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Analysis of a Capacitor Modelled with Conformable Fractional Derivative under DC and Sinusoidal Signals

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Abstract

Fractional order circuit elements are successfully used to model circuits and systems in the last few decades. There are different types of fractional derivatives. Recently, another one named "the conformable fractional derivative" (CFD) has been introduced and shown to give good results for modeling supercapacitors. It is imperative to know how circuit elements behave for different current and voltage waveforms in circuit theory so that they can be exploited at their full potential. A CFD capacitor is not a well-known element, and its usage and circuit solutions are rarely addressed in the literature. In this study, it is examined how a CFD capacitor would behave under DC and AC excitations when it is fed by not only a current source but also a voltage source.

1. Introduction

Fractional derivative (FD) is a branch of mathematical analysis [1, 2]. A differential operator can be of any arbitrary order within it. It has first appeared in the 17th century [3]. In the last decades, Fractional Calculus has emerged as a popular research area because of its applicability in many different fields [4-5]. The self-taught Oliver Heaviside has used fractional calculus to find the solution of the telegrapher's equation around 1890 [3]. The fractional-order circuit elements are used to model circuit elements such as capacitors, inductors and memristors [6-10]. Filters, controllers and oscillators which are based on fractional-order circuit elements are made or can be used to model systems [4-5, 8-9, 11-14]. Another FD is suggested in [15]. It is named as "the conformable fractional derivative" (CFD). Its definition is built on the conventional limit definition of the derivative of a function. It casts off the other FD definitions [15]. This makes it simpler and advantageous than the other FDs. The CFD is elaborated in [16]. However, a CFD is actually not an FD: it can be described as a first-order derivative time a power function of the independent variable [15, 16]. This new definition is a broadening of the ordinary derivative. It is also distinct from the other FDs. The CFD is able to accommodate the common features of

FDs. Well-known calculus theorems such as product rule, Rolle's Theorem, Average Value Theorem, partial integration, Taylor series can be easily extended or applied to the CFD. The conformal fractional derivative has a very important property: while the Riemann-Liouville FD of a constant is not zero, the CFD of a constant is zero. Due to these properties, the conformal fractional derivative has become a hot research area. The conformal derivative has also the advantage of being physically interpretable compared to the other types of fractional derivatives [17]. Usage of the FDs in electric circuits has been examined in [18]. Supercapacitors have been modelled using fractional-order models in [19-22]. The oscillation of impulsive conformable fractional differential equations has been inspected in [23]. Electric circuits modelled with FD circuit elements under sinusoidal excitation have been analyzed with the enhanced fractional derivative method, which is called Caputo FD generalizing the differential equations and ordinary integrals are not necessary to describe the fractional-order initial conditions like Riemann-Liouville FD [24-25]. Several electric circuits characterized by CFDs have been solved in [26]. Other circuits modelled with the CFD have been examined in [27]. The conformable fractional derivative has been used to analyze an electric circuit containing a supercapacitor in [28]. Analytical solutions of electrical circuits described by fractional conformable

derivatives in Liouville-Caputo sense is given in [29]. Electric circuits of the CFDs with and without singular kernels have been solved in [30].

It is important to analyze new circuit elements for different current and voltage waveforms so that they can be exploited at full potential. In this paper, the conformal fractional derivative capacitor model for DC and sinusoidal waveforms have been solved. CFD capacitor has been fed with not only voltage sources but also for current sources. The analytical solutions were given with incomplete gamma function for sinusoidal voltage case. The discussions are provided in the conclusion section.

The rest of the paper is structured as follows. The CFD capacitor model is given in the second section. Its analysis for DC and AC signals are given in the third section. Op-amp-based differentiator and integrator circuits with a CFD capacitor are examined in the fourth section. Finally, the paper is concluded in "Conclusions" section.

2. Conformal Fractional Derivative and CFD Capacitor Constitutional Law

The CFD is described in [15] as the follows:
Definition 1. Let $f : [0, +\infty) \to R$ and $t > 0$. The CFD for $0 < \alpha \leq 1$ is described as

$$D_\alpha f(t) = \lim_{p \to 0} \frac{f(t + p^{(1-\alpha)}) - f(t)}{p} \qquad (2.1)$$

For t>0 and the conformable fractional derivative at 0 is defined as $D_\alpha f(0) = \lim_{t \to 0^+} (D_\alpha f)(t)$. If it is differentiable then $D_\alpha f(t) = t^{1-\alpha} f'(t)$ (2.2)

Definition 2. Let $\alpha \in (0,1]$. The conformable fractional integral of a function $f : [0, +\infty) \to R$ of order a is denoted by $I_\alpha f(t)$ and is defined as

$$I_\alpha f(t) = \int_0^t s^{(\alpha-1)} f(s).ds \qquad (2.3)$$

$$\frac{d^\alpha f(t)}{dt^\alpha} = f'(t)t^{1-\alpha} = \frac{df(t)}{dt}t^{1-\alpha} \qquad (2.4)$$

More information about CFD can be found in [15-17]. If a capacitor can be modelled using CFD [28], its constitutive law can be written as

$$i_C(t) = C_\alpha \frac{d^\alpha v_C(t)}{dt^\alpha} \qquad (2.5)$$

Where $i_C(t)$, $v_C(t)$ and C_α are CFD capacitor current, CFD capacitor voltage and CFD capacitor coefficient, respectively.

3. The CFD Capacitor Fed by a Current Source

In this section, the solutions of a CFD capacitor is found if it is fed by a current source as shown in Figure 1 for the cases: (a) the current source being constant and (b) it being sinusoidal.
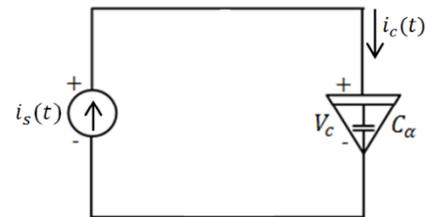
Figure 1: The CFD capacitor fed by a current source

3.1 Constant Current Solution

If the current source has a constant magnitude or the CFD capacitor current is constant;

$$i_s(t) = i_c(t) = I_{dc} = C_\alpha \frac{dv_C(t)}{dt} t^{1-\alpha} \qquad (3.1)$$

By arranging both sides,

$$\frac{dv_C(t)}{dt} = \frac{I_{dc}}{C_\alpha t^{1-\alpha}} \to dv_C(t) = \frac{I_{dc}}{C_\alpha t^{1-\alpha}} dt \qquad (3.2)$$

Its voltage can be found as

$$v_C(t) = \int \frac{I_{dc}}{C_\alpha t^{1-\alpha}} dt = \frac{I_{dc}}{C_\alpha} \int t^{\alpha-1} dt \qquad (3.3)$$

$$v_C(t) = \frac{I_{dc} t^\alpha}{C_\alpha \alpha} + K \qquad (3.4)$$

Where K is the integration constant and the capacitor voltage at $t = 0$ is used to find K;

$$v_C(0) = \frac{I_{dc} 0^\alpha}{C_\alpha \alpha} + K \to K = v_C(0) \qquad (3.5)$$

Using the integration constant, the CFD capacitor current can be obtained as;

$$v_C(t) = \frac{I_{dc} t^\alpha}{C_\alpha \alpha} + v_C(0) \qquad (3.6)$$

3.2 Sinusoidal Current Solution

If a sinusoidal current of $i_s(t) = I_m \cos(\omega t + \varphi)$ is applied to the CFD capacitor, its voltage can be calculated with the following steps:

$$i_s(t) = i_c(t) = I_m \cos(\omega t + \varphi) = C_\alpha \frac{dv_C(t)}{dt} t^{1-\alpha} \quad \textbf{(3.7)}$$

By arranging both sides of the equation, the CFD capacitor voltage is written as

$$\frac{dv_C(t)}{dt} = \frac{I_m \cos(\omega t + \varphi)}{C_\alpha t^{1-\alpha}} \quad \textbf{(3.8)}$$

$$v_c(t) = \frac{I_m}{C_\alpha} \int \cos(\omega t + \varphi) t^{(\alpha-1)} dt \quad \textbf{(3.9)}$$

Using Wolfram Alpha [31], the solution of the integral is found as

$$v_c(t) = \frac{I_m}{C_\alpha} \left(\begin{array}{l} -\frac{t^\alpha (\omega^2 t^2)^{-\alpha}}{2}((-i\omega t)^\alpha (\cos(\varphi) - i\sin(\varphi))\Gamma(a, i\omega t) + \\ (i\omega t)^\alpha (\cos(\varphi) + i\sin(\varphi))\Gamma(a, -i\omega t)) + K \end{array} \right) \textbf{(3.10)}$$

Using the following identities:
$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$$
$$e^{-i\varphi} = \cos(\varphi) - i\sin(\varphi)$$
$$\quad \textbf{(3.11)}$$

The capacitor voltage turns into;

$$v_c(t) = \frac{I_m}{C_\alpha} \left(\begin{array}{l} -\frac{t^\alpha (\omega^2 t^2)^{-\alpha}}{2}((-i\omega t)^\alpha e^{-i\varphi})\Gamma(a, i\omega t) + \\ (i\omega t)^\alpha e^{i\varphi})\Gamma(a, -i\omega t)) + K \end{array} \right) \textbf{(3.12)}$$

In mathematics, the upper incomplete gamma function is a transcendental function that appears as solutions to diverse problems such as definite integrals.

When splitting the incomplete gamma function at s point $x \geq 0$, two types of the incomplete gamma functions called upper and lower are obtained.
The definition of the upper incomplete gamma function intervals are explained from x to ∞.

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt \quad \textbf{(3.13)}$$

If the initial condition is used at t=0,

$$v_c(0) = \frac{I_m}{C_\alpha}[0 + K] \rightarrow K = v_c(0)\frac{C_\alpha}{I_m} \quad \textbf{(3.14)}$$

Using the integration constant, the CFD capacitor voltage can be obtained as

$$v_c(t) = -\frac{I_m t^\alpha (\omega^2 t^2)^{-\alpha}}{2C_\alpha}((-i\omega t)^\alpha e^{-i\varphi})\Gamma(a, i\omega t) + \quad \textbf{(3.15)}$$
$$((i\omega t)^\alpha e^{i\varphi})\Gamma(a, -i\omega t)) + v_c(0)$$

Numerical methods can be used to calculate the CFD capacitor voltage since the solution requires the upper incomplete gamma function which is also evaluated numerically.

4. The CFD Capacitor Fed by a Voltage Source

In this section, the solution of the CFD capacitor fed by a voltage source shown in Figure 2 is found for both of the cases: the voltage source being a constant (a) and a sinusoidal (b).

Figure 2. The CFD capacitor fed by a voltage source

4.1. Constant Voltage Solution

If a step function voltage is applied to the CFD capacitor at time being t_0:

$$v_C(t) = V_{dc} u(t - t_0) \quad \textbf{(4.1)}$$

The CFD capacitor current becomes

$$i_c(t) = C_\alpha \frac{dv_C(t)}{dt} t^{1-\alpha} = C_\alpha \frac{d(V_{dc} u(t - t_0))}{dt} t^{1-\alpha} \quad \textbf{(4.2)}$$

$$i_c(t) = C_\alpha V_{dc} \delta(t - t_0) t^{1-\alpha} = C_\alpha V_{dc} \delta(t - t_0) t_0^{1-\alpha} \quad \textbf{(4.3)}$$

Where $\delta(t - t_0)$ is the Dirac-delta function [32] shifted to time being t_0.
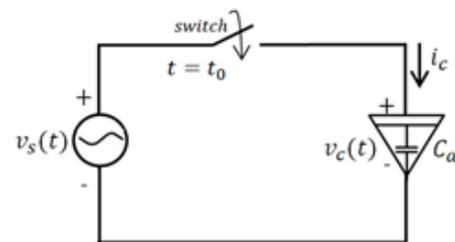
Figure 3. The CFD capacitor fed by a voltage source and depends on the time.

A CFD capacitor without a series resistor withdraws a Dirac pulse as a linear time-invariant resistor does. However, its current magnitude depends on the time the voltage is applied opposite to the case in which an LTI (Linear Time-Invariant) capacitor current pulse

magnitude is constant not depending on the time the step function is applied as shown in Figure 3.
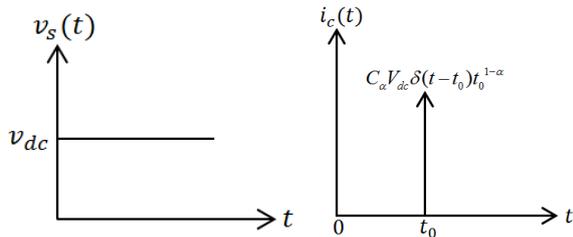
Figure 4. The CFD capacitor voltage and current vs. time when it is fed by a constant voltage

4.2.   Sinusoidal Voltage Solution

If a sinusoidal voltage of $v_C(t) = V_m \sin(\omega t + \varphi) u(t - t_0)$ is applied to the CFD capacitor at time being t_0, its current is found as

$$i_c(t) = C_\alpha \frac{dv_C(t)}{dt} t^{1-\alpha} = C_\alpha \frac{d\left(V_m \sin(\omega t) u(t - t_0)\right)}{dt} t^{1-\alpha} \quad \textbf{(4.4)}$$

$$i_c(t) = C_\alpha V_m t^{1-\alpha} (\omega \cos(\omega t) + \sin(\omega t) \delta(t - t_0)) \quad \textbf{(4.5)}$$

When the Dirac-delta function rule $f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)$ is used, the CFD capacitor current can be written as:

$$i_c(t) = C_\alpha V_m \omega \cos(\omega t) t^{1-\alpha} + C_\alpha V_m \sin(\omega t_0) t_0^{1-\alpha} \delta(t - t_0) \quad \textbf{(4.6)}$$

The circuit waveforms are drawn with Matlab™. The CFD capacitor current and voltage for this case is shown in Figure 5 and 6. The CFD capacitor current magnitude is not constant and varies with time due to the term $t^{1-\alpha}$ as shown in Figure 4. When $V_m = 5V$, $\omega = 1$ rad/s, $\varphi = 0$ rad/s and $C_\alpha = 1 \ F/s^{1-\alpha}$ values are used, the CFD capacitor current graph in Figure 6 is sketched for three different alpha values.
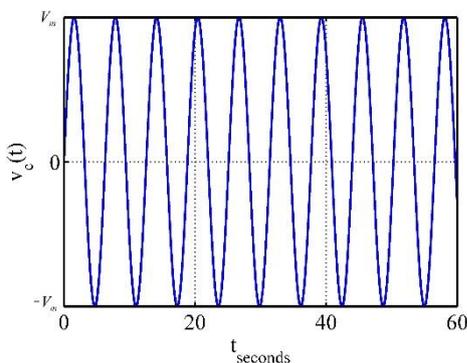
Figure 5. The CFD capacitor voltage vs. time when it is fed by a sinusoidal voltage
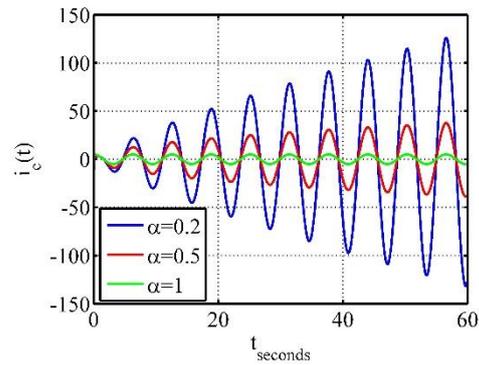
Figure 6. The CFD current vs. time when it is fed by a sinusoidal voltage

All real capacitor models have series and parallel resistors used in their equivalent circuit to model their losses. In this study, the capacitors are assumed to be ideal, ie. without losses and that's why no resistor is used in the analysis. The CFD capacitor current goes to infinity due to the term $t^{1-\alpha}$ as seen in Figure 6 because of this. Such a situation would not appear in a more realistic capacitor model when a series resistor is used.

5. Conclusions

Supercapacitors or ultra-capacitors are becoming more common each day. They cannot be modelled with capacitor constitutive law. Therefore, easier and robust models are needed for their usage and analysis in the circuits. Only then, such capacitors can be fully exploited.

The conformal fractional derivative provides an easier solution than the fractional derivatives such as Caputo and Riemann Liouville. Some capacitors can be modelled using fractional-order derivatives. In this study, the conformal fractional derivative is used to model a capacitor. Every circuit element should be examined for the basic waveforms such as DC and sinusoidal signals. The solutions of the CFD capacitor for the waveforms are given. Only in the case when it is fed by a current source, its voltage is found using a transcendental function the incomplete gamma functions. The solutions can be used in the analysis of the circuits with a CFD capacitor if a supercapacitor is found to be modelled with a good accuracy using experimental data. In comparison to the prior studies, the CFD derivative is more practical and analytical solutions are found to be possible while it has been not possible for other fractional derivatives. The analysis given here can also be used in modelling renewable energy systems or inverters with supercapacitors in the future. This paper can also be used as a tutorial for the researchers who have just been introduced to the research area.

Author's Contributions

Utku Palaz: Made literature search, prepared computer codes for numerical solutions of the sample problems, made simulations, and wrote the manuscript.

Reşat Mutlu: Made literature search, assisted in analytical analysis on the structure, supervised the experiment's progress, result in interpretation, and helped in manuscript preparation.

Ethics

There are no ethical issues after the publication of this manuscript.

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