



Some Remarks on Riemannian Submersions Admitting An Almost Yamabe Soliton

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Abstract

In this paper, we study the Riemannian submersions $\pi : M \rightarrow B$ whose total manifolds admit an almost Yamabe soliton. Here, we give some necessary conditions for which any fiber of π or B are almost Yamabe soliton or Yamabe soliton. Also, we calculate the scalar curvatures of any fiber and B and using them, we present the relations between the scalar curvatures of them and obtain some characterizations of such a soliton (that is, shrinking, steady or expanding).

Keywords: Riemannian manifold; Riemannian submersion; Almost Yamabe soliton.

Hemen Hemen Yamabe Soliton Kabul Eden Riemann Submersiyonlar Üzerine Bazı Notlar

Öz

Bu çalışmada total uzayı hemen hemen Yamabe soliton olan Riemann submersiyonlar ele alındı. Burada π 'nin herhangi bir lifinin veya B manifoldunun birer Yamabe soliton veya hemen hemen Yamabe soliton olması için bazı gerekli koşullar verildi. Ayrıca liflerin ve B manifoldunun skalar eğrilikleri hesaplandı ve bunlar arasındaki ilişkiler ortaya koyularak söz konusu solitonun bazı karakterizasyonları (yani daralan, durgun veya genişleyen) elde edildi.



Anahtar Kelimeler: Riemann manifold; Riemann submersiyon; Hemen hemen Yamabe soliton.

1. Introduction

The concept of Yamabe flow was defined by Hamilton to solve the Yamabe problem in [1]. Yamabe solitons are self-similar solutions for Yamabe flows and they seem to be as singularity models. More clearly, the Yamabe soliton comes from the blow-up procedure along the Yamabe flow, so such solitons have been studied intensively (see [2-8]).

A generalization of Yamabe solitons was given in [2] as follows:

A Riemannian manifold M is said to be an almost Yamabe soliton, if there exists a vector field ν on M which satisfies

$$\frac{1}{2}L_{\nu}g = (\tau - \mu)g, \quad (1)$$

where τ is the scalar curvature of M , μ is a smooth function, ν is a soliton field for (M, g) and L is the Lie-derivative. An almost Yamabe soliton is denoted by (M, g, ν, μ) . Also, we say that an almost Yamabe soliton is steady, expanding or shrinking, if $\mu = 0$, $\mu < 0$ or $\mu > 0$.

From the above definition given by Eqn. (1), if μ is a constant, the almost Yamabe soliton is said to be the Yamabe soliton. It is obvious that Einstein manifolds are almost Yamabe solitons.

On the other hand, submersions are very interesting topic not only in differential geometry but also physics and mechanics, especially Riemannian submersions which are defined between Riemannian manifolds. Because, Riemannian submersions have many applications there. (for details, we refer to [9]).

Riemannian submersions were firstly studied by A. Gray [10] and B. O'Neill [11], independently. They presented some fundamental properties and formulas for Riemannian submersions and recently, such a theory has been intensively studied (we refer to [12-15]).

Considering above notions of almost Yamabe soliton and Riemannian submersion, the present paper contains some notes for Riemannian submersion in Section 2. The next section is about a Riemannian submersion π between Riemannian manifolds whose total space admits an almost Yamabe soliton. According to the soliton field (that is, such a field is vertical or horizontal), we obtain some characterizations for which any fiber of π or the target manifold is

a Yamabe soliton or an almost Yamabe soliton. Here, we calculate the extrinsic vertical and horizontal scalar curvatures and using them, we present the relationships between the scalar curvatures of any fiber or the target manifold and the characterization of almost Yamabe soliton (that is such a soliton is shrinking, expanding or steady).

2. Preliminaries

We recall here some basic notions from [9], as follows:

A differentiable map $\pi : (M, g) \rightarrow (B, g')$ is said to be a Riemannian submersion from Riemannian manifold M onto the Riemannian manifold B if it satisfies

(i) the derivative map π_* is onto,

(ii) $g_p(U, V) = g'_{\pi(p)}(\pi_*U, \pi_*V)$,

for any $U, V \in \Gamma(TM)$ and $p \in M$. Note that, for any $x \in B$, if $m - n = r$, $\pi^{-1}(x)$ of dimension r which is also a submanifold of M .

For any $p \in M$, we denote $\mathcal{V}_p = \ker \pi_{*p}$ such that it corresponds to the foliation of M determined by the fibers of Riemannian submersion π , since each \mathcal{V}_p coincides with the tangent space of $\pi^{-1}(x)$ at p , $\pi(p) = x$. Hence, \mathcal{V}_p is said to be the vertical space.

Denoting the complementary of the vertical distribution \mathcal{V} by \mathcal{H} , we have

$$T_pM = \mathcal{V}_p \oplus \mathcal{H}_p.$$

Here \mathcal{H}_p is said to be a horizontal space, for $p \in M$.

Some properties on a Riemannian submersion are given with the following:

Let π be a Riemannian submersion from M to B . Denoting the Levi-Civita connections by ∇ and ∇' of M and B , respectively. Then, for X, Z are basic vector fields π -related to $X', Z' \in \Gamma(TB)$ (i.e. $\pi_*(X) = X'$, $\pi_*(Z) = Z'$), the followings are hold:

(1) $g(X, Z) = g'(X', Z') \circ \pi$,

(2) $h[X, Z]$ is π -related to $[X', Z']$,

(3) $h(\nabla_X Z)$ is π -related to $\nabla'_{X'} Z'$,

(4) $[X, V]$ is the vertical, for any vertical vector field V .

Note that, two tensor fields \mathcal{A} and \mathcal{T} is determined by Riemannian submersion π on M . Also, these tensor fields are given by

$$\mathcal{A}(E, F) = \mathcal{A}_E F = v\nabla_{hE} hF + h\nabla_{hE} vF,$$

$$\mathcal{T}(E, F) = \mathcal{T}_E F = h\nabla_{vE} vF + v\nabla_{vE} hF,$$

for the vector fields E, F are tangent to M . Here the horizontal projection and vertical projection are respectively denoted by h and v and ∇ is the Levi-Civita connection of the total space.

We remark that the tensor field \mathcal{T} vanishes on M if and only if any fiber is totally geodesic. Similarly, the tensor field \mathcal{A} vanishes on M if and only if \mathcal{H} is integrable.

Using O'Neill tensors \mathcal{A} and \mathcal{T} , the followings are hold:

$$\nabla_U W = \mathcal{T}_U W + \hat{\nabla}_U W, \tag{2}$$

$$\nabla_U X = h(\nabla_U X) + \mathcal{T}_U X, \tag{3}$$

$$\nabla_X U = \mathcal{A}_X U + v(\nabla_X U), \tag{4}$$

$$\nabla_X Y = \mathcal{A}_X Y + h(\nabla_X Y), \tag{5}$$

where $\hat{\nabla}_U W = \nabla_U W$, X, Y are tangent to \mathcal{H} and U, W are tangent to \mathcal{V} . Indeed, such tensor fields satisfy

$$g(\mathcal{A}_E F, G) = -g(\mathcal{A}_E G, F), \tag{6}$$

$$g(\mathcal{T}_E F, G) = -g(\mathcal{T}_E G, F), \tag{7}$$

for any E, F, G are tangent to M .

Furthermore, the O'Neill tensors \mathcal{T} and \mathcal{A} satisfy

$$\sum_{j=1}^r g(\mathcal{T}_U U_j, \mathcal{T}_V U_j) = \sum_{i=1}^n g(\mathcal{T}_U X_i, \mathcal{T}_V X_i), \tag{8}$$

$$\sum_{j=1}^r g(\mathcal{A}_X U_j, \mathcal{A}_Y U_j) = \sum_{i=1}^n g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i), \tag{9}$$

$$\sum_{j=1}^r g(\mathcal{A}_X U_j, \mathcal{T}_U U_j) = \sum_{i=1}^n g(\mathcal{A}_X X_i, \mathcal{T}_U X_i), \tag{10}$$

where $\{U_j\}_{1 \leq j \leq r}$ and $\{X_i\}_{1 \leq i \leq n}$ are orthonormal frames of vertical distribution \mathcal{V} and horizontal distribution \mathcal{H} , respectively for any $X, Y \in \mathcal{H}$ and $U, V \in \mathcal{V}$.

Denote the Riemannian curvature tensors of B , M and any fiber respectively by R' , R and \hat{R} . Then, we get

$$R(U, V, F, W) = \hat{R}(U, V, F, W) + g(\mathcal{T}_V F, \mathcal{T}_U W) - g(\mathcal{T}_U F, \mathcal{T}_V W),$$

$$R(X, Y, G, Z) = R'(X', Y', G', Z') \circ \pi - g(\mathcal{A}_Y G, \mathcal{A}_X Z) + 2g(\mathcal{A}_X Y, \mathcal{A}_G Z) + g(\mathcal{A}_X G, \mathcal{A}_Y Z),$$

for any horizontal vectors X, Y, G, Z and vertical vectors U, V, F, W .

Moreover, denoting the mean curvature vector of any fiber by H which is given as

$$N = rH, \tag{11}$$

such that

$$N = \sum_{j=1}^r \mathcal{T}_{V_j} V_j, \tag{12}$$

where $\{V_1, V_2, \dots, V_r\}$ is an orthonormal frame of \mathcal{V} . Also, remark that

$$\mathcal{T}_U V = g(U, V)H \tag{13}$$

is satisfied for U, V are tangent to \mathcal{V} if and only if any fiber is totally umbilical submanifold.

Indeed, the vector field N is zero on M if and only if any fiber is minimal.

Using Eqn. (12), one has

$$g(\nabla_E N, Z) = \sum_{j=1}^r g((\nabla_E \mathcal{T})(U_j, U_j), Z),$$

where any horizontal vector field Z and any vector field E .

Denoting the horizontal divergence of the horizontal vector field Z by $\tilde{\delta}(Z)$ which holds

$$\tilde{\delta}(Z) = \sum_{k=1}^n g(\nabla_{X_k} Z, X_k). \tag{14}$$

Here $\{X_1, X_2, \dots, X_n\}$ is an orthonormal frame of \mathcal{H} . Therefore, from Eqn. (14), one has

$$\tilde{\delta}(N) = \sum_{k=1}^n \sum_{j=1}^r g((\nabla_{X_k} \mathcal{T})(U_j, U_j), X_k).$$

(See [16], pp. 243).

The Ricci tensor on the total space of π is given as

$$\begin{aligned} Ric(U, W) &= \hat{Ric}(U, W) - \sum_{k=1}^n g((\nabla_{X_k} \mathcal{T})(U, W), X_k) + g(N, \mathcal{T}_U W) \\ &\quad - \sum_{k=1}^n g(\mathcal{A}_{X_k} U, \mathcal{A}_{X_k} W) \end{aligned} \tag{15}$$

$$\begin{aligned} Ric(X, Y) &= Ric'(X', Y') \circ \pi + 2 \sum_{k=1}^n g(\mathcal{A}_X X_k, \mathcal{A}_Y X_k) \\ &\quad + \sum_{k=1}^n g(\mathcal{T}_{U_j} X, \mathcal{T}_{U_j} Y), \end{aligned} \tag{16}$$

$$\begin{aligned} Ric(U, X) &= -g(\nabla_U N, X) - \sum_{k=1}^n \{g((\nabla_{X_k} \mathcal{A})(X_k, X), U) \\ &\quad + 2g(\mathcal{A}_{X_k} X, \mathcal{T}_U X_k)\} + \sum_{j=1}^r g((\nabla_{U_j} \mathcal{T})(U_j, U), X) \end{aligned} \tag{17}$$

where \hat{Ric} and Ric' denote the Ricci tensors of fiber and B respectively. Here $\{U_j\}$ and $\{X_k\}$ are the orthonormal bases of the vertical and horizontal distributions respectively, for any $X, Y \in \mathcal{H}$ and $U, W \in \mathcal{V}$.

Taking into account the equalities Eqn. (15)-Eqn. (16), the extrinsic vertical scalar curvature $\tau|_{\mathcal{V}}$ and the extrinsic horizontal scalar curvature $\tau|_{\mathcal{H}}$ are given by

$$\begin{aligned} \tau|_{\mathcal{V}} &= \sum_{j=1}^r Ric(U_j, U_j) = \sum_{j=1}^r \{ \hat{Ric}(U_j, U_j) + g(N, \mathcal{T}_{U_j} U_j) \\ &\quad - \sum_{i=1}^n g((\nabla_{X_i} \mathcal{T})(U_j, U_j), X_i) - g(\mathcal{A}_{X_i} U_j, \mathcal{A}_{X_i} U_j) \} \end{aligned} \tag{18}$$

$$\begin{aligned} \tau|_{\mathcal{H}} &= \sum_{i=1}^n Ric(X_i, X_i) = \sum_{i=1}^n \{ (Ric'(X_i', X_i') \circ \pi) - g(\nabla_{X_i} N, X_i) \\ &\quad + 2 \sum_{k=1}^n g(\mathcal{A}_{X_i} X_k, \mathcal{A}_{X_i} X_k) + \sum_{j=1}^r g(\mathcal{T}_{U_j} X_i, \mathcal{T}_{U_j} X_i) \}. \end{aligned} \tag{19}$$

Above equalities Eqn. (18)-Eqn. (19) imply

$$\tau|_{\mathcal{V}} = \hat{\tau} + \|N\|^2 - \|\mathcal{A}\|^2 - \check{\delta}(N), \tag{20}$$

$$\tau|_{\mathcal{H}} = (\tau' \circ \pi) - \check{\delta}(N) + 2\|\mathcal{A}\|^2 + \|\mathcal{T}\|^2, \tag{21}$$

where $\hat{\tau}$ and τ' are the scalar curvatures of any fiber of π and B respectively, such that $\|\mathcal{A}\|^2 = \sum_{i,j} g(\mathcal{A}_{X_i} U_j, \mathcal{A}_{X_i} U_j)$ and $\|\mathcal{T}\|^2 = \sum_{i,j} g(\mathcal{T}_{U_j} X_i, \mathcal{T}_{U_j} X_i)$.

Finally using Eqn. (20)-Eqn. (21), the scalar curvature τ on the base manifold M is given by

$$\tau = \hat{\tau} + (\tau' \circ \pi) + \|N\|^2 + \|\mathcal{T}\|^2 + \|\mathcal{A}\|^2 - 2\check{\delta}(N).$$

3. Riemannian Submersions Admitting an Almost Yamabe Soliton

In the present part, we investigate the Riemannian submersion $\pi: M \rightarrow B$ between Riemannian manifolds whose total space M admits an almost Yamabe soliton. Here, we give some characterizations about any fiber or B is an almost Yamabe soliton or a Yamabe soliton.

We recall the following lemma from [14]:

Lemma 1. Let π be a Riemannian submersion from M onto B . The horizontal distribution \mathcal{H} is parallel with respect to ∇ on M if and only if the O'Neill tensors \mathcal{A} and \mathcal{T} vanish, identically.

Theorem 2. Let π be a Riemannian submersion admitting an almost Yamabe soliton (M, g, ν, μ) such that ν is vertical. Then, any fiber of π is an almost Yamabe soliton.

Proof. Because M is an almost Yamabe soliton, from Eqn. (1) we have

$$\frac{1}{2}\{g(\nabla_{\nu} \nu, U) + g(\nabla_U \nu, V)\} = (\tau - \mu)g(V, U), \tag{22}$$

for any $U, V \in \mathcal{V}$. Using Eqn. (2), one has

$$\frac{1}{2}\{g(\hat{\nabla}_{\nu} \nu, U) + g(\hat{\nabla}_U \nu, V)\} = (\tau|_{\mathcal{V}} - \mu)g(V, U), \tag{23}$$

where $\hat{\nabla}$ is the Levi-Civita connection on any fiber of π . Putting the extrinsic vertical scalar curvature $\tau|_{\mathcal{V}}$ of Eqn. (20) in Eqn. (23), it gives

$$\frac{1}{2}\{g(\hat{\nabla}_{\nu} \nu, U) + g(\hat{\nabla}_U \nu, V)\} = (\hat{\tau} + \|N\|^2 - \|\mathcal{A}\|^2 - \check{\delta}(N) - \mu)g(V, U). \tag{24}$$

If we denote $\sigma = -\|N\|^2 + \|\mathcal{A}\|^2 + \check{\delta}(N) + \mu$, then the equality Eqn. (24) follows

$$\frac{1}{2}(L_{\nu} g)(U, V) = (\hat{\tau} - \sigma)\hat{g}(U, V),$$

which means such a fiber of π is an almost Yamabe soliton.

Using Lemma 1, as particular case of Theorem 2, one has:

Remark 3. Let π be a Riemannian submersion admitting a Yamabe soliton (M, g, ν, μ) such that ν is vertical. Then, any fiber becomes a Yamabe soliton.

Theorem 4. Let π be a Riemannian submersion admitting an almost Yamabe soliton (M, g, ν, μ) such that ν is vertical. Then, the extrinsic horizontal scalar curvature $\tau|_{\mathcal{H}}$ satisfies

$$\tau|_{\mathcal{H}} - \mu = 0. \tag{25}$$

Proof. Because the total space M is an almost Yamabe soliton, from Eqn. (1) we get

$$\frac{1}{2}\{g(\nabla_X \nu, Z) + g(\nabla_Z \nu, X)\} = (\tau - \mu)g(X, Z), \tag{26}$$

for any the horizontal vectors X, Z . Considering Eqn. (4) in Eqn. (26), we have

$$\frac{1}{2}\{g(\mathcal{A}_X \nu, Z) + g(\mathcal{A}_Z \nu, X)\} = (\tau|_{\mathcal{H}} - \mu)g(X, Z). \tag{27}$$

Also, considering the properties of the tensor field \mathcal{A} in the equality Eqn. (6), the left hand side of Eqn. (27) vanishes identically. For any $X, Z \in \mathcal{H}$, we have

$$(\tau|_{\mathcal{H}} - \mu)g(X, Z) = 0,$$

which gives Eqn. (25).

As a consequence of Theorem 4, we conclude the following:

Corollary 5. Let π be a Riemannian submersion admitting an almost Yamabe soliton (M, g, ν, μ) be such that ν is vertical. If \mathcal{H} is parallel, the followings are hold:

- (i) (M, g, ν, μ) is shrinking if and only if the manifold B has positive scalar curvature.
- (ii) (M, g, ν, μ) is expanding if and only if the manifold B has negative scalar curvature.
- (iii) (M, g, ν, μ) is steady if and only if the manifold B has zero scalar curvature.

In this section, from now on, we suppose that the potential field of the almost Yamabe soliton is horizontal. Then, we have some theorems as follows:

Theorem 6. Let π be a Riemannian submersion admitting an almost Yamabe soliton (M, g, ξ, μ) such that ξ is horizontal. Then, the Riemannian manifold B is an almost Yamabe soliton.

Proof. Since (M, g) is an almost Yamabe soliton, from Eqn. (1), then we get

$$\frac{1}{2}\{g(\nabla_X \xi, Z) + g(\nabla_Z \xi, X)\} = (\tau - \mu)g(X, Z),$$

for any the horizontal vectors X, Z . Using Eqn. (5), one has

$$\frac{1}{2}\{g(h(\nabla_x \xi), Z) + g(h(\nabla_Z \xi), X)\} = (\tau|_{\mathcal{V}} - \mu)g(X, Z). \tag{28}$$

Putting Eqn. (21) in Eqn. (28), it follows

$$\begin{aligned} \frac{1}{2}\{g(h(\nabla_x \xi), Z) + g(h(\nabla_Z \xi), X)\} &= \{(\tau' \circ \pi) + \|\mathcal{T}\|^2 + 2\|\mathcal{A}\|^2 \\ &\quad - \check{\delta}(N) - \mu\}g(X, Z). \end{aligned} \tag{29}$$

If we denote $\gamma = -\|\mathcal{T}\|^2 - 2\|\mathcal{A}\|^2 + \check{\delta}(N) + \mu$, then Eqn. (29) is equivalent to

$$\frac{1}{2}\{g(h(\nabla_x \xi), Z) + g(h(\nabla_Z \xi), X)\} = \{(\tau' \circ \pi) - \gamma\}g(X, Z).$$

Here we note that $h(\nabla_x \xi)$ and $h(\nabla_Z \xi)$ are π -related to $\nabla'_x \xi'$ and $\nabla'_{Z'} \xi'$, respectively. It follows

$$\frac{1}{2}\{g'(\nabla'_x \xi', Z') + g'(\nabla'_{Z'} \xi', X')\} = (\tau' - \gamma)g'(X', Z'), \tag{30}$$

for any the vector fields X', Z' are tangent to B . Then, Eqn. (30) gives

$$\frac{1}{2}L_{\xi'} g' = (\tau' - \gamma)g', \tag{31}$$

which means the Riemannian manifold B is an almost Yamabe soliton with potential vector field ξ' , such that $\pi_*(\xi) = \xi'$.

Considering Lemma 1, in particular case we have the following:

Remark 7. Let π be a Riemannian submersion admitting an almost Yamabe soliton (M, g, ξ, μ) such that ξ is horizontal. If \mathcal{H} is parallel, then B becomes a Yamabe soliton.

Theorem 8. Let π be a Riemannian submersion with totally umbilical fibres admitting an almost Yamabe soliton (M, g, ξ, μ) such that ξ is horizontal. Then, the extrinsic vertical scalar curvature $\tau|_{\mathcal{V}}$ on \mathcal{V} satisfies

$$\tau|_{\mathcal{V}} = \mu + g(H, \xi). \tag{32}$$

Here H is the mean curvature vector field of fiber.

Proof. Since the total space M of π is an almost Yamabe soliton, then from Eqn. (1), one has

$$\frac{1}{2}\{g(\nabla_U \xi, W) + g(\nabla_W \xi, U)\} = (\tau - \mu)g(U, W),$$

for any $U, W \in \mathcal{V}$. By using Eqn. (3), the last equation gives

$$\frac{1}{2}\{g(\mathcal{T}_U \xi, W) + g(\mathcal{T}_W \xi, U)\} = (\tau - \mu)g(U, W). \quad (33)$$

Also, using Eqn. (7) in the left hand side of Eqn. (33), it follows

$$g(\mathcal{T}_U W, \xi) = (\tau|_{\mathcal{V}} - \mu)g(U, W). \quad (34)$$

Finally, since π has totally umbilical fibres, using Eqn. (13) we have

$$g(H, \xi) = \tau|_{\mathcal{V}} - \mu, \quad (35)$$

which gives Eqn. (32).

Considering Theorem 8, we have the followings immediately:

Remark 9. Let π be a Riemannian submersion with minimal fibers admitting an almost Yamabe soliton (M, g, ξ, μ) such that ξ is horizontal. Then, the extrinsic vertical scalar curvature $\tau|_{\mathcal{V}}$ satisfies

$$\tau|_{\mathcal{V}} = \mu.$$

Using Remark 9, we infer the following:

Corollary 10. Let π be a Riemannian submersion with minimal fibres admitting an almost Yamabe soliton (M, g, ξ, μ) such that ξ is horizontal. If \mathcal{H} is parallel, then we have the following:

- (i) (M, g, ξ, μ) is shrinking if and only if any fiber has positive scalar curvature.
- (ii) (M, g, ξ, μ) is expanding if and only if any fiber has negative scalar curvature.

(iii) (M, g, ξ, μ) is steady if and only if any fiber has zero scalar curvature.

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