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Comparison of Jordan (σ,τ) - Derivations and Jordan Triple (σ,τ) - Derivations in Semiprime Rings

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Abstract

Let R be a 3!-torsion free semiprime ring, τ, σ two endomorphisms of R, $d: R \to R$ be an additive mapping and L be a noncentral square-closed Lie ideal of R. An additive mapping $d: R \to R$ is said to be a Jordan (σ,τ) −derivation if $d(x^2) = d(x)\sigma(x) + \tau(x)d(x)$ holds for all $x, y \in R$. Also, d is called a Jordan triple (σ,τ) −derivation if $d(xyx) = d(x)\sigma(yx) + \tau(x)d(y)\sigma(x) + \tau(xy)d(x)$, for all $x, y \in R$. In this paper, we proved the following result: d is a Jordan (σ,τ) −derivation on L if and only if d is a Jordan triple (σ,τ) −derivation on L.

Keywords: Semiprime ring; Jordan derivation; Jordan triple derivation; (σ,τ) -derivation; Jordan (σ,τ) derivation; Jordan triple (σ,τ)-derivation.

Yarıasal halkalarda Jordan (σ,τ)- Türevler ve Jordan Üçlü (σ,τ)-Türevlerin Karşılaştırılması

Öz

R bir 3!-torsion free yarıasal halka, τ ve σ iki endomorfizm, $d: R \to R$ toplamsal dönüşüm ve L merkez tarafından kapsanmayan R halkasının bir kare kapalı Lie ideali olsun. $d: R \to R$

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toplamsal dönüşümü her $x, y \in R$ için $d(x^2) = d(x)\sigma(x) + \tau(x)d(x)$ koşulunu sağlıyorsa d dönüşümüne Jordan (σ, τ) −türev denir. Ayrıca, $d: R \to R$ toplamsal dönüşümü her $x, y \in R$ için $d(xyx) = d(x)\sigma(yx) + \tau(x)d(y)\sigma(x) + \tau(xy)d(x)$ koşulunu sağlıyorsa d dönüşümüne Jordan üçlü (σ, τ) −türev denir. Bu çalışmada, d bir L üzerinde Jordan (σ, τ) −türev olması için gerek ve yeter koşul d dönüşümünün L üzerinde Jordan üçlü (σ, τ) −türev olmasıdır sonucu ispatlanmıştır.

Anahtar kelimeler: Yarıasal halka; Jordan türev; Jordan üçlü türev; (σ, τ) −türev; Jordan (σ, τ) −türev; Jordan üçlü (σ, τ) −türev.

1. Introduction

R is an associative ring with center Z. A ring R is prime ring if $xRy = (0)$ implies $x = 0$ or $y = 0$, and semiprime ring if $xRx = (0)$ implies $x = 0$. An additive subgroup L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$. A Lie ideal L is said to be square-closed if $a^2 \in L$ for all $a \in L$. An additive mapping $d: R \to R$ is called a derivation (resp. Jordan derivation) if $d(u_1u_2) = d(u_1)u_2 + u_1d(u_2)$ (resp. $d({u_1}^2) = d(u_1)u_1 + u_1d(u_1)$) holds for all $u_1, u_2 \in R$. Let σ and τ be endomorphisms of R. An additive mapping $d: R \to R$ is said to be a (σ, τ) −derivation (resp. Jordan (σ, τ) −derivation) if $d(uv) = d(u)\sigma(v) + \tau(u)d(v)$ (resp. $d(u^2) = d(u)\sigma(u) + \tau(u)d(v)$) holds for all $u, v \in R$. A Jordan triple derivation $d: R \to R$ is an additive mapping satisfying $d(u_1u_2u_1) = d(u_1)u_2u_1 + u_1d(u_2)u_1 + u_1u_2d(u_1)$, for all $u_1, u_2 \in R$. Also, d is called a Jordan triple (σ, τ) −derivation if $d(u_1u_2u_1) = d(u_1)\sigma(u_2u_1) + \tau(u_1)d(u_2)\sigma(u_1) + \tau(u_1u_2)d(u_1)$, for all $u_1, u_2 \in R$.

We can invetigate that every derivation is a Jordan derivation, but the opposite is not usually true. This result by Herstein [1] was shown in a prime ring of 2- torsion free. The same result was proved by Cusack in [2] to the semiprime rings. The same result was generalized to the Lie ideal of the semiprime ring in [4].

In [4], Jing and Lu, has proven to be a derivation of any of the Jordan triple derivation on prime rings. Vukman [5], examined the results for the semiprime rings. Hence, Rehman and Koç Sögütcü has transferred it Lie ideal of the semiprime ring in [6].

In [7] Herstein proved that each Jordan derivation of the prime ring is a Jordan triple derivation, while Bresar is a derivation of each Jordan triple derivation of a semiprime ring in [8]. In [9], the relation between Jordan triple derivation and Jordan derivation is given.

In this paper, we present the results corresponding to Jordan triple (σ,τ) −derivation and Jordan (σ,τ) −derivation.

2. Results

Lemma 1. [10, Corollary 2.1] Let R be a 2-torsion free semiprime ring, L be a Lie ideal of R such that $L \nsubseteq Z(R)$ and a,b∈L.

i) If $aLa = (0)$, then $a = 0$.

ii) If $aL = (0)$ (or $La = (0)$), then $a = 0$.

iii) If L is square-closed and $aLb = (0)$, then $ab = 0$ and $ba = 0$.

Lemma 2. [6, Theorem 2.1] Let R be a 2-torsion free semiprime ring, $\alpha,\beta \epsilon Aut(R)$ and $L \nsubseteq Z(R)$ be a nonzero square-closed Lie ideal of R. If an additive mapping $d: R \to R$ satisfying

$d(u_1u_2u_1) = d(u_1)\alpha(u_2u_1) + \beta(u_1)d(u_2)\alpha(u_1) + \beta(u_1u_2)d(u_1)$, for all $u_1, u_2 \in L$.

and $d(u_1), \beta(u_2) \in L$, then d is a (α,β) −derivation on L.

Lemma 3. Let R be a 2-torsion free semiprime ring, $L \nsubseteq Z(R)$ is a square-closed Lie ideal of R, τ, σ two endomorphisms of R, $\sigma(L) = L$ and $a, b \in L$. If $a\sigma(ub) + \tau(bu)a = 0$, for all $u \in L$ then $a\sigma(ub) = 0$.

Proof. By the hypothesis, we have

$$a\sigma(ub) + \tau(bu)a = 0. \quad (1)$$

Then replacing u by $4ubv, v \in U$ in Eqn. (1) and by 2-torsion freeness, we get $a\sigma(ubvb) + \tau(bubv)a = 0$.

Application ofEqn. (1) yields that

$-\tau(bu)a\sigma(vb) + \tau(bu)\tau(bv)a = 0$.

Again using Eqn. (1), we get $a\sigma(ub)\sigma(vb) = 0$.

Using $\sigma(L) = L$, we obtain that $a\sigma(ub)L\sigma(b) = 0$, and so $a\sigma(ub)La\sigma(ub) = 0$. By Lemma 1, we get $a\sigma(ub) = 0$, for all $u \in L$.

Theorem 4. Let R be a 3!-torsion free semiprime ring, τ, σ two endomorphisms of R, $d: R \to R$ an additive mapping, $L \nsubseteq Z(R)$ be a nonzero square-closed Lie ideal of R and

$d(L), \tau(L) \subseteq L, \sigma(L) = L$. Then d is a Jordan (σ, τ) −derivation on L if and only if d is a Jordan triple (σ, τ) −derivation on L.

Proof. We obtain that

$$d(u_1{}^2) = d(u_1)\sigma(u_1) + \tau(u_1)d(u_1), \text{ for all } u_1 \in L. \quad (2)$$

Replacing u_1 by $u_1 + u_2$ in Eqn. (2), using d is an additive mapping and $u_1 \circ u_2 = u_1u_2 + u_2u_1$, we see that

$$d(u_1{}^2) + d(u_1 \circ u_2) + d(u_2{}^2) = d(u_1)\sigma(u_1) + \tau(u_1)d(u_1) + d(u_1)\sigma(u_2) + d(u_2)\sigma(u_1) + +\tau(u_1)d(u_2) + \tau(u_2)d(u_1) + d(u_2)\sigma(u_2) + \tau(u_2)d(u_2).$$

By the Eqn. (2), we have

$$d(u_1 \circ u_2) = d(u_1)\sigma(u_2) + d(u_2)\sigma(u_1) + \tau(u_1)d(u_2) + \tau(u_2)d(u_1), \quad (3)$$

for all $u_1, u_2 \in$L. Since $u_1{}^2 \circ u_2 + 2u_1u_2u_1 = u_1 \circ (u_1 \circ u_2)$, we find

$$d(u_1{}^2 \circ u_2 + 2u_1u_2u_1) = d(u_1 \circ (u_1 \circ u_2)), \text{ for all } u_1, u_2 \in \text{L}.$$

By the Eqn. (3), we see that

$$\begin{aligned} d(u_1{}^2 \circ u_2 + 2u_1u_2u_1) &= d(u_1)\sigma(u_1)\sigma(u_2) + \tau(u_1)d(u_1)\sigma(u_2) + d(u_2)\sigma(u_1{}^2) \\ &+\tau(u_1{}^2)d(u_2) + \tau(u_2)d(u_1)\sigma(u_1) + \tau(u_2)\tau(u_1)d(u_1) \\ &+d(2u_1u_2u_1). \end{aligned}$$

On the other hand,

$$\begin{aligned} d\big(u_1 \circ (u_1 \circ u_2)\big) &= d(u_1)\sigma(u_1u_2) + d(u_1)\sigma(u_2u_1) + d(u_1)\sigma(u_2)\sigma(u_1) \\ &+d(u_2)\sigma(u_1)\sigma(u_1) + \tau(u_1)d(u_2)\sigma(u_1) + \tau(u_2)d(u_1)\sigma(u_1) \\ &+ \tau(u_1)d(u_1)\sigma(u_2) + \tau(u_1)d(u_2)\sigma(u_1) + \tau(u_1)\tau(u_1)d(u_2) \\ &+ \tau(u_1)\tau(u_2)d(u_1) + \tau(u_1u_2)d(u_1) + \tau(u_2u_1)d(u_1). \end{aligned}$$

After comparing the above two equations, we get

$$2d(u_1u_2u_1) = 2d(u_1)\sigma(u_2u_1) + 2\tau(u_1)d(u_2)\sigma(u_1) + 2\tau(u_1u_2)d(u_1),$$

for all $u_1, u_2 \in L$.

That is,

$d(u_1u_2u_1) = d(u_1)\sigma(u_2u_1) + \tau(u_1)d(u_2)\sigma(u_1) + \tau(u_1u_2)d(u_1).$

Reverse, we see that

$$d(u_1u_2u_1) = d(u_1)\sigma(u_2u_1) + \tau(u_1)d(u_2)\sigma(u_1) + \tau(u_1u_2)d(u_1), \quad (4)$$

for all $u_1, u_2 \in L$.

Replacing u_2 by $4u_1u_2u_1$ in Eqn. (4), using Eqn. (4) and 2-torsion freeness of R, this implies that

$d({u_1}^2u_2{u_1}^2) = d(u_1)\sigma(u_1u_2{u_1}^2) + \tau(u_1)d(u_1u_2u_1)\sigma(u_1) + \tau({u_1}^2u_2u_1)d(u_1)$

$= d(u_1)\sigma(u_1u_2{u_1}^2) + \tau(u_1)d(u_1)\sigma(u_2u_1)\sigma(u_1) + \tau(u_1)\tau(u_1)d(u_2)\sigma(u_1)\sigma(u_1)$

$+\tau(u_1)\tau(u_1u_2)d(u_1)\sigma(u_1) + \tau({u_1}^2u_2u_1)d(u_1).$

On the other hand, replacing u_1 by ${u_1}^2$ in Eqn. (4), we have

$d({u_1}^2u_2{u_1}^2) = d({u_1}^2)\sigma(u_2{u_1}^2) + \tau({u_1}^2)d(u_2)\sigma({u_1}^2) + \tau({u_1}^2u_2)d({u_1}^2).$

Comparing the expressions and let us write $A(u_1) = d({u_1}^2) - d(u_1)\sigma(u_1) - \tau(u_1)d(u_1)$ for brevity, we get

$A(u_1)\sigma(u_2{u_1}^2) + \tau({u_1}^2u_2)A(u_1) = 0.$

By Lemma 3, we get $A(u_1)\sigma(u_2)\sigma({u_1}^2) = 0$, for all $u_1, u_2 \in L$and using $\sigma(L) = L$, we have

$$A(u_1)u_2\sigma({u_1}^2) = 0, \text{ for all } u_1, u_2 \in L. \quad (5)$$

Multiplying $\sigma({u_1}^2)$ on the left and $A(u_1)$ on the right hand side of Eqn. (5), we see that

$\sigma({u_1}^2)A(u_1)u_2\sigma({u_1}^2)A(u_1) = 0$, for all $u_1, u_2 \in L$.

Lemma 1 leads to

$$\sigma({u_1}^2)A(u_1) = 0, \text{ for all } u_1 \in L. \quad (6)$$

Replacing u_2 by $4\sigma({u_1}^2)u_2A(u_1)$ in Eqn. (5) and by 2-torsion freeness, we have

$A(u_1)\sigma({u_1}^2)u_2A(u_1)\sigma({u_1}^2) = 0$, for all $u_1, u_2 \in L$.

Using Lemma 1, we have

$A(u_1)\sigma(u_1{}^2) = 0$ for all $u_1 \in L$. (7)

Replacing u_1 by u_1+u_2 in Eqn. (7), we obtain that

$$\begin{aligned} 0 &= A(u_1+u_2)\sigma((u_1+u_2)^2) \\ &= (d(u_1{}^2) - d(u_1)\sigma(u_1) - \tau(u_1)d(u_1) + d(u_2{}^2) - d(u_2)\sigma(u_2) - \tau(u_2)d(u_2) \\ &\quad + d(u_1 \circ u_2) - d(u_1)\sigma(u_2) - d(u_2)\sigma(u_1) \\ &\quad - \tau(u_1)d(u_2) - \tau(u_2)d(u_1))\sigma((u_1+u_2)^2). \end{aligned}$$

Let us write $B(u_1,u_2) = d(u_1 \circ u_2) - d(u_1)\sigma(u_2) - d(u_2)\sigma(u_1) - \tau(u_1)d(u_2) - \tau(v)d(u_1)$, for brevity. For all $u_1, u_2 \in L$,

$$(A(u_1) + A(u_2) + B(u_1,u_2))\sigma((u_1+u_2)^2) = 0.$$

Using Eqn. (7) and $(u_1+u_2)^2 = u_1{}^2 + u_1 \circ u_2 + u_2{}^2$, we have

$$\begin{aligned} 0 &= A(u_2)\sigma(u_1{}^2) + A(u_1)\sigma(u_2{}^2) + A(u_1)\sigma(u_1 \circ u_2) + A(u_2)\sigma(u_1 \circ u_2) \\ &\quad + B(u_1,u_2)\sigma(u_1{}^2) + B(u_1,u_2)\sigma(u_2{}^2) + B(u_1,u_2)\sigma(u_1 \circ u_2). \end{aligned} \tag{8}$$

Replacing u_1 with -u_1 in Eqn. (8) and using $A(-u_1) = A(u_1)$ and $B(-u_1,u_2) = -B(u_1,u_2)$, we get

$$\begin{aligned} 0 &= A(u_2)\sigma(u_1^2) + A(u_1)\sigma(u_2^2) - A(u_1)\sigma(u_1 \circ u_2) - A(u_2)\sigma(u_1 \circ u_2) \\ &\quad - B(u_1,u_2)\sigma(u_1{}^2) - B(u_1,u_2)\sigma(u_2{}^2) + B(u_1,u_2)\sigma(u_1 \circ u_2). \end{aligned} \tag{9}$$

Combining Eqn. (8) with Eqn. (9), we have

$$2A(u_1)\sigma(u_1 \circ u_2) + 2A(u_2)\sigma(u_1 \circ u_2) + 2B(u_1,u_2)\sigma(u_1{}^2) + 2B(u_1,u_2)\sigma(u_2{}^2) = 0.$$

By 2-torsion freeness, we have

$$A(u_1)\sigma(u_1 \circ u_2) + A(u_2)\sigma(u_1 \circ u_2) + B(u_1,u_2)\sigma(u_1{}^2) + B(u_1,u_2)\sigma(u_2{}^2) = 0. \tag{10}$$

Replacing u_1 by $2u_1$ in Eqn. (8), we find

$$\begin{aligned} 0 &= 4A(u_2)\sigma(u_1^2) + 4A(u_1)\sigma(u_2^2) + 8A(u_1)\sigma(u_1 \circ u_2) + 2A(u_2)\sigma(u_1 \circ u_2) \\ &\quad + 8B(u_1,u_2)\sigma(u_1{}^2) + 2B(u_1,u_2)\sigma(u_2{}^2) + 4B(u_1,u_2)\sigma(u_1 \circ u_2). \end{aligned}$$

Using Eqn. (8) and Eqn. (9) in the last equation, we get

$6\,A(u_1)\sigma(u_1 \circ u_2) + 6B(u_1,u_2)\sigma({u_1}^2) = 0$, for all $u_1, u_2 \in L$.

By 3!-torsion freeness, we have

$$A(u_1)\sigma(u_1 \circ u_2) + B(u_1,u_2)\sigma({u_1}^2) = 0, \tag{11}$$

for all $u_1, u_2 \in L$.

Right multiplication of Eqn.(11) by $A(u_1)$, we have

$A(u_1)\sigma(u_1 \circ u_2)A(u_1) + B(u_1,u_2)\sigma({u_1}^2)A(u_1) = 0$.

Using Eqn.(6), we find that

$A(u_1)\sigma(u_1u_2)A(u_1) + A(u_1)\sigma(u_2u_1)A(u_1) = 0$, for all $u_1, u_2 \in L$.

Since $\sigma(L) = L$, we have

$$A(u_1)\sigma(u_1)u_2A(u_1) + A(u_1)u_2\sigma(u_1)A(u_1) = 0, \tag{12}$$

for all $u_1, u_2 \in L$.

Replacing u_2 by $2u_2\sigma(u_1)$ in the above relation and by 2-torsion freeness, we get

$A(u_1)\sigma(u_1)u_2\sigma(u_1)A(u_1) + A(u_1)u_2\sigma({u_1}^2)A(u_1) = 0$, for all $u_1, u_2 \in L$.

Again using Eqn. (6), we get

$A(u_1)\sigma(u_1)u_2\sigma(u_1)A(u_1) = 0$, for all $u_1, u_2 \in L$.

and so

$\sigma(u_1)A(u_1)\sigma(u_1)u_2\sigma(u_1)A(u_1)\sigma(u_1) = 0$, for all $u_1, u_2 \in L$.

By Lemma 1, we have

$\sigma(u_1)A(u_1)\sigma(u_1) = 0$, for all $u_1 \in L$.

Right multiplication of Eqn. (12) by $\sigma(u_1)$ and using the last equation, we see that

$A(u_1)\sigma(u_1)u_2A(u_1)\sigma(u_1) = 0$, for all $u_1, u_2 \in L$.

Again using Lemma 1, we have

$$A(u_1)\sigma(u_1) = 0, \text{ for all } u_1 \in L. \tag{13}$$

Replacing u_1 by u_1+u_2, we have

$$0 = A(u_1 + u_2)\sigma(u_1 + u_2) = (A(u_1) + A(u_2) + B(u_1, u_2))\sigma(u_1 + u_2).$$

Using Eqn. (13), we get

$$A(u_1)\sigma(u_2) + A(u_2)\sigma(u_1) + B(u_1, u_2)\sigma(u_1) + B(u_1, u_2)\sigma(u_2) = 0.$$

Replacing u_1 by -u_1 in the above relation, we have

$$A(u_1)\sigma(u_2) + B(u_1, u_2)\sigma(u_1) = 0, \qquad (14)$$

for all $u_1, u_2 \in L$.

Right multiplication of Eqn. (14) by $\sigma(u_1)A(u_1)$, we find

$$A(u_1)\sigma(u_2)\sigma(u_1)A(u_1) + B(u_1, u_2)\sigma({u_1}^2)A(u_1) = 0.$$

Using Eqn. (6), we see

$$A(u_1)\sigma(u_2)\sigma(u_1)A(u_1) = 0$$

and so $\sigma(u_1)A(u_1)u_2\sigma(u_1)A(u_1) = 0$.

By Lemma 1, we have

$\sigma(u_1)A(u_1) = 0$, for all $u_1 \in L$.

Right multiplication of Eqn. (14) by A(u_1) and using the last equation

$A(u_1)\sigma(u_2)A(u_1) = 0$, for all $u_1, u_2 \in L$.

By Lemma 1 and $\sigma(L) = L$, we get $A(u_1) = 0$, for all $u_1 \in L$. We conclude that d is a Jordan (σ, τ)-derivation.

Corollary 5. Let R be a 3!-torsion free semiprime ring, τ, σ two endomorphisms of R, $d: R \to R$ an additive mapping, $L \nsubseteq Z(R)$ be a nonzero square-closed Lie ideal of R, $\sigma, \tau \epsilon Aut(R)$ and $d(L), \tau(L) \subseteq L, \sigma(L) = L$. If d is a Jordan $(\sigma, \tau)-$derivation on L, then d is $(\sigma, \tau)-$derivation on L.

Proof. By Theorem 4 and Lemma 2, we get the required results.

3. Conclusions

Our study is about the comparison of Jordan triple (σ,τ) -derivation and Jordan (σ,τ)-derivation. Using this theorem, each Jordan (σ,τ)-derivation has been shown to be a (σ,τ)-derivation.

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