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Analyzing of Quantum Entanglement with Concurrence in the Deep Lamb-Dicke Regime

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Abstract

The entangled states of three-level of trapped ion and two phonons (into coherent state) in Λ configuration forming a Hilbert space of 12-dimensional (D) are analyzed. The concurrence is considered "such as a quantum measure" in trapped ion-coherent state system. Four values of Lamb-Dicke parameter (LDP), η = 0.005, 0.07, 0.08 and 0.09 are investigated by the worths of concurrence. We demonstrate that as η is increased, sudden birth of entangled states in the system is increased. Thus, establishing of entangled states can be tuned by LDP.

Keywords: Trapped ion-coherent system; Interaction Hamiltonians; Probability amplitudes; Frequency of harmonic trap.

Derin Lamb-Dicke Rejiminde Eş Zamanlılık ile Kuantum Dolanıklığın Analiz Edilmesi

Öz

12-boyutlu Hilbert uzayında **Λ** konfigürasyonunda iki fonon (kohorent durum içinde) ve üç-düzeyli tuzaklanmiş iyon sisteminde dolanık durumlar analiz edildi. Eş zamanlılık;

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tuzaklanmış iyon-koherent durum sisteminde "bir kuantum ölçümü olarak" düşünüldü. LDP'nin $\boldsymbol{\eta}$ = 0.005, 0.07, 0.08 ve 0.09 ile verilen dört değeri için dolanıklığın değerleri araştırıldı. LDP ($\boldsymbol{\eta}$) arttıkça, sistemdeki dolanık durumların ani doğumunun arttığı gösterildi. Böylece dolanık durumların oluşturulması LDP ile ayarlanabilir.

Anahtar Kelimeler: Tuzaklanmış iyon-koherent sistemi; Etkileşim Hamiltonyenleri; Olasılık genlikleri; Harmonik tuzak frekansı.

1. Introduction

Entangled state is one of the central themes making characteristic between classical and quantum mechanics. Entanglement is an attractive physical phenomenon in which the overlap of two separable states is entangled state with any photon. Thus, entanglement shows a feature of non-local of quantum mechanics [1]. So far, most of the articles have been investigated on information theory, in fact, Einstein, Podolsky and Rosen (EPR) published that famous article to criticize quantum mechanics [2]. In the 1935, Niels Bohr published an article [3] with the same name as the EPR article. This famous paper introduced the entanglement with conversations on quantum theory. J. Bell conceived the quantum theory forecasts are in conflict with the local realistic theory [1]. In the 1935, Erwin Schrödinger acquired the EPR reasoning twice, first in his paper in Naturwissenschaften introducing "Verschrankung", where he advises the Schrödinger cat state, and secondly in a paper for the Cambridge Philosophical Society introducing entanglement [4]. Trapped ions systems are important for the entangled states works [5-8], concurrence C [9], negativity N [10, 11].

Pure qudit states and quantum correlation of one trapped ion system with respect to C and N are discussed by R. Dermez et al. [11]. In the internal states, very small optical wavelength of coupling of vibrational phonons have been achieved. This is named the deep Lamb-Dicke regime (LDR) described by LDP of small, $\eta << 1$, such as this study. LD limit is not necessarily appointed in typical experiments [12]. Such experiments act in named as beyond LDR where $\eta < 1$, for examples $\eta \approx 0.2$ [13] and $\eta \approx 0.25$ [14].

Product base and entangled base are shown generalization of Schmidt coefficients [15]. Entanglement of pure qudit states [16] and of mixed qudit states [17] are demonstrated by a quantum system with single-step dynamics under the Λ configuration. The calculations and graphs in these three articles differ from the traditional theoretical approach. Quantum entropy is offered for pure qudit states [18]. All entanglement measurements are known to test whether any given state is separable and entangled. Therefore, C and N are applied for pure states [19]. To perform quantum computations in trapped ion it is advocated to use deeply phonon nonclassical states. For Fock states, squeezed [20], coherent states of odd-even [21], and their superpositions [22] were suggested.

In current quantum information theory, *C* is one of quantum entanglement measures. *C* is a useful measurement in quantum optics, especially in trapped ion. We report analytical results of the quantum entanglement for system via *C* for the deep LD regime and the Hilbert space of 12-D. We focus on the quantum dynamics in concurrence [11, 16] with respect to the total and the reduced density matrix. With help of Ref. [16] we have plotted the evolutions of *C* in the Figs. 2-6 for trapped ion-coherent system.

The rest of the paper is organized as follows. Section 2 discusses the growth of two unentangled qubits and analytical solutions in the trapped ion system. Section 3 describes how to obtain highly concurrence of system by the deep LD regime. The results and comments are given in Section 4.

2. Model and Solution of Trapped Ion-Coherent System

We study a trapped ion in a harmonic potential and two coherent states. Theoretical physical system is emerged via previous investigation [23, 24]. The Hamiltonian of system is $H_{total} = H_{ion} + H_1 + H_2$ and H_{ion} of trapped ion:

$$H_{ion} = \omega_g |g\rangle\langle g| + \omega_r |r\rangle\langle r| + 0|e\rangle\langle e| + \frac{p_x^2}{2m} + \frac{1}{2} m \nu^2 x_{ion}^2. \tag{1}$$

Here H_1 and H_2 are Hamiltonians of these interactions for excited-ground and excited-raman:

$$H_{e-g} = H_1 = \frac{\Omega}{2} e^{i(k_1 x_{ion} - \omega t)} |e\rangle\langle g| + hermitian\ \ c. \tag{2}$$

$$H_{e-r} = H_2 = \frac{\Omega}{2} e^{i(-k_2 x_{ion} - \omega t)} |e\rangle\langle r| + hermitian\ \ c., \tag{3}$$

where $\hbar = 1$, p_x and x_{ion} are momentum and the *x*-component of position of ion center of mass movement. The movement of the ion in the system is along the *x*-axis (one-D). Atomic levels of ion are given: $|e\rangle \rightarrow$trapped ion excited level, $|r\rangle \rightarrow$raman level energy, and $|g\rangle \rightarrow$ground level in Fig. 1. The excited level energy is $\omega_e = 0$, raman level is ω_r, and ground level is ω_g in Eqn. (1). The mass center of trapped ion has been performed by the standard harmonic-oscillator of

H_{ion} in $p_x = i\sqrt{\frac{1}{2}m\nu}\left(a^+ - a\right)$ and $x_{ion} = \sqrt{\frac{1}{2}m\nu}\left(a + a^+\right)$. Here, a is annihilation operator and a^+ creation operator of the vibrational phonons. Two laser frequencies are ω_1 and ω_2, and Rabi frequency is Ω in Fig. 1. Ion-phonon total Hamiltonian is written as:

$$H = \left(\frac{\Omega}{2} e^{i\eta\left(a^+ + a\right)} |e\rangle\langle g| + \nu a^+ a - \delta |e\rangle\langle e| + \frac{\Omega}{2} e^{-i\eta\left(a + a^+\right)} |e\rangle\langle r| \right) + h.c., \tag{4}$$

here LPD is $\eta = k/2m\nu$, ν is harmonic trap frequency, and $\delta = \nu\eta^2$. We have taken the base vectors as follow:

$$|e\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ |r\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ |g\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5}$$

$\widetilde{H} = U^+ H U$ is transformed Hamiltonian. The Hamiltonian in Eqn. (4) is found as follows after the transmission process. The optical Λ configuration can be equal to a cascade Ξ scheme for two coherent states. Trapped ion-coherent state is covered by unitary transformation. Matrix of transformation, namely U is written as [23],

$$U = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ -\sqrt{2}B[\eta] & B[\eta] & -B[\eta] \\ \sqrt{2}B[-\eta] & B[-\eta] & -B[-\eta] \end{pmatrix}. \tag{6}$$

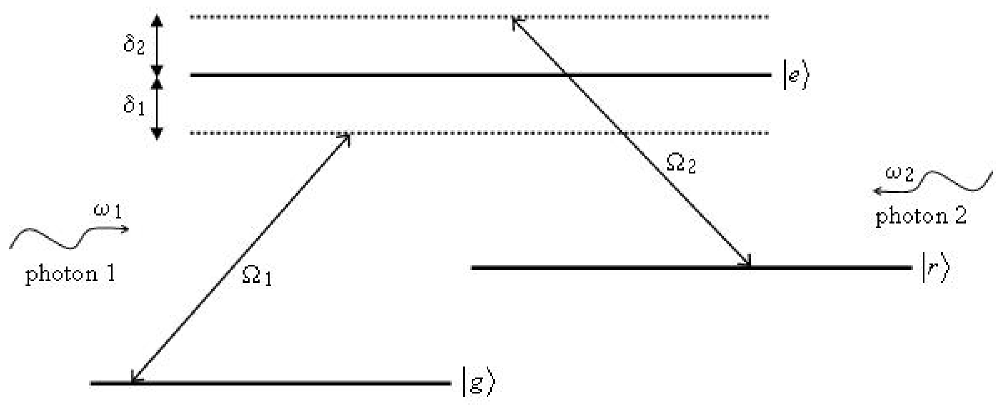

Figure 1: Figure of three internal electronic levels of trapped ion-coherent system. Time is given by dimensionless for ion-coherent system, the other coupling parameters are allowed to be $\Omega = \Omega_1 = \Omega_2$, $\omega = \omega_1 = \omega_2$, and $\delta = \delta_1 = \delta_2$

Here displacement operators of Glauber, $B(\eta)=e^{(i\eta(a+a^{+}))}$, $B(-\eta)=e^{(-i\eta(a+a^{+}))}$ are achieved. $\widetilde{H}$ is written as $\widetilde{H}=\widetilde{H}_0+\widetilde{V}$, here

$$\widetilde{H}_0=\nu(|r\rangle\langle r|-|g\rangle\langle g|)+\upsilon\eta^2+\upsilon a^{+}a, \tag{7}$$

$$\widetilde{V}=-i\frac{\sqrt{2}\nu\eta}{2}\left(a^{+}|e\rangle\langle r|-a^{+}|e\rangle\langle g|+hermitian\ c.\right) \tag{8}$$

The deep LD regime is allowed between the 0.005 and 0.9. Under the unitary transformation method [23], an early state $|\psi(0)\rangle$ is given by

$$|\psi(t)\rangle=U_0^{+}Ue^{-it\widetilde{H}_0}K(t)U^{+}|\psi(0)\rangle, \tag{9}$$

where $K(t)$ is a propagator vector; $e^{(-it\widetilde{H}_0)}$ is the transformation of Shrödinger picture, and $U_0=\exp(-i\omega t|e\rangle\langle e|)$ is the transformation matrix [23]. Trapped ion-coherent state system evolves in the Λ scheme. The propagator is written as:

$$K(t)=\frac{1}{2}\begin{pmatrix} Cos(\Lambda t) & -\varepsilon Sa^{+} & -\varepsilon Sa \\ \varepsilon aS & 1+\varepsilon^2 aGa^{+} & \varepsilon^2 aGa \\ \varepsilon a^{+}S & \varepsilon^2 a^{+}Ga^{+} & 1+\varepsilon^2 a^{+}Ga \end{pmatrix}, \tag{10}$$

here $\varepsilon=\nu\eta/\sqrt{2}$, $\Lambda=\varepsilon\sqrt{2a^{+}a+1}$, $G=\frac{Cos(\Lambda t)-1}{\Lambda^2}$ and $S=\frac{Sin(\Lambda t)}{\Lambda}$. We consider the parameters for trapped ion-coherent state with an application. We take $\nu=10^7$ Hz and $\omega_{eg}=5\times10^{15}$ Hz With first order terms of our system, we have taken $a=1$ and $b=0.005$ for two coherent states. Trapped ion normalization condition is exactly $\left[\frac{1}{\sqrt{2}}\right]^2+\left[-\frac{1}{\sqrt{2}}\right]^2=1$, and two phonons normalization condition is approximately $\|a\|^2+\|b\|^2=|1|^2+|0.005|^5\cong 1$. So, the earliest of ion-coherent state system is given as:

$$|\psi(0)\rangle=\frac{1}{\sqrt{2}}[|g\rangle-|r\rangle]\otimes(a|0\rangle+b|1\rangle), \tag{11}$$

here the phonon levels are $\langle 0|=(1,0)$, and $\langle 1|=(0,1)$. The new formula of system is given as

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}[|g\rangle - |r\rangle] \otimes \left(\sum_{n=0}^{\infty} F_n(b)|n\rangle \right). \tag{12}$$

We have used the coherent state by means of the zero-order and first-order terms of LDP such as η^0, η^1. This system is transformed to an early separable state,

$$|\psi K(t)\rangle = |\tilde{\psi}(0)\rangle = U^{+}|\psi(0)\rangle = \sum_{\sigma,m} N_{\sigma,m}(t)|\sigma,m\rangle. \tag{13}$$

In Eqn. (11), our system is produced in respect of $\sum_{\sigma,m} N_{\sigma,m}(t)|\sigma,m\rangle$. The twelve of system's probability amplitudes are written as:

$$N_{e0}(t) = \left[\cos\left(\sqrt{\frac{1}{2}}t \right) + \frac{\eta i}{\sqrt{2}} \sin\left(\sqrt{\frac{1}{2}}t \right) \right] \exp[-ti/\eta] \tag{14}$$

$$N_{e1}(t) = b\cos\left(\sqrt{\frac{3}{2}}t \right) \exp[-ti/\eta] \tag{15}$$

$$N_{e2}(t) = -\frac{\eta i}{\sqrt{5}} \sin\left(\sqrt{\frac{5}{2}}t \right) \exp[-2ti/\eta] \tag{16}$$

$$N_{r0}(t) = \frac{b}{\sqrt{3}} \sin\left(\sqrt{\frac{3}{2}}t \right) \exp[-ti/\eta] \tag{17}$$

$$N_{r1}(t) = \frac{\eta i}{\sqrt{2}} \left[\frac{3}{2} + \frac{2}{5} \cos\left(\sqrt{\frac{5}{2}}t \right) \right] \exp[-2ti/\eta] \tag{18}$$

$$N_{g1}(t) = \left[\sin\left(\sqrt{\frac{1}{2}}t \right) - \frac{\eta i}{\sqrt{2}} \cos\left(\sqrt{\frac{1}{2}}t \right) \right] \exp[-ti/\eta] \tag{19}$$

$$N_{g2}(t) = b\sqrt{\frac{2}{3}} \sin\left(\sqrt{\frac{3}{2}}t \right) \exp[-ti/\eta] \tag{20}$$

$$N_{g3}(t) = -\frac{\sqrt{3}}{5} \eta i \left[1 - \cos\left(\sqrt{\frac{5}{2}}t \right) \right] \exp[-2ti/\eta] \tag{21}$$

and four amplitudes are zero: $N_{e3}(t)=N_{r2}(t)=N_{r3}(t)=N_{g0}(t)=0$. In Eqns. (14) to (21), the first index σ is the states of atomic (*g, r, e*), second index *m* is the vibrational quantum numbers (0, 1, 2, 3). The vibrational coherent state is indicated by a four-dimensional (four-D) Hilbert space $\mathbf{H}_p$ and subsystem of trapped ion is shown by a three-D Hilbert space $\mathbf{H}_i$. Therefore, the trapped ion-coherent states are C^{12} D of Hilbert space. In the equations, *t* is dimensionless time scaled by $\nu\eta$ (harmonic trap frequency-LDP). For the analytical expressions in Eqn. (22), we have ignored the second-order, third-order and forth-order terms. For trapped ion-two coherent state, the final state vector is given as:

$$\left|\psi_{final}(t)\right\rangle=\sum_{m=0}^{3}\left(A_m(t)\left|e,m\right\rangle+B_m(t)\left|r,m\right\rangle+C_m(t)\left|g,m\right\rangle\right). \tag{22}$$

These coefficients of the final vector are

$$A_m(t)=\frac{1}{\sqrt{2}}e^{-i\omega t/\nu\eta}\left[N_{rm}(t)+N_{gm}(t)\right],\ (m=0,1,2,3), \tag{23}$$

$$B_0(t)=-\frac{1}{\sqrt{2}}N_{e0}(t)+\frac{1}{2}N_{r0}(t)-\frac{i\eta}{2}N_{g1}(t) \tag{24}$$

$$B_1(t)=-\frac{i\eta}{\sqrt{2}}N_{e0}(t)-\frac{1}{2}N_{r1}(t)+\frac{1}{2}N_{r1}(t)-\frac{1}{2}N_{g1}(t) \tag{25}$$

$$B_2(t)=-\frac{1}{\sqrt{2}}N_{e2}(t)-\frac{i\eta}{\sqrt{2}}N_{g1}(t)-\frac{1}{2}N_{g2}(t) \tag{26}$$

$$B_3(t)=-\frac{1}{2}N_{g3}(t) \tag{27}$$

$$C_0(t)=\frac{1}{\sqrt{2}}N_{e0}(t)+\frac{1}{2}N_{r0}(t)-\frac{i\eta}{2}N_{g1}(t) \tag{28}$$

$$C_1(t)=-\frac{i\eta}{\sqrt{2}}N_{e0}(t)+\frac{1}{2}N_{r1}(t)+\frac{1}{2}N_{r1}(t)-\frac{1}{2}N_{g1}(t) \tag{29}$$

$$C_2(t)=\frac{1}{\sqrt{2}}N_{e2}(t)+\frac{i\eta}{\sqrt{2}}N_{g1}(t)-\frac{1}{2}N_{g2}(t) \tag{30}$$

$$C_3(t) = -\frac{1}{2} N_{g3}(t) \tag{31}$$

here ω_{eg} is the frequency of resonance of excited-ground level, $\omega = \omega_{eg} - \eta^2 \nu$, *i* is ion index, and *i* is a complex number. We have illustrated the *C* of pure qudit states as $l \otimes l'(l \leq l')$ in Figs. 2-6. We found that the final state vector $\left|\psi_{final}(t)\right\rangle$ is superposition of twelve function in Eqn. (23-31).

In our model, these dimensions (D) of Hilbert space are $l = 4$ for the two-phonons and $l' = 3$ for the trapped ion subsystem. We have used a reduced density operator $\rho_{ion} = Tr_{phonon}\left(\rho_{ion-p}\right)$ by the help of Eqn. (32). A measurement for pure qudit states can be optimized by a generalized *C* [6, 18, 19]. The total density operator ρ_{ion-p} is shown by a 12×12 D matrix in the base of $\left|i, p\right\rangle$. Taking trace over the phonon system, 3×3 reduced density operator, ρ_{ion} is given as,

$$\rho_{ion} = Tr_{phonon}\left(\rho_{ion-phonon}\right) = \begin{pmatrix} Tr\left|e\right\rangle\left\langle e\right| & Tr\left|e\right\rangle\left\langle r\right| & Tr\left|e\right\rangle\left\langle g\right| \\ Tr\left|r\right\rangle\left\langle e\right| & Tr\left|r\right\rangle\left\langle r\right| & Tr\left|r\right\rangle\left\langle g\right| \\ Tr\left|g\right\rangle\left\langle e\right| & Tr\left|g\right\rangle\left\langle r\right| & Tr\left|g\right\rangle\left\langle g\right| \end{pmatrix} \tag{32}$$

where diagonal terms, $\left|e\right\rangle\left\langle e\right|$, $\left|r\right\rangle\left\langle r\right|$ and $\left|g\right\rangle\left\langle g\right|$ are a 4×4 square matrix. With respect to Eqn. (32), fully density operator of the system is written as:

$$\rho_{ion-phonon} = \left(\left|Z\right\rangle\left\langle Z\right|\right) \tag{33}$$

where $\left|Z\right\rangle\left\langle Z\right|$ is a 12×12 D square matrix in the 12-D Hilbert space. With Eqn. (33), the analytic solution of the total density operator is

$$\rho_{ion-phonon} = U^{+}(t)[\rho^{i}(0) \otimes \rho^{p}(0)]U(t). \tag{34}$$

3. The Measure of Concurrence, Deep LD Regime and Discussion

Concurrence is important in quantum entanglement measures. We investigate the entanglement by the concurrence for deep LD regime in the system and explain it. The coherent state is shown by a four D H_p of a quadrit. So, the early state in Section 2 derives in the Hilbert space $H = H_i \otimes H_p$ D. In a pure qudit state $|\psi(t)\rangle$ the density matrix of the system is shown by $\rho_{ion-phonon} = |\psi(t)\rangle\langle\psi(t)| = |Z\rangle\langle Z|$ in Eqns. (32) and (33). Concurrence is first reported in the literature as a quantum entanglement measurement in Ref [25] for two qubits. For any arbitrary bipartite pure state is written [25]

$$Concurrence = \sqrt{2(Tr(\rho_{ion-phonon}) - Tr(\rho_{ion}^2))}. \tag{35}$$

where $\rho_{ion} = Tr_{phonon}\left(\rho_{ion-p}\right)$ is the reduced density matrix in Eqn. (32). So that, in Figs. 2-5 time evolution of C have been plotted by η = 0.005, 0.07, 0.08, and 0.09. We have obtained high amount of entanglement for four values of LDP. This is called "deep LD regime" is qualified by η= 0.005, the other way, it is known that "beyond LD regime" is η= 0.5.

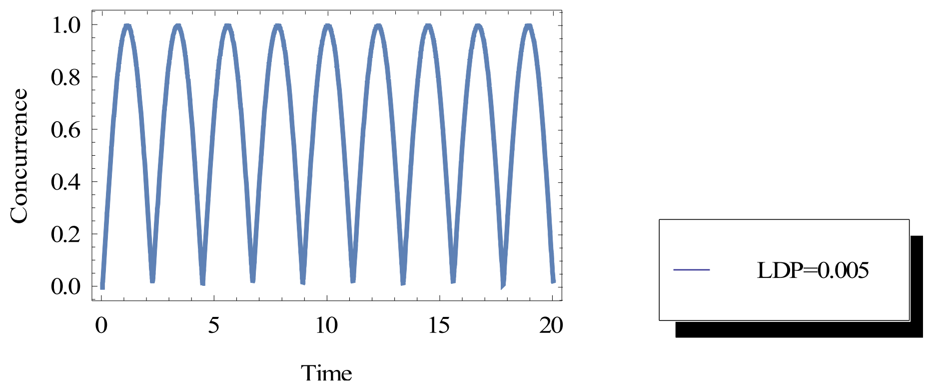

Figure 2: The time change of C is given by η = 0.005, first value of deep LD regime. The earliest state of trapped ion-coherent state system is $|\psi(0)\rangle = \frac{1}{\sqrt{2}}[|g\rangle - |r\rangle] \otimes (a|0\rangle + b|1\rangle)$, with a =1, b =0.005. Assumption parameters are given for $\nu = 1\times10^7$ Hertz and $\omega_{eg} = 5\times10^{15}$ Hertz

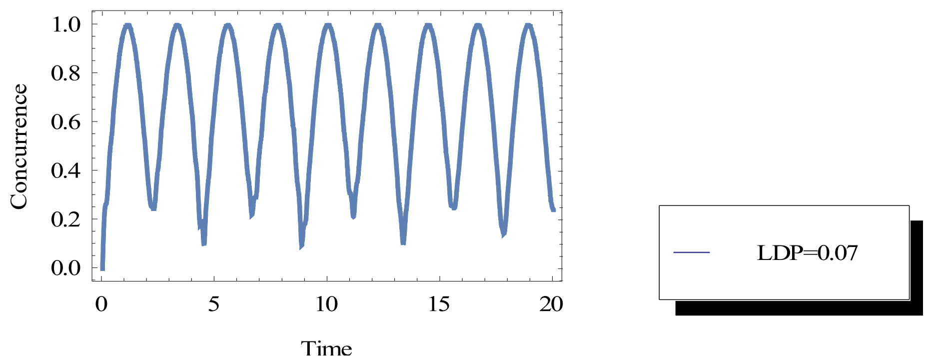

Figure 3: The time change of C is given by η = 0.07 second value of deep LD regime. Other assumptions parameters are the same as Fig. 2 in the system

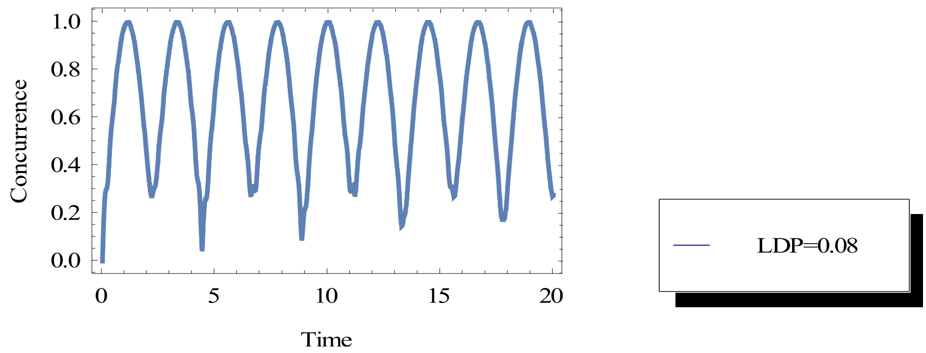

Figure 4: The time change of C is given by η = 0.08, third value of deep LD regime. Other assumption parameters are the same as Fig. 2 in the system

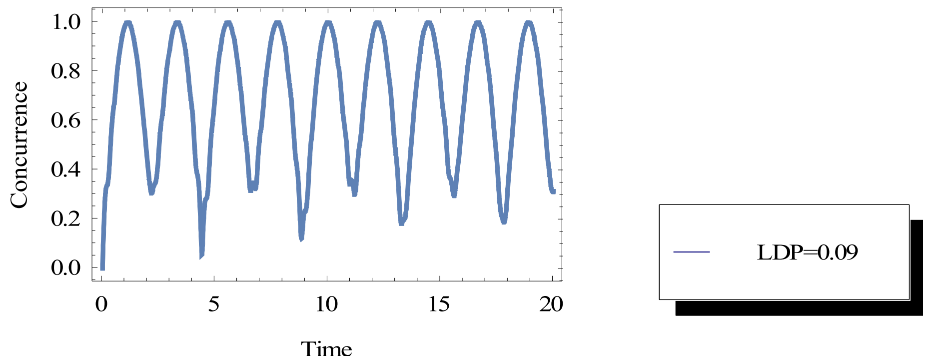

Figure 5: The time change of C is given by η = 0.09, fourth value of deep LD regime. Other assumption parameters are the same as Fig. 2 in the system

Table 1: The eight values of concurrence for two optimum time, t = 4.2 and t = 3.50 via Figs. 2-5

	Fig. 2, b=0.005, $\eta = 0.005$	Fig. 3, b=0.005, $\eta = 0.07$	Fig. 4, b=0.005, $\eta = 0.08$	Fig. 5, b=0.005, $\eta = 0.09$
Concurrence, t=4.2	0.336	0.381	0.349	0.383
Concurrence, t=3.50	0.999	0.971	0.971	0.969

In this paper, we measured quantum entanglement with concurrence in the deep LD regime distinctively from the other paper [16]. The values of concurrence for two optimum time are given in Table 1. We have taken the harmonic trap frequency as 10^7 Hz in Figs. 2-6. The maximum value of concurrence is interestingly recorded C = 0.999 for η = 0.005 in Table 1. The four different values of η are determined and taken into consideration in this study. When we checked the literature we didn't see that it has been worked with 0.005 value. We report the quantum dynamics of C as a function of LDP in Fig. 6 with two graphics for two optimum time. Our previous analytical results [16, 26, 27] are in similar with Figs. 2-6. Negativity and quantum entropy, which are the other elaborated measurements from the measurement family defining the entanglement concept, have also been studied in the literature [6, 8, 11, 18, 26, 27, 29].

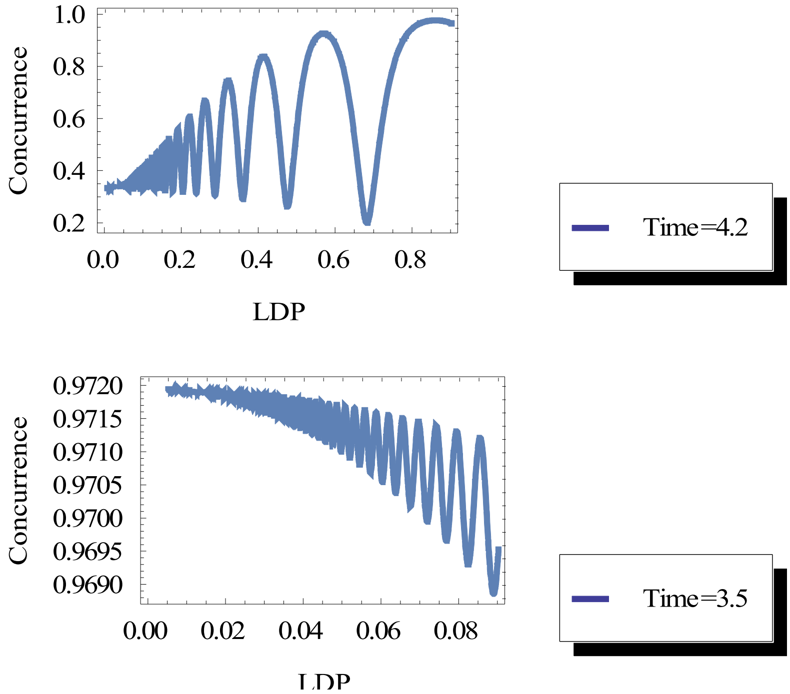

Figure 6: The η evolution of C is given for $t = 4.2$ and $t = 3.5$ (from $\eta = 0.005$ to $\eta = 0.09$). The other coupling parameters are the same as Fig. 2 in the system

We propose the quantum dynamics with concurrence for coupling parameters. We have detected different dynamical features in C in reaction to increasing η. In Fig. 3, C oscillates between minimum value $C = 0$ and the highest value $C = 0.999$ at $t = 3.50$ for $\eta = 0.005$. This time is maximally entangled state at the optimum time in Table 1. By this method, we have observed that the maximally entangled states do not collapse in system. The presence of robust entanglement in trapped ion-coherent state the system has been recognized by Fig. 6.

If parameters of the system are $\eta = 0.09$ and at $t = 3.50$, a concurrence measurement becomes 0.969 in Table 1. We explore with C that the measurement degrees have a sudden birth of entangled state in parallel with raising η, and this is in comparison to the previous observations [6, 26-30]. In Fig. 1, two internal electronic levels of trapped ion can be built by a qubit, such as three internal levels can play a role of a qutrit in the quantum Rabi model. They suggested the quantum simulations of the Rabi model in single trapped ion with deep $\eta = 0.06$ [31], similar to our suggestion in this study for LDP. These articles similar to our model [32, 33] have been proposed a design for creating odd and even coherent states of trapped ion with respect to laser excitation of two vibration modes.

4. Conclusions

In summary, we have presented quantum entanglement of trapped ion-coherent system in the Λ scheme by the unitary transformation. Quantum mechanical expressions such as: the final state, density operator and reduced density operator have been normalized in analytical high-level calculations. We concentrate the concurrence through the definition of variance deep LD regime. These plots are obtained by concurrence which is consistent with quantum corrections. Quantum entanglement is discussed by an elaborated measure which is C. The quantum dynamics of interactions between trapped ion and coherent state determine the extent of coupling parameters such as Rabi and harmonic trap frequencies. Entangled states of the trapped ion-coherent system can be called pure qudit states. Because, strong quantum correlations can arise between higher dimensional such as qubits or qutrits.

These pure qudit states are calculated and analyzed by η. When concurrence is given sudden birth, quantum entanglement among trapped ion qutrit-phonon quadrit states is generalized. The results are summarized as follows: (1) quantum entanglement is seen to have the capacity in our system and the amount of concurrence is $C = 0.999$; (2) C depends on four deep different LDPs; (3) This extracts that such entanglement is connected with η.

We achieved the long-lived entanglement in the deep LD regime. Maximally entangled states as demonstrated through our system can be important for researchers with trapped ions. The extending of life time can be succeeded with Rabi frequences and η. These new approaches and results can provide useful data for identification of the life time of entanglement in future theoretical studies.

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