Eastern Anatolian Journal of Science
Volume VI, Issue I, 2020, 11-20

Eastern Anatolian Journal of Science

Analysis of a Fractional Plant-Nectar-Pollinator Model with the Exponential Kernel

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Abstract

This paper extends the plant-nectar-pollination model to the Caputo-Fabrizio fractional derivative, following which the existence and singularity resolutions of the new model are studies with the Picard-Lindelöf method. Afterwards Hyers-Ulam stability is utilized to analyse the stability of the PNP model. Lastly, Adams-Bashforth numerical approach is used for numerical resolutions.

Keywords: PNP model, Caputo-Fabrizio fractional derivative, Adams-Bashforth numerical method, Picard-Lindelöf method.

1. Introduction

Many natural and flowering plants survive thanks to the accessibility of appropriate actors that pollenate or disperse seeds. Flowering plants require mechanisms, which will introduce pollen to their roots that will enable them to breed. In this respect, pollen transposition is termed pollination. The occurrence of pollination and the appropriateness of pollen and stigma lead to the formation of a pollen tube from a particle of pollen and this pollen tube transmits sperm into the ovule in the ovary. The life of the species necessitates the existence of seeds in most seed plants.

In this respect, a mathematical model has been brought forth. One can list the crucial studies in the literature as follows:

Received: 23.05.2020
Accepted: 19.06.2020
Published:24.06.2020
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Cite this article as: M.A. Dokuyucu, Analysis of a Fractional Plant-Nectar-Pollinator Model with the Exponential Kernel, *Eastern Anatolian Journal of Science, Vol. 6, Issue 1, 11-20, 2020*

Wang (2019) made use of pollination-mutuality in his analysis of the impact of nectar. He showed the mechanism through which the pollinator can lead to nectar consumption ratio, nectar degeneration ratio and through which the nectar production ratio can result in the perpetuity/demolition of pollination-mutuality in his examination of the model.

Wang (2018) has examined the global dynamics of plant-pollinator-robber systems, which comprise two opponent consumers being pollinator and nectar robber. In this example, whereas the nectar robber is a plant parasite pollinator is reciprocity. Vanbergen et al. (2017) examined the hardiness of insect flower visitor networks, the impacts of terrain disruption among species and the degree of mutuality among species. They were found to be in accord with the network structure in the integral and troubled terrain. In addition, they patterned whether the internal dependency of species on reciprocity affects the inclination of extinction cascades in the network. Khan et al. (2020) have popularized a PNP model that comprises Atangana-Baleanu gradual order differentials. They have acquired crucial data regarding the variables used in the complex system thanks to this new differential type. The presence and singularity of the Atangana-Baleanu differential were examined with the PL method and the stability analysis was conducted with Picard's stability technique for the fractional-order plant-nectar-pollinator model. The model brought forth also revealed different schemes of the plant-nectar-pollinator (PNP) model with numerical examples. In addition, there are many studies in the literature on the expansion of new fractional derivative operators to mathematical models (Dokuyucu et al. 2018a; Dokuyucu et al. 2018b; Dokuyucu 2020a; Dokuyucu and Dutta 2020; Dokuyucu 2020b; Rashid et al. 2019a; Nie et al. 2019; Rashid et al 2019b, Ekinci and Ozdemir 2019).

There are five chapters in this article. The first part presents general information concerning the plant-nectar-pollinator (PNP) model. The second part offers essential definitions and theorems concerning the new Caputo differential. The third part comprises the analysis of the existence and singularity of the mathematical model. The Picard- Lindelöf theorem helped in revealing the existence of the model. In the fourth part, the stability of the model is studied with Hyers-Ulam stability theorem. In the last part, the model's numerical solution is performed by integrating Adams-Bashforth's numerical approach into the Caputo-Fabrizio fractional differential. Simulations were also performed and the model was examined thoroughly.

2. *Preliminaries*

Descriptions and theorems regarding the non-singular fractional Caputo-Fabrizio operator are presented in this part. Please see (Caputo 1967; Caputo and Fabrizio 2015; Losada and Nieto 2015) articles for more detailed information.

Definition 2.1. The well-known fractional order Caputo derivative is defined as follows (Caputo 1967), let $f \in H^1(a,b)$

$$ {}_a^C D_t^\rho f(t) = \frac{1}{\Gamma(n-\rho)} \int_a^t \frac{f^{(n)}(r)}{(t-r)^{\rho+1-n}}, \tag{1}$$

where $n-1 < \rho < n \in N$.

Definition 2.2. Let $f \in H^1(a,b), 0 < \rho < 1$. The new Caputo fractional derivative is defined as follows (Caputo and Fabrizio 2015),

$$ {}_a^{CF} D_t^\rho f(t) = \frac{\rho M(\rho)}{1-\rho} \int_a^t \frac{df(x)}{dx}\, exp\left[\rho \frac{x-t}{1-\rho}\right] dx, \tag{2}$$

Here $M(\rho)$ is a normalization constant. Also $M(0)$ and $M(1)$ are equal to 1. Further it can be written below, if the f does not belong to $H^1(a,b)$.

$$ {}_a^{CF} D_t^\rho f(t) = \frac{\rho M(\rho)}{1-\rho} \int_a^t \begin{matrix} (f(t)-f(x)) \\ \times exp\left[\rho \frac{x-t}{1-\rho}\right] \end{matrix} dx, \tag{3}$$

Definition 2.3. Let $f \in H^1(a,b),\ 0 < \rho < 1$. The Caputo-Fabrizio fractional derivative of order f is as follows (Losada and Nieto 2015),

$$ {}^{CF}D_\star^\rho f(t) = \frac{1}{1-\rho} \int_a^t f'(x) exp\left[\rho \frac{x-t}{1-\rho}\right] dx. \tag{4}$$

Definition 2.4. Let $0 < \rho < 1$. The fractional integral order ρ of a function f is defined by (Losada and Nieto 2015),

$$ \begin{aligned} I^\rho f(t) &= \frac{2(1-\rho)}{(2-\rho)M(\rho)} u(t) \\ &+ \frac{2\rho}{(2-\rho)M(\rho)} \int_a^t u(s) ds \end{aligned} \tag{5}$$

3. *Analysis of the existence and uniqueness of the new system*

3.1. Existence Solution for the Plant-Nectar-Pollinator Model

Our system of equations comprises two types, one being the first plant and the other, a pollinator that interacts with the plant. As Revilla (2015) puts forth, one can describe the dynamic equations generated for these two types as follows.

$$ \begin{aligned} (\mathcal{N}_1)_t &= \mathcal{G}_1(.)\mathcal{N}_1 + \sigma_1\beta_0\mathcal{F}\mathcal{N}_0 + \sigma_1\beta_2\mathcal{F}\mathcal{N}_2, \\ (\mathcal{N}_2)_t &= \mathcal{G}_2(.)\mathcal{N}_2 + \sigma_2\beta_2\mathcal{F}\mathcal{N}_2, \\ \mathcal{F}_t &= \alpha\mathcal{N}_1 - (\omega + \beta_0\mathcal{N}_0 + \beta_2\mathcal{N}_2)\mathcal{F}. \end{aligned} \tag{6}$$

The term "$\sigma_1\beta_0\mathcal{N}_0$" in the first equation indicates that pollination can be accomplished by abiotic factors. We can give the wind flow at the very beginning of the abiotic factors. For convenience, let's assume that $\mathcal{G}_1\mathcal{N}_1 = b_1 N_1, \mathcal{G}_2\mathcal{N}_2 = -b_2$. Here, the r_1 parameter represents the plant's internal growth rate, and c_1 indicates the intraspecific competition level of r_1, while r_2 indicates the pollinator's mortality rate. Also, all parameters are positive in the system (6).

In the equation system (6), N_1 indicates the plant population density, while N_2 indicates the population density of the pollinator. Also, F represents the number of fruit or flowers produced by the plant. In addition, the explanation of each parameter is given in the table below.

Parameter	Explanation
$\mathcal{G}_i(.)$	Percentage change of species i per person when it does not interact with j species by mutualism
σ_1	Parameter of plant conversion rate from a flower or fruit to new adult plants
σ_2	Conversion rate to biomass
β_1	Pollination rate by pollinator or nectar consumption rate by pollinator
α	Per capita rate of plants in nectar production
ω	Rate of loss or deterioration of nectar

Now let $N_1 = A$, $\beta_2 N_2 = B$, $\alpha = a$, $\sigma_1 = d_1$, $\sigma_2\beta_2 = d_2$, $\beta_0 N_0 = e_1$ and $\omega + \beta_0 N_0 = e_2$. this case, if the system (6) is regulated, the following system is obtained.

$$\begin{aligned} A_t &= A(t)\big(b_1 - c_1 A(t)\big) + d_1(e_1 + B(t))C(t),\\ B_t &= B(t)(-b_2 + d_2 C(t)),\\ C_t &= aA(t) - (e_2 + B(t))C(t), \end{aligned} \tag{7}$$

with initial conditions

$$A(0) \geq 0,\ \ B(0) \geq 0,\ \ C(0) \geq 0.$$

When the equation system (7) is extended to the Caputo-Fabrizio fractional derivative obtained using the exponential kernel, the following system is obtained.

$$\begin{aligned} {}^{CF}_{a}D^{\rho}_{t}\, A(t) &= A(t)\big(b_1 - c_1 A(t)\big)\\ &\quad + d_1(e_1 + B(t))C(t),\\ {}^{CF}_{a}D^{\rho}_{t}\, B(t) &= B(t)(-b_2 + d_2 C(t)),\\ {}^{CF}_{a}D^{\rho}_{t}\, C(t) &= aA(t) - \big(e_2 + B(t)\big)C(t). \end{aligned} \tag{8}$$

It is crucial to verify the existence and singularity of the solution for the equation or system of equations in derivative calculations. Hence, this part initially aims to demonstrate the presence of the system (8) of equations. System A, B and C are created and the constants used are just the same as Wang (2019). In the light of the integral form acquired with the help of Laplace transform of the new Caputo fractional differential, we can initially write the following theorem.

Theorem 3.1. Fractional differential equation below,

$${}^{CF}_{a}D^{\rho}_{t}\, f(t) = u(t) - u(0),$$

has a unique solution that takes the inverse Laplace transform and uses the following convolution theorem.

$$\begin{aligned} f(t) - f(0) &= \frac{2(1-\rho)}{(2-\rho)M(\rho)} u(t)\\ &\quad + \frac{2\rho}{(2-\rho)M(\rho)} \int_a^t u(s)ds \end{aligned} \tag{9}$$

With the help of the above theorem, the following system of equations can be obtained.

$$\begin{aligned} &A(t) - g_1(t)\\ &= \frac{2(1-\rho)}{(2-\rho)M(\rho)} \Big(A(t)\big(b_1 - c_1 A(t)\big) + d_1\big(e_1\\ &\qquad + B(t)\big)C(t)\Big)\\ &+ \frac{2\rho}{(2-\rho)M(\rho)} \int_0^t \Big(A(s)\big(b_1 - c_1 A(s)\big)\\ &\qquad + d_1(e_1 + B(s))C(s)\Big)ds\\ &B(t) - g_2(t)\\ &= \frac{2(1-\rho)}{(2-\rho)M(\rho)} \Big(B(t)\big(-b_2 + d_2 C(t)\big)\Big)\\ &+ \frac{2\rho}{(2-\rho)M(\rho)} \int_0^t \Big(B(s)\big(-b_2 + d_2 C(s)\big)\Big)\, ds \end{aligned} \tag{10}$$

$$C(t)-g_3(t) = \frac{2(1-\rho)}{(2-\rho)M(\rho)}\big(aA(t)-\big(e_2+B(t)\big)C(t)\big) + \frac{2\rho}{(2-\rho)M(\rho)}\int_0^t (aA(s) - (e_2+B(s))C(s))\,ds$$

If it is taken as follows for iteration application for the system (10), we have

$$\begin{aligned} A_0(t) &= g_1(t)\\ B_0(t) &= g_2(t)\\ C_0(t) &= g_3(t)\end{aligned} \tag{11}$$

and

$$\begin{aligned} A_{n+1}(t) &= \frac{2(1-\rho)}{(2-\rho)M(\rho)}\Big(A_n(t)\big(b_1-c_1A_n(t)\big)+d_1\big(e_1+B_n(t)\big)C_n(t)\Big)\\ &+\frac{2\rho}{(2-\rho)M(\rho)}\int_0^t \big(A_n(s)\big(b_1-c_1A_n(s)\big)+d_1\big(e_1+B_n(s)\big)C_n(s)\big)ds\\ B_{n+1}(t) &= \frac{2(1-\rho)}{(2-\rho)M(\rho)}\Big(B_n(t)\big(-b_2+d_2C_n(t)\big)\Big)\\ &+\frac{2\rho}{(2-\rho)M(\rho)}\int_0^t \Big(B_n(s)\big(-b_2+d_2C_n(s)\big)\Big)ds\\ C_{n+1}(t) &= \frac{2(1-\rho)}{(2-\rho)M(\rho)}\Big(aA_n(t)-\big(e_2+B_n(t)\big)C_n(t)\Big)\\ &+\frac{2\rho}{(2-\rho)M(\rho)}\int_0^t \Big(aA_n(s)-\big(e_2+B_n(s)\big)C_n(s)\Big)\,ds\end{aligned} \tag{12}$$

We will try to find the exact solution by limiting a large enough n value.

$$\begin{aligned} g_1(t,x) &= A(t)\big(b_1-c_1A(t)\big)+d_1\big(e_1+B(t)\big)C(t),\\ g_2(t,x) &= B(t)\big(-b_2+d_2C(t)\big),\\ g_3(t,x) &= aA(t)-\big(e_2+B(t)\big)C(t),\end{aligned} \tag{13}$$

If the kernels are taken as above, it is clear that the $g_1, g_2$and g_3 functions have a contraction of p, r and s, respectively. Let

$$\begin{aligned} C_1 &= \sup_{N_{y,z_!}}\|g_1(t,p)\|,\\ C_2 &= \sup_{N_{y,z_2}}\|g_2(t,r)\|,\\ C_3 &= \sup_{N_{y,z_3}}\|g_3(t,s)\|,\end{aligned} \tag{14}$$

where

$$\begin{aligned} N_{y,z_i} &= [t-y,t+y]\times[x-z_i,x+z_i]\\ &= Y_i\times Z_i,\quad i=1,2,3.\end{aligned} \tag{15}$$

The equation system can also be used together with the metric in Banach space by means of the fixed-point theorem.

$$\|g(t)\|_\infty = \sup_{t\in[t-y,t+y]}|f(t)|. \tag{16}$$

Another operator is described among uninterrupted functions and is identified by the Picard operator.

$$T: N(Y_1,Z_1,Z_2,Z_3)\to N(Y_1,Z_1,Z_2,Z_3) \tag{17}$$

Defined as follows

$$T\Phi(t) = \Phi_0(t)+\frac{2(1-\rho)}{(2-\rho)M(\rho)}X(t) + \frac{2\rho}{(2-\rho)M(\rho)}\int_0^t F\big(s,X(s)\big)ds, \tag{18}$$

where Φ is the given matrix

$$\Phi(t)=\begin{cases}A(t)\\ B(t)\\ C(t)\end{cases}\qquad \Phi_0(t)=\begin{cases}g_1(t)\\ g_2(t)\\ g_3(t)\end{cases} \tag{19}$$

$$\mathrm{G}(\mathrm{t},\Phi(t))=\begin{cases}g_1(t,x)\\ g_2(t,x)\\ g_3(t,x)\end{cases} \tag{20}$$

Since all plants are unlikely to be pollinated, the solutions can be considered to be limited in a time frame.

$$\|x(t)\|_\infty \le \max\{b_1,b_2,b_3\}, \tag{21}$$

$$\begin{aligned}
&\|T\Phi(t)-\Phi_0(t)\| \\
&= \left\|\frac{2(1-\rho)}{(2-\rho)M(\rho)}F(t,\Phi(t))\right\| \\
&+ \left\|\frac{2\rho}{(2-\rho)M(\rho)}\int_0^t F(s,\Phi(s))ds\right\| \\
&\leq \frac{2(1-\rho)}{(2-\rho)M(\rho)}\left\|F\big(t,\Phi(t)\big)\right\| \\
&+ \frac{2\rho}{(2-\rho)M(\rho)}\left\|\int_0^t F(s,\Phi(s))ds\right\| \\
&\leq \frac{2(1-\rho)}{(2-\rho)M(\rho)}M+\frac{2\rho}{(2-\rho)M(\rho)}M \\
&\leq aM \leq b = max\{b_1,b_2,b_3\},
\end{aligned} \tag{22}$$

where $M=\max\{M_1,M_2,M_3\}$. As a result,

$$a<\frac{b}{M}$$

In addition to the above, the following inequality can be found.

$$\|T\Phi_1-T\Phi_2\|_\infty = \sup_{t\in A}|\Phi_1-\Phi_2| \tag{23}$$

With the definition of the defined operator in hand, we produce the following

$$\begin{aligned}
&\|T\Phi_1-T\Phi_2\| \\
&= \left\|\frac{2(1-\rho)}{(2-\rho)M(\rho)}\Big(F\big(t,\Phi_1(t)\big)\right. \\
&- F\big(t,\Phi_2(t)\big)\Big) \\
&+ \frac{2\rho}{(2-\rho)M(\rho)}\int_0^t \Big(F\big(s,\Phi_1(s)\big) \\
&\left.- F(s,\Phi_2(s))\Big)ds\right\| \\
&\leq \frac{2(1-\rho)}{(2-\rho)M(\rho)}\Big\|\Big(F\big(t,\Phi_1(t)\big) \\
&- F\big(t,\Phi_2(t)\big)\Big)\Big\| \\
&+ \frac{2\rho}{(2-\rho)M(\rho)}\int_0^t \Big\|\Big(F\big(s,\Phi_1(s)\big) \\
&- F(s,\Phi_2(s))\Big)\Big\|ds \\
&\leq \frac{2(1-\rho)}{(2-\rho)M(\rho)}q+\frac{2\rho q}{(2-\rho)M(\rho)}\|\Phi_1(t)-\Phi_2(t)\| \\
&\leq aq\|\Phi_1(t)-\Phi_2(t)\|.
\end{aligned} \tag{24}$$

According to the last inequality, q less than 1. Namely, G has a contraction. At the same time, T has a contraction since $aq<1$. This result shows us that there is a unique solution set.

3.2. Uniqueness Solution for the Plant-Nectar-Pollinator Model

In this section we will show you the unique solution of Plant-Nectar-Pollinator mathematical model.

Theorem 3.2. The Plant-Nectar-Pollinator mathematical model shown in system (10) will have a unique solution if the following inequality hold true:

$$\left(\frac{2(1-\rho)}{2M(\rho)-\rho\, M(\rho)}+\frac{2\rho}{2M(\rho)-\rho\, M(\rho)}\right)\Xi_i \leq 1 \tag{25}$$

where $i=1,2,3$.

Proof Let us assume that the system (10) has solutions $A(t),B(t),C(t)$, as well as $\bar{A}(t),\bar{B}(t),\bar{C}(t)$. that, the following system can be written,

$$\begin{aligned}
\bar{A}(t) &= \frac{2(1-\rho)}{2M(\rho)-\rho\, M(\rho)}\mathfrak{I}_1\big(t,\bar{A}(t)\big) \\
&+ \frac{2\rho}{2M(\rho)-\rho\, M(\rho)}\int_0^t \mathfrak{I}_1\big(y,\bar{A}(y)\big)\,dy \\
\bar{B}(t) &= \frac{2(1-\rho)}{2M(\rho)-\rho\, M(\rho)}\mathfrak{I}_1\big(t,\bar{B}(t)\big) \\
&+ \frac{2\rho}{2M(\rho)-\rho\, M(\rho)}\int_0^t \mathfrak{I}_1\big(y,\bar{B}(y)\big)\,dy \\
\bar{C}(t) &= \frac{2(1-\rho)}{2M(\rho)-\rho\, M(\rho)}\mathfrak{I}_1\big(t,\bar{C}(t)\big) \\
&+ \frac{2\rho}{2M(\rho)-\rho\, M(\rho)}\int_0^t \mathfrak{I}_1\big(y,\bar{C}(y)\big)\,dy
\end{aligned} \tag{26}$$

When the norm is taken from both sides of the system of equations above, firstly

$$\begin{aligned}
&\|A(t)-\bar{A}(t)\| \\
&\leq \frac{2(1-\rho)}{2M(\rho)-\rho\, M(\rho)}\left\|\mathfrak{I}_1\big(t,A(t)\big)-\mathfrak{I}_1\big(t,\bar{A}(t)\big)\right\| + \\
&\frac{2\rho}{2M(\rho)-\rho\, M(\rho)}\int_0^t\left\|\mathfrak{I}_1\big(y,A(y)\big)-\mathfrak{I}_1\big(y,\bar{A}(y)\big)\right\|dy \\
&\leq \frac{2(1-\rho)}{2M(\rho)-\rho\, M(\rho)}\Xi_1\|A-\bar{A}\|+\frac{2\rho\Xi_1}{2M(\rho)-\rho\, M(\rho)}\|A-\bar{A}\|
\end{aligned} \tag{27}$$

The following inequality can be written,

$$\begin{pmatrix} \frac{2(1-\rho)}{2M(\rho)-\rho\,M(\rho)}\Xi_1\|A-\bar{A}\| \\ +\frac{2\rho\Xi_1}{2M(\rho)-\rho\,M(\rho)}\|A-\bar{A}\| \end{pmatrix} \geq 0 \tag{28}$$

Thus $\|A-\bar{A}\|=0$. This implies $A(t)=\bar{A}(t)$. When the same method is applied that $B(t)=\bar{B}(t),\ C(t)=\bar{C}(t)$. According to these results, the model has a unique solution.

4. *Stability Analysis*

Exploring the stability of a mathematical model is as crucial as discovering resolutions. The stability of the Plant-Nectar-Pollinator model will be studied in this part. The following description should initially be provided.

Definition 4.1. The system (30) Hyers-Ulam stable if exists constants $\Theta_i, i=1,2,3$ satisfying for every $\varsigma_i>0, i=1,2,3$.

$$\begin{aligned}
&\left|A(t)-\frac{2(1-\rho)}{2M(\rho)-\rho\,M(\rho)}\Im_1\big(t,A(t)\big)+\frac{2\rho}{2M(\rho)-\rho\,M(\rho)}\int_0^t \Im_1\big(y,A(y)\big)dy\right|\leq\varsigma_1,\\
&\left|B(t)-\frac{2(1-\rho)}{2M(\rho)-\rho\,M(\rho)}\Im_2\big(t,B(t)\big)+\frac{2\rho}{2M(\rho)-\rho\,M(\rho)}\int_0^t \Im_2\big(y,B(y)\big)dy\right|\leq\varsigma_2,\\
&\left|C(t)-\frac{2(1-\rho)}{2M(\rho)-\rho\,M(\rho)}\Im_3\big(t,C(t)\big)+\frac{2\rho}{2M(\rho)-\rho\,M(\rho)}\int_0^t \Im_3\big(y,C(y)\big)dy\right|\leq\varsigma_3.
\end{aligned}\tag{29}$$

There exist $\bar{A}(t),\bar{B}(t),\bar{C}(t)$ are satisfying,

$$\begin{aligned}
&\left|\bar{A}(t)-\frac{2(1-\rho)}{2M(\rho)-\rho\,M(\rho)}\Im_1\big(t,\bar{A}(t)\big)+\frac{2\rho}{2M(\rho)-\rho\,M(\rho)}\int_0^t \Im_1\big(y,\bar{A}(y)\big)dy\right|\leq\varsigma_1,\\
&\left|\bar{B}(t)-\frac{2(1-\rho)}{2M(\rho)-\rho\,M(\rho)}\Im_2\big(t,\bar{B}(t)\big)+\frac{2\rho}{2M(\rho)-\rho\,M(\rho)}\int_0^t \Im_2\big(y,\bar{B}(y)\big)dy\right|\leq\varsigma_2,\\
&\left|\bar{C}(t)-\frac{2(1-\rho)}{2M(\rho)-\rho\,M(\rho)}\Im_3\big(t,\bar{C}(t)\big)+\frac{2\rho}{2M(\rho)-\rho\,M(\rho)}\int_0^t \Im_3\big(y,\bar{C}(y)\big)dy\right|\leq\varsigma_3,
\end{aligned}\tag{30}$$

such that,

$$\begin{aligned}
|A(t)-\bar{A}(t)| &\leq \Theta_1\varsigma_1\\
|B(t)-\bar{B}(t)| &\leq \Theta_2\varsigma_2\\
|C(t)-\bar{C}(t)| &\leq \Theta_3\varsigma_3
\end{aligned}\tag{31}$$

Theorem 4.2. The fractional system (8) is Hyers-Ulam stable with assumption H.

Proof. In theorem (3.2) $A(t),B(t),C(t)$, were shown to have a unique solution. Let $\bar{A}(t),\bar{B}(t),\bar{C}(t)$ be an approximate solution of system (8) satisfying system (25). After, we can say that

$$\begin{aligned}
&\|A(t)-\bar{A}(t)\|\\
&\leq \tfrac{2(1-\rho)}{2M(\rho)-\rho\,M(\rho)}\left\|\Im_1\big(t,A(t)\big)-\Im_1\big(t,\bar{A}(t)\big)\right\|+\\
&\tfrac{2\rho}{2M(\rho)-\rho\,M(\rho)}\int_0^t\left\|\Im_1\big(y,A(y)\big)-\Im_1\big(y,\bar{A}(y)\big)\right\|dy\\
&\leq \tfrac{2(1-\rho)}{2M(\rho)-\rho\,M(\rho)}\Xi_1\|A-\bar{A}\|+\tfrac{2\rho\Xi_1}{2M(\rho)-\rho\,M(\rho)}\|A-\bar{A}\|.
\end{aligned}\tag{32}$$

When we take $\varsigma_1=\Xi_1$ and $\Theta_1=\frac{2(1-\rho)}{2M(\rho)-\rho\,M(\rho)}+\frac{2\rho}{2M(\rho)-\rho\,M(\rho)}$, we have,

$$\|A(t)-\bar{A}(t)\|\leq\varsigma_1\Theta_1. \tag{33}$$

In this way, the following inequalities can be easily written.

$$\|B(t)-\bar{B}(t)\| \leq \varsigma_2 \Theta_2$$
$$\|C(t)-\bar{C}(t)\| \leq \varsigma_3 \Theta_3 \tag{34}$$

With the help of inequalities (33) and (34), the system (18) Hyers-Ulam is stable. Thus, the theorem is proved.

5. Numerical Simulations

Atangana and Owolabi (2018) resorted to the Adams-Bashforth numeric approach to distinguish fractional differential equations, utilized the new Caputo fractional derivative and acquired a new numeric approach.

$${}_{0}^{CF}D_t^{\rho} x(t) = f\big(t, x(t)\big), \tag{35}$$

or

$$f\big(t,x(t)\big) = \frac{M(\rho)}{1-\rho}\int_0^t x'(\tau) exp\left[-\rho\frac{t-\tau}{1-\rho}\right] d\tau. \tag{36}$$

When the above equation is edited with the help of basic analysis theorem, we have,

$$x(t)-x(0) = \frac{1-\rho}{M(\rho)} f\big(t,x(t)\big) + \frac{\rho}{M(\rho)}\int_0^t f\big(\tau,x(\tau)\big)d\tau, \tag{37}$$

consequently,

$$x(t_{n+1})-x(0) = \frac{1-\rho}{M(\rho)} f\big(t_n,x(t_n)\big) + \frac{\rho}{M(\rho)}\int_0^{t_{n+1}} f\big(t,x(t)\big)dt, \tag{38}$$

and

$$x(t_n)-x(0) = \frac{1-\rho}{M(\rho)} f\big(t_{n-1},x(t_{n-1})\big) + \frac{\rho}{M(\rho)}\int_0^{t_n} f\big(t,x(t)\big)dt, \tag{39}$$

using the equations (38) and (39),

$$x(t_{n+1})-x(t_n) = \frac{1-\rho}{M(\rho)}\left\{f\big(t_n,x(t_n)\big) - f\big(t_{n-1},x(t_{n-1})\big)\right\} + \frac{\rho}{M(\rho)}\int_0^{t_n} f\big(t,x(t)\big)dt, \tag{40}$$

where

$$\int_{t_n}^{t_{n+1}} f\big(t,x(t)\big)dt = \int_{t_n}^{t_{n+1}} \left\{ \begin{array}{l} \frac{f(t_n,x_n)}{h}(t-t_{n-1}) \\ -\frac{f(t_{n-1},x_{n-1})}{h}(t-t_n) \end{array} \right\} dt = \frac{3h}{2} f(t_n,x_n) - \frac{h}{2} f(t_{n-1},x_{n-1}). \tag{41}$$

Thus,

$$x(t_{n+1})-x(t_n) = \frac{1-\rho}{M(\rho)}\left\{f\big(t_n,x(t_n)\big) - f\big(t_{n-1},x(t_{n-1})\big)\right\} + \frac{3\rho h}{2M(\rho)} f(t_n,x_n) - \frac{\rho h}{2M(\rho)} f(t_{n-1},x_{n-1}), \tag{42}$$

which implies that

$$x(t_{n+1})-x(t_n) = \left(\frac{1-\rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)}\right)\left\{f\big(t_n,x(t_n)\big) + \left(\frac{1-\rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)}\right) f\big(t_{n-1},x(t_{n-1})\big)\right\} \tag{43}$$

Hence,

$$x(t_{n+1}) = x(t_n) + \left(\frac{1-\rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)}\right)\left\{f\big(t_n,x(t_n)\big) + \left(\frac{1-\rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)}\right) f\big(t_{n-1},x(t_{n-1})\big)\right\} \tag{44}$$

Theorem 5.1. Let f is a continuous function and $x(t)$ be a solution of

$${}_0^{CF}D_t^{\rho}x(t)=f\big(t,x(t)\big)$$

for the Caputo-Fabrizio fractional derivative (Atangana and Owolabi 2018).

$$\begin{aligned}x(t_{n+1})=x(t_n)+\Big(&\frac{1-\rho}{M(\rho)}\\&+\frac{3\rho h}{2M(\rho)}\Big)\Big\{f\big(t_n,x(t_n)\big)\\&+\Big(\frac{1-\rho}{M(\rho)}\\&+\frac{3\rho h}{2M(\rho)}\Big)f\big(t_{n-1},x(t_{n-1})\big)\Big\}\\&+R_\rho^n\end{aligned}\tag{45}$$

where $\|R_\rho^n\|\le M$.

5.1. Numerical Simulations

The extended Plant-Nectar-Pollinator model for the new Caputo fractional differential was brought in the system (8). The next system of equations is acquired for $\Upsilon_i, i=1,2,3$.

$$\begin{aligned}A(t)-A(0)&=\frac{1-\rho}{M(\rho)}\Upsilon_1\big(t,A(t)\big)\\&\quad+\frac{\rho}{M(\rho)}\int_0^t\Upsilon_1\big(\tau,A(\tau)\big)\,d\tau,\\B(t)-B(0)&=\frac{1-\rho}{M(\rho)}\Upsilon_2\big(t,B(t)\big)\\&\quad+\frac{\rho}{M(\rho)}\int_0^t\Upsilon_2\big(\tau,B(\tau)\big)\,d\tau,\\C(t)-C(0)&=\frac{1-\rho}{M(\rho)}\Upsilon_3\big(t,C(t)\big)\\&\quad+\frac{\rho}{M(\rho)}\int_0^t\Upsilon_3\big(\tau,C(\tau)\big)\,d\tau.\end{aligned}\tag{46}$$

Thus,

$$\begin{aligned}A(t_{n+1})-A(0)&=\frac{1-\rho}{M(\rho)}\Upsilon_1\big(t_n,A(t_n)\big)\\&\quad+\frac{\rho}{M(\rho)}\int_0^{t_{n+1}}\Upsilon_1\big(t,A(t)\big)\,dt,\\B(t_{n+1})-B(0)&=\frac{1-\rho}{M(\rho)}\Upsilon_2\big(t_n,B(t_n)\big)\\&\quad+\frac{\rho}{M(\rho)}\int_0^{t_{n+1}}\Upsilon_2\big(t,B(t)\big)\,dt,\\C(t_{n+1})-C(0)&=\frac{1-\rho}{M(\rho)}\Upsilon_3\big(t_n,C(t_n)\big)\\&\quad+\frac{\rho}{M(\rho)}\int_0^{t_{n+1}}\Upsilon_3\big(t,C(t)\big)\,dt,\end{aligned}\tag{47}$$

and

$$\begin{aligned}A(t_n)-A(0)&=\frac{1-\rho}{M(\rho)}\Upsilon_1\big(t_{n-1},A(t_{n-1})\big)\\&\quad+\frac{\rho}{M(\rho)}\int_0^{t_n}\Upsilon_1\big(t,A(t)\big)\,dt,\\B(t_n)-B(0)&=\frac{1-\rho}{M(\rho)}\Upsilon_2\big(t_{n-1},B(t_{n-1})\big)\\&\quad+\frac{\rho}{M(\rho)}\int_0^{t_n}\Upsilon_2\big(t,B(t)\big)\,dt,\\C(t_n)-C(0)&=\frac{1-\rho}{M(\rho)}\Upsilon_3\big(t_{n-1},C(t_{n-1})\big)\\&\quad+\frac{\rho}{M(\rho)}\int_0^{t_n}\Upsilon_3\big(t,C(t)\big)\,dt.\end{aligned}\tag{48}$$

When we removing (48) from (47), the following equation system is obtained.

$$\begin{aligned}A(t_{n+1})-A(0)&=\frac{1-\rho}{M(\rho)}\big\{\Upsilon_1\big(t_n,A(t_n)\big)\\&\quad-\Upsilon_1\big(t_{n-1},A(t_{n-1})\big)\big\}\\&\quad+\frac{\rho}{M(\rho)}\int_{t_n}^{t_{n+1}}\Upsilon_1\big(t,A(t)\big)\,dt,\\B(t_{n+1})-B(0)&=\frac{1-\rho}{M(\rho)}\big\{\Upsilon_2\big(t_n,B(t_n)\big)\\&\quad-\Upsilon_2\big(t_{n-1},B(t_{n-1})\big)\big\}\\&\quad+\frac{\rho}{M(\rho)}\int_{t_n}^{t_{n+1}}\Upsilon_2\big(t,B(t)\big)dt,\\C(t_{n+1})-C(0)&=\frac{1-\rho}{M(\rho)}\big\{\Upsilon_3\big(t_n,C(t_n)\big)\\&\quad-\Upsilon_3\big(t_{n-1},C(t_{n-1})\big)\big\}\\&\quad+\frac{\rho}{M(\rho)}\int_{t_n}^{t_{n+1}}\Upsilon_3\big(t,C(t)\big)dt,\end{aligned}\tag{49}$$

where

$\int_{t_n}^{t_{n+1}}\Upsilon_1(t,A(t))dt=\int_{t_n}^{t_{n+1}}\Big\{\frac{\Upsilon_1(t_n,A_n)}{h}(t-$

$$t_{n-1}) - \frac{\Upsilon_1(t_{n-1},A_{n-1})}{h}(t-t_n)\Big\}dt = \frac{3h}{2}\Upsilon_1(t_n,A_n) - \frac{h}{2}\Upsilon_1(t_{n-1},A_{n-1}),$$

$$\int_{t_n}^{t_{n+1}}\Upsilon_2(t,B(t))dt = \int_{t_n}^{t_{n+1}}\Big\{\frac{\Upsilon_2(t_n,B_n)}{h}(t - t_{n-1}) - \frac{\Upsilon_2(t_{n-1},B_{n-1})}{h}(t-t_n)\Big\}dt = \frac{3h}{2}\Upsilon_2(t_n,B_n) - \frac{h}{2}\Upsilon_2(t_{n-1},B_{n-1}), \tag{50}$$

$$\int_{t_n}^{t_{n+1}}\Upsilon_3(t,C(t))dt = \int_{t_n}^{t_{n+1}}\Big\{\frac{\Upsilon_3(t_n,C_n)}{h}(t - t_{n-1}) - \frac{\Upsilon_3(t_{n-1},C_{n-1})}{h}(t-t_n)\Big\}dt = \frac{3h}{2}\Upsilon_3(t_n,C_n) - \frac{h}{2}\Upsilon_3(t_{n-1},C_{n-1}),$$

Therefore,

$$\begin{aligned} A(t_{n+1}) - A(0) &= \frac{1-\rho}{M(\rho)}\Upsilon_1\big(t_n,A(t_n)\big) + \frac{\rho}{M(\rho)}\int_0^{t_{n+1}}\Upsilon_1\big(t,A(t)\big)\,dt,\\ B(t_{n+1}) - B(0) &= \frac{1-\rho}{M(\rho)}\Upsilon_2\big(t_n,B(t_n)\big) + \frac{\rho}{M(\rho)}\int_0^{t_{n+1}}\Upsilon_2\big(t,B(t)\big)\,dt,\\ C(t_{n+1}) - C(0) &= \frac{1-\rho}{M(\rho)}\Upsilon_3\big(t_n,C(t_n)\big) + \frac{\rho}{M(\rho)}\int_0^{t_{n+1}}\Upsilon_3\big(t,C(t)\big)\,dt, \end{aligned} \tag{51}$$

which implies that,

$$\begin{aligned} A_{n+1} &= A_n + \left(\frac{1-\rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)}\right)\Upsilon_1(t_n,A_n) + \left(\frac{1-\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right)\Upsilon_1(t_{n-1},A_{n-1}),\\ B_{n+1} &= B_n + \left(\frac{1-\rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)}\right)\Upsilon_2(t_n,B_n) + \left(\frac{1-\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right)\Upsilon_2(t_{n-1},B_{n-1}),\\ C_{n+1} &= C_n + \left(\frac{1-\rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)}\right)\Upsilon_3(t_n,C_n) + \left(\frac{1-\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right)\Upsilon_3(t_{n-1},C_{n-1}). \end{aligned} \tag{52}$$

According to theorem (5.1), we get,

$$\begin{aligned} A_{n+1} &= A_n + \left(\frac{1-\rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)}\right)\Upsilon_1(t_n,A_n) + \left(\frac{1-\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right)\Upsilon_1(t_{n-1},A_{n-1}) + {}^{1}R_\rho^n,\\ B_{n+1} &= B_n + \left(\frac{1-\rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)}\right)\Upsilon_2(t_n,B_n) + \left(\frac{1-\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right)\Upsilon_2(t_{n-1},B_{n-1}) + {}^{2}R_\rho^n,\\ C_{n+1} &= C_n + \left(\frac{1-\rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)}\right)\Upsilon_3(t_n,C_n) + \left(\frac{1-\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right)\Upsilon_3(t_{n-1},C_{n-1}) + {}^{3}R_\rho^n. \end{aligned} \tag{53}$$

where

$$\left\| {}^{i}R_\rho^n \right\|_\infty < \frac{\rho}{M(\rho)}(n+1)!\,h^{n+1}, \quad i = 1,2,3. \tag{54}$$

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