)-y(v))|}{|u-v|^\alpha}\|x\|_\infty K \\ & + \|u-v\|_\infty \|x\|_\infty \int_0^1 \frac{|k(u,\tau)-k(v,\tau)|}{|u-v|^\alpha} d\tau + \frac{|y(u)-y(v)|}{|u-v|^\alpha}\|x-y\|_\infty K \\ & + \|y\|_\infty \|x-y\|_\infty \int_0^1 \frac{|k(u,\tau)-k(v,\tau)|}{|u-v|^\alpha} d\tau \qquad (2.13)\end{aligned}$$

for all $u, v \in [0,1]$ with $u \neq v$. Therefore the equality

$$\begin{aligned}(Nx)(0)-(Ny)(0) &= (Gx)(0)-(Gy)(0)+x(0)\int_0^1 k(0,\tau)x(q(\tau))d\tau - y(0)\int_0^1 k(0,\tau)y(q(\tau))d\tau \\ &= (Gx)(0)-(Gy)(0)+(x(0)-y(0))\int_0^1 k(0,\tau)x(q(\tau))d\tau \\ &\quad + y(0)\int_0^1 k(0,\tau)[x(q(\tau))-y(q(\tau))]d\tau\end{aligned}$$

holds. So, we get the inequality

$$|(Nx)(0)-(Ny)(0)| \leq |(Gx)(0)-(Gy)(0)| + |x(0)-y(0)|K\|x\|_\infty + |y(0)|\|x-y\|_\infty K. \qquad (2.14)$$

Moreover, since $\|x\|_\infty \leq \|x\|_\alpha \leq r_0$, $\|y\|_\infty \leq \|y\|_\alpha \leq r_0$ and $\|x-y\|_\infty \leq \|x-y\|_\alpha$, from (2.13) and (2.14), we have that

$$\begin{aligned}\|Nx-Ny\|_\alpha &= |(Nx-Ny)(0)| + H^\alpha_{Nx-Ny} \\ &= |(Nx)(0)-(Ny)(0)| \\ &\quad + \sup\left\{\frac{|(Nx)(u)-(Ny)(u)-((Nx)(v)-(Ny)(v))|}{|u-v|^\alpha} : u,v\in[0,1] \text{ and } u\neq v\right\} \\ &\leq \|Gx-Gy\|_\alpha + \|x-y\|_\alpha K\|x\|_\infty + \|y\|_\alpha\|x-y\|_\infty K \\ &\quad + \|x-y\|_\infty(\|x\|_\infty+\|y\|_\infty) \\ &\leq \|Gx-Gy\|_\alpha + \|x-y\|_\alpha(\|x\|_\alpha+\|y\|_\alpha)(K+1) \\ &\leq \|Gx-Gy\|_\alpha + \|x-y\|_\alpha 2r_0(K+1). \qquad (2.15)\end{aligned}$$

Since the operator $G : H_\beta[0,1] \to H_\beta[0,1]$ is continuous on $H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$, it is also continuous at the point $y \in B^\beta_{r_0}$. Let us take an arbitrary $\varepsilon > 0$. There exists $\delta > 0$ such that the inequality:

$$\|Gx-Gy\|_\alpha < \frac{\varepsilon}{2}$$

where $\|x-y\|_\alpha < \delta$ and

$$0 < \delta < \frac{\varepsilon}{4r_0(K+1)}.$$

Then, taking into account (2.15), we derive the following inequality:

$$\|Nx-Ny\|_\alpha < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

As a results, we infer that the operator N is continuous at the point $y \in B^\beta_{r_0}$. Because y was chosen arbitrarily, we deduce that N is continuous on $B^\beta_{r_0}$ with respect to the norm $\|\cdot\|_\alpha$. As $B^\beta_{r_0}$ is compact in $H_\alpha[0,1]$, from the classical Schauder fixed point theorem, we get the desired result. □

3. An Example

Example 3.1. Let us consider the following nonlinear quadratic integral equation:

$$x(u)=\frac{1}{3}\left(x^2(u)+x(u)+\frac{3}{2^{10}}\right)+x(u)\int_0^1 \sqrt[7]{mu^4+\tau x}\left(\sqrt{\frac{1}{\tau+1}}\right)d\tau, \tag{3.1}$$

where $u\in I=[0,1]$ and m is the real number.

Set $(Gx)(u)=\frac{1}{3}\left(x^2(u)+x(u)+\frac{3}{2^{10}}\right)$, $k(u,\tau)=\sqrt[7]{mu^4+\tau}$ and $q(\tau)=\sqrt{\frac{1}{\tau+1}}$. We will show that the operator G continuous according to be norm with $\|.\|_\alpha$. To do this, fix arbitrarily $\varepsilon>0$ and $y\in H_\beta[0,1]$. Assume that $x\in H_\beta[0,1]$ is an arbitrary function and $\|x-y\|_\alpha<\delta$, where δ is a positive number such that

$$0<\delta\le\frac{1}{12}\left(-(6\|y\|_\alpha+2)+\sqrt{(6\|y\|_\alpha+2)^2+36\varepsilon}\right).$$

Then, for arbitrary $u,v\in[0,1]$ we obtain

$$\begin{aligned}
3(Gx-Gy)(u)-3(Gx-Gy)(v) &= x^2(u)+x(u)+\frac{3}{2^{10}}-y^2(u)-y(u)-\frac{3}{2^{10}}\\
&\quad -\left(x^2(v)+x(v)+\frac{3}{2^{10}}-y^2(v)-y(v)-\frac{3}{2^{10}}\right)\\
&= x^2(u)-y^2(u)-(x^2(v)-y^2(v))+x(u)-y(u)-(x(v)-y(v))\\
&= (x(u)-y(u))(x(u)+y(u))-(x(u)-y(v))(x(v)+y(v))\\
&\quad +(x(u)-y(u)-(x(v)-y(v)))\\
&= [x(u)-y(v)-(x(v)-y(v))](x(u)+y(u))\\
&\quad +(x(v)-y(v))(x(u)+y(u)-x(v)-y(v))\\
&\quad +x(u)-y(u)-(x(v)-y(v))\\
&= [x(u)-y(u)-(x(v)-y(v))](x(u)+y(u)+1)\\
&\quad +(x(v)-y(v))(x(u)+y(u)-x(v)-y(v)).
\end{aligned} \tag{3.2}$$

By (3.2), we get

$$\begin{aligned}
3|(Gx-Gy)(u)-(Gx-Gy)(v)| &\le (\|x+y\|_\infty+1)|x(u)-y(u)-(x(v)-y(v))|\\
&\quad +\|x-y\|_\infty|x(u)+y(u)-x(v)-y(v)|\\
&\le (\|x+y\|_\alpha+1)|x(u)-y(u)-(x(v)-y(v))|\\
&\quad +\|x-y\|_\alpha|x(u)+y(u)-x(v)-y(v)|.
\end{aligned} \tag{3.3}$$

By (3.3), we have:

$$3\sup\left\{\frac{|(Gx-Gy)(u)-(Gx-Gy)(v)|}{|u-v|^\alpha}:u,v\in[0,1] \text{ and } u\ne v\right\}$$

$$\begin{aligned}
&\le (\|x+y\|_\alpha+1)\sup\left\{\frac{|x(u)-y(u)-(x(v)-y(v))|}{|u-v|^\alpha}:u,v\in[0,1]\text{ and } u\ne v\right\}\\
&\quad +\|x-y\|_\alpha\sup\left\{\frac{|x(u)+y(u)-(x(v)+y(v))|}{|u-v|^\alpha}:u,v\in[0,1]\text{ and } u\ne v\right\}\\
&\le (\|x+y\|_\alpha+1)\|x-y\|_\alpha+\|x-y\|_\alpha\|x+y\|_\alpha\\
&\le \|x-y\|_\alpha(2\|x+y\|_\alpha+1).
\end{aligned} \tag{3.4}$$

From (3.4), we obtain the following inequality:

$$\begin{aligned}
3\|Gx - Gy\|_\alpha &= 3|(Gx-Gy)(0)| + 3\sup\left\{\frac{|(Gx-Gy)(u)-(Gx-Gy)(v)|}{|u-v|^\alpha} : u,v\in[0,1] \text{ and } u\neq v\right\}\\
&\leq |x^2(0)-y^2(0)| + |x(0)-y(0)| + \|x-y\|_\alpha(2\|x+y\|_\alpha+1)\\
&\leq |x(0)-y(0)||x(0)+y(0)| + |x(0)-y(0)| + \|x-y\|_\alpha(2\|x+y\|_\alpha+1)\\
&\leq \|x-y\|_\infty(\|x+y\|_\infty+1) + \|x-y\|_\alpha(2\|x+y\|_\alpha+1)\\
&\leq \|x-y\|_\alpha(\|x+y\|_\alpha+1) + \|x-y\|_\alpha(2\|x+y\|_\alpha+1)\\
&\leq \|x-y\|_\alpha(3\|x+y\|_\alpha+2)\\
&\leq \|x-y\|_\alpha(3\|x-y\|_\alpha+6\|y\|_\alpha+2)\\
&< 3\varepsilon
\end{aligned}$$

which yields that the operator G is continuous on $H_\beta[0,1]$ with respect to the norm $\|.\|_\alpha$. Also,

$$\begin{aligned}
3|(Gx)(0)| &= |x^2(0)+x(0)| + \frac{3}{2^{10}}\\
&\leq |x^2(0)| + |x(0)| + \frac{3}{2^{10}}\\
&\leq \|x\|_\infty^2 + \|x\|_\infty + \frac{3}{2^{10}}\\
&\leq \|x\|_\beta^2 + \|x\|_\beta + \frac{3}{2^{10}}
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
3\sup\left\{\frac{|(Gx)(u)-(Gx)(v)|}{|u-v|^\beta}\right\} &= \frac{\left|x^2(u)+x(u)+\frac{3}{2^{10}}-x^2(v)-x(v)-\frac{3}{2^{10}}\right|}{|u-v|^\beta}\\
&= \frac{|x^2(u)-x^2(v)+x(u)-x(v)|}{|u-v|^\beta}\\
&= \frac{|(x(u)-x(v))(x(u)+x(v))+x(u)-x(v)|}{|u-v|^\beta}\\
&= \frac{|(x(u)-x(v))||(x(u)+x(v)+1)|}{|u-v|^\beta}\\
&\leq \sup\left\{\frac{|(x(u)-x(v))|}{|u-v|^\beta}\right\}(2\|x\|_\infty+1)\\
&\leq \|x\|_\beta(2\|x\|_\beta+1)\\
&\leq 2\|x\|_\beta^2+\|x\|_\beta.
\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we get

$$\begin{aligned}
3\|Gx\|_\beta &\leq \|x\|_\beta^2 + \|x\|_\beta + \frac{3}{2^{10}} + 2\|x\|_\beta^2 + \|x\|_\beta\\
&= 3\|x\|_\beta^2 + 2\|x\|_\beta + \frac{3}{2^{10}}
\end{aligned}$$

which implies

$$\|Gx\|_\beta \leq \|x\|_\beta^2 + \frac{2}{3}\|x\|_\beta + \frac{1}{2^{10}}.$$

Therefore, there exists the function $w:\mathbb{R}_+\to\mathbb{R}_+,\ w(x)=x^2+\frac{2}{3}x+\frac{1}{2^{10}}$ which is non-decreasing such that it holds the inequality

$$\|Gx\|_\beta \leq w\left(\|x\|_\beta\right)$$

for any $x \in H_\beta[0,1]$. So, the assumption (i) of Theorem 2.1 holds.

Further, we have

$$\begin{aligned}
|k(u,\tau)-k(v,\tau)| &= \left|\sqrt[7]{mu^4+\tau}-\sqrt[7]{mv^4+\tau}\right| \\
&\leq \left|\sqrt[7]{m(u^4-v^4)}\right| \\
&\leq \sqrt[7]{m}\,\sqrt[7]{|(u^4-v^4)|} \\
&\leq \sqrt[7]{4m}|u-v|^{\frac{1}{7}}
\end{aligned}$$

for all $u,v,\tau \in [0,1]$. The assumption (ii) of Theorem 2.1 holds with the constant $k_\beta = k_{\frac{1}{7}} = \sqrt[7]{4m}$.

The function $q : [0,1] \to [0,1], q(\tau) = \sqrt{\frac{1}{\tau+1}}$ decreasing function is measurable and this satisfies assumption (iii).

Further, we can calculate that

$$\begin{aligned}
\sup\left\{\int_0^1 |k(u,\tau)|d\tau : u \in [0,1]\right\} &= \sup\left\{\int_0^1 \left|\sqrt[7]{mu^4+\tau}\right| d\tau : u \in [0,1]\right\} \\
&= \sup\left\{\frac{7}{8}\left(\sqrt[7]{(mu^4+1)^8}-\sqrt[7]{(mu^4)^8}\right) : u \in [0,1]\right\} \\
&\leq \sup\left\{\frac{7}{8}\sqrt[7]{(mu^4+1)^8} : u \in [0,1]\right\} \\
&= \frac{7}{8}\sqrt[7]{(m+1)^8} \\
&\leq \sqrt[7]{(m+1)^8} \\
&= K.
\end{aligned}$$

In this case, the inequality appearing in assumption (vi) of Theorem 2.1 takes the following form:

$$w(r) + (2K + k_\beta)r^2 \leq r$$

which is equivalent to

$$r^2 + \frac{2}{3}r + \frac{1}{2^{10}} + \left(2\sqrt[7]{(m+1)^8} + \sqrt[7]{4m}\right) r^2 \leq r. \tag{3.7}$$

There exists a positive number r_0 satisfying (3.7) provided that the constant m is chosen as suitable. For example, if one chose $m = \frac{1}{10^{49}}$, then the inequality

$$r^2 + \frac{2}{3}r + \frac{1}{2^{10}} + \left(2\sqrt[7]{\left(\frac{1}{10^{49}}+1\right)^8} + \sqrt[7]{\frac{4}{10^{49}}}\right) r^2 \leq r$$

holds for $r = r_0 = 0.10 \in [0.0030113,\ 0.1081]$. Therefore, using Theorem 2.1, we infer that there is at least one solution x of the equation (3.1) in the space $H_\alpha[0,1]$ with $0 < \alpha < \frac{1}{7}$.

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