MANAS Journal of Engineering
ISSN 1694-7398 | e-ISSN 1694-7398

Volume 8 (Issue 1) (2020) Pages 68-76

Stability of the third order rational difference equation

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ABSTRACT

In this paper, we examine the global stability and boundedness of the difference equation

$$x_{n+1}=\frac{\alpha x_n x_{n-1}+\beta x_n x_{n-2}}{\gamma x_{n-1}+\theta x_{n-2}}$$

where the initial conditions x_{-2}, x_{-1}, x_0 are non zero real numbers and α, β, γ and θ are positive constants such that

$$\alpha+\beta \leq \gamma+\theta.$$

Also, we discuss and illustrate the stability of the solutions of the considered equation via MATLAB at the end of study to support our results.

ARTICLE INFO

Research article

Received: 02.04.2020
Accepted: 22.06.2020

Keywords:
Asymptotic stability, difference equation, global behavior

AMS: 39A10, 39A21, 39A30

1. Introduction

Some mathematical models we construct to better understand and analyze daily life includes amounts that are bound to known values at equal intervals at certain times. These amounts are expressed through sequences as a function of time. These sequences can be expressed as difference equations for the assumptions in the models. Difference equations are used in some modeling such as population growth models, Finance and economic models, business models, consumer behavior models etc. As this topic is interesting for researchers, it will continue to increase researches by using difference equations in the following years. The problems in these studies will be aimed at finding the stability of nonlinear difference equations. Finding solutions and analyzing stability in these studies is a challenging task. Until now, many studies have been done on the stability of nonlinear difference equations. For example:

Yang et al. [1] investigated the global asymptotic stability of the difference equation

$$x_{n+1}=\frac{x_{n-1}x_{n-2}+x_{n-3}+a}{x_{n-1}+x_{n-2}x_{n-3}+a}.$$

Kulenovic, Ladas and Sizer et al. [2] studied the behavior of rational recursive sequence

$$x_{n+1}=\frac{\alpha x_n+\beta x_{n-1}}{\gamma x_n+\delta x_{n-1}}.$$

Elabbasy and colleagues et al. [3] investigated and study some special cases of the difference equation

$$x_{n+1}=\frac{a x_{n-l}x_{n-k}}{b x_{n-p}-c x_{n-q}}.$$

Abdul Khaliq and Elsayed et al. [4] studied behavior and obtained some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_n x_{n-l}}{\beta x_{n-m} + \gamma x_{n-l}}.$$

See also $[5-14]$. Our aim is examine the global behavior of the following third-order rational difference equation that will serve as the basis for such modeling

$$x_{n+1} = \frac{\alpha x_n x_{n-1} + \beta x_n x_{n-2}}{\gamma x_{n-1} + \theta x_{n-2}} \tag{0.1}$$

where the initial conditions x_{-2}, x_{-1}, x_0 are non zero real numbers and $\alpha, \beta, \gamma, \theta$ are positive constants such that

$$\alpha + \beta \le \gamma + \theta.$$

A computational examples given at the end of study and simulated solutions of some problems via MATLAB. We hope that the results of this study contribute to the development of the theory on the global stability of nonlinear rational differential equations.

Let us give some definitions and theorems that we need.

Definition 1.1. [15] A difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0,1,... \tag{0.2}$$

where F is a function that maps some set I^{k+1} into I. The set I is usually an interval of real numbers, or a union of intervals, or a discrete set such as the set of integers $\mathbb{Z} = \dots, -1, 0, 1, \dots$.

A solution of (1.2) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies (1.2) for all $n \ge 0$.

A solution of (1.2) that is constant for all $n \ge -k$ is called an equilibium solution of (1.2). If

$$x_n = \overline{x}, \quad \text{for all} \quad n \ge -k$$

is an equilibrium solution of (1.2), then $\overline{x}$ is called an **equilibrium point**, or simply an **equilibrium** of (1.2).

Definition 1.2. [15] Let $\overline{x}$ be an equilibrium point of (1.2).

i. An equilibrium point $\overline{x}$ of (1.2) is called **locally stable** if, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of (1.2) with

$$|x_{-k} - \overline{x}| + |x_{1-k} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta,$$

then

$$|x_n - \overline{x}| < \epsilon \text{ for all } n \ge 0.$$

ii. An equilibrium point $\overline{x}$ of (1.2) is called **locally asymptotically stable** if, it is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of (1.2) with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma$$

then

$$\lim_{n\to\infty} x_n = \bar{x}.$$

iii. An equilibrium point $\bar{x}$ of (1.2) is called **global attractor** if, for every solution $\{x_n\}_{n=-k}^{\infty}$ of (1.2), we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

iv. An equilibrium point $\bar{x}$ of (1.2) is called **global asymptotically stable** if $\bar{x}$ is locally stable and $\bar{x}$ is also global attractor of (1.2).

v. An equilibrium $\bar{x}$ of (1.2) is called **unstable** if $\bar{x}$ is not locally stable.

Definition 1.3. [15] Suppose that the function F is continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Let

$$q_i = \frac{\partial F}{\partial u_i}(\bar{x}, \bar{x}, \ldots, \bar{x}), \text{ for } i = 0,1,\ldots,k$$

denote the partial derivative of $F(u_0, u_1, \ldots, u_k)$ with respect to u_i evaluated at the equilibrium point $\bar{x}$ of (1.2). Then the equation

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + \cdots + q_k y_{n-k}, n = 0,1,... \tag{0.3}$$

is called the linearized equation of (1.2) about the equilibrium point $\bar{x}$, and the equation

$$\lambda^{k+1} - q_0 \lambda^k - \cdots - q_{k-1}\lambda - q_k = 0 \tag{0.4}$$

is called the characteristic equation of (1.2) about $\bar{x}$.

The following theorem state necessary and sufficient conditions to determine the local asymptotic stability of the equilibrium points of the (1.2).

Theorem 1.1. [15] Assume that a_3, a_2, a_1 and a_0 are real numbers. Then a necessary and sufficient condition for all roots of the equation

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

to lie inside the unit disk is

$$|a_2 + a_0| < 1 + a_1, \quad |a_2 - 3a_0| < 3 - a_1, \quad \text{and} \quad a_0^2 + a_1 - a_0 a_2 < 1.$$

Theorem 1.2. [15] Let $[p,q]$ be an interval of real numbers and assume that $f:[p,q]^3 \to [p,q]$ is a continuous function satisfying the following properties:

a) $f(x,y,z)$ is non-decreasing in $y,z \in [p,q]$ for each $x \in [p,q]$, and non-increasing in $x \in [p,q]$ for each $y,z \in [p,q]$;

b) If $(m,M) \in [p,q] \times [p,q]$ is a solution of the system

$$M = f(m,M,M) \text{ and } m = f(M,m,m)$$

then $m = M$.

Then (1.2) has a unique equilibrium $\bar{x} \in [p, q]$ and every solution of (1.2) converges to $\bar{x}$.

2. Dynamics of (1.1)

In this section, we investigate the dynamics of (1.1) under the assumptions that all parameters in the equation are positive and the initial conditions are non-negative.

2.1. Local Stability of (1.1)

(1.1) has a unique equilibrium point and is given by the equation

$$\bar{x} = \frac{\alpha \bar{x}^2 + \beta \bar{x}^2}{\gamma \bar{x} + \theta \bar{x}},$$

so,

$$\bar{x}^2 (\gamma + \theta) = \bar{x}^2 (\alpha + \beta).$$

If $\alpha + \beta < \gamma + \theta$, $\bar{x}_1 = 0$ is the equilibrium point of (1.1),
if $\alpha + \beta = \gamma + \theta$ then $\bar{x}_2 \in \mathbb{R}$ is the equilibrium point of (1.1).

Let $f : (0, \infty)^3 \to (0, \infty)$ be a function defined by

$$f(u, v, t) = \frac{\alpha u v + \beta u t}{\gamma v + \theta t}. \tag{1.1}$$

So,

$$\frac{\partial f}{\partial u}(\bar{x}, \bar{x}, \bar{x}) = \frac{\alpha + \beta}{\gamma + \theta},$$

$$\frac{\partial f}{\partial v}(\bar{x}, \bar{x}, \bar{x}) = \frac{\alpha\theta - \beta\gamma}{(\gamma + \theta)^2},$$

$$\frac{\partial f}{\partial t}(\bar{x}, \bar{x}, \bar{x}) = \frac{\beta\gamma - \alpha\theta}{(\gamma + \theta)^2}.$$

The linearized equation of (1.1) is

$$y_{n+1} - \frac{\alpha + \beta}{\gamma + \theta} y_n - \frac{\alpha\theta - \beta\gamma}{(\gamma + \theta)^2} y_{n-1} - \frac{\beta\gamma - \alpha\theta}{(\gamma + \theta)^2} y_{n-2} = 0, \tag{1.2}$$

and the characteristic equation of (1.1) is

$$\lambda^3 - \frac{\alpha + \beta}{\gamma + \theta} \lambda^2 - \frac{\alpha\theta - \beta\gamma}{(\gamma + \theta)^2} \lambda - \frac{\beta\gamma - \alpha\theta}{(\gamma + \theta)^2} = 0. \tag{1.3}$$

Theorem 2.1. The equilibrium points $\bar{x}_1 = 0$ for $\alpha + \beta < \gamma + \theta$ and $\bar{x}_2 \in \mathbb{R}$ for $\alpha + \beta = \gamma + \theta$ of (1.1) are local asymptotically stable.

Proof. From Theorem 1.1 and (2.3),

$$a_2 = -\frac{\alpha + \beta}{\gamma + \theta}, \quad a_1 = \frac{\beta\gamma - \alpha\theta}{(\gamma + \theta)^2}, \quad a_0 = \frac{\alpha\theta - \beta\gamma}{(\gamma + \theta)^2}.$$

So

$$\begin{aligned}|a_2+a_0|-a_1 &= \left|\frac{\alpha\theta-\beta\gamma}{(\gamma+\theta)^2}-\frac{\alpha+\beta}{\gamma+\theta}\right|-\frac{\beta\gamma-\alpha\theta}{(\gamma+\theta)^2}\\ &= \frac{\alpha\gamma+\alpha\theta+\beta\theta+\beta\gamma}{(\gamma+\theta)^2}\\ &= \frac{(\alpha+\beta)(\gamma+\theta)}{(\gamma+\theta)^2}\\ &\le 1\end{aligned}$$

since $\alpha+\beta\le\gamma+\theta \;\Rightarrow\; \frac{\alpha+\beta}{\gamma+\theta}\le 1$.

$$\begin{aligned}|a_2-3a_0|+a_1 &= \left|-\frac{\alpha+\beta}{\gamma+\theta}-3\frac{\alpha\theta-\beta\gamma}{(\gamma+\theta)^2}\right|+\frac{\beta\gamma-\alpha\theta}{(\gamma+\theta)^2}\\ &= \frac{\alpha\gamma+3\alpha\theta+\beta\theta-\beta\gamma}{(\gamma+\theta)^2}\\ &= \frac{\alpha(\gamma+\theta)+\beta(\gamma+\theta)+2\alpha\theta+2\beta\gamma}{(\gamma+\theta)^2}\\ &= \frac{\alpha+\beta}{\gamma+\theta}+2\frac{\alpha\theta+\beta\gamma+\alpha\gamma-\alpha\gamma+\beta\theta-\beta\theta}{(\gamma+\theta)^2}\\ &= 3\frac{\alpha+\beta}{\gamma+\theta}-2\frac{\alpha\gamma+\beta\theta}{(\gamma+\theta)^2}\\ &< 3\end{aligned}$$

and

$$\begin{aligned}a_0^2+a_1-a_0a_2 &= \left(\frac{\alpha\theta-\beta\gamma}{(\gamma+\theta)^2}\right)^2+\frac{\beta\gamma-\alpha\theta}{(\gamma+\theta)^2}+\frac{\alpha\theta-\beta\gamma}{(\gamma+\theta)^2}\cdot\frac{\alpha+\beta}{\gamma+\theta}\\ &= \frac{\beta\gamma-\alpha\theta}{(\gamma+\theta)^2}\cdot\left(\frac{\beta\gamma-\alpha\theta}{(\gamma+\theta)^2}-\frac{\alpha+\beta}{\gamma+\theta}+1\right)\\ &= \frac{\beta\gamma-\alpha\theta}{(\gamma+\theta)^2}\cdot\left(1-\frac{\alpha\gamma+2\alpha\theta+\beta\theta}{(\gamma+\theta)^2}\right)\\ &< 1\end{aligned}$$

from

$$\begin{aligned}\alpha+\beta\le\gamma+\theta \;&\Rightarrow\; \alpha\gamma+\alpha\theta+\beta\gamma+\beta\theta\le(\gamma+\theta)^2\\ &\Rightarrow\; \beta\gamma-\alpha\theta<\alpha\gamma+\alpha\theta+\beta\gamma+\beta\theta\le(\gamma+\theta)^2\\ &\Rightarrow\; \frac{\beta\gamma-\alpha\theta}{(\gamma+\theta)^2}<1.\end{aligned}$$

By Theorem 1.1, $\overline{x}_1$ *and* $\overline{x}_2$ *are local asymptotically stable of* (1.1) *and the proof is complete.*

2.2. Global Attractor of $\bar{x}$ of (1.1)

Theorem 2.2. The equilibrium points of (1.1) is global attractor.

Proof. Let $[p,q]$ be real numbers and assume that $f:[p,q]^3 \to [p,q]$ is a function defined by $f(u,v,t)=\frac{\alpha uv+\beta ut}{\gamma v+\theta t}$. Then we can easily see that the function is increasing in u and decreasing in v,t.

Suppose that (m,M) *is a solution of the system*

$$M = f(m,M,M) \text{ and } m = f(M,m,m).$$

Since from (1.1)

$$m=\frac{\alpha Mm+\beta Mm}{\gamma m+\theta m} \text{ and } M=\frac{\alpha mM+\beta mM}{\gamma M+\theta M}$$

we have

$$(\gamma+\theta)m^2 = Mm(\alpha+\beta) \text{ and } (\gamma+\theta)M^2 = Mm(\alpha+\beta)$$

then

$$(\gamma+\theta)(M^2-m^2)=0.$$

Thus

$$M=m.$$

By Theorem 1.2, $\bar{x}_1$ *and* $\bar{x}_2$ *are global attractor of* (1.1) *and the proof is complete.*

2.3. Boundedness of Solutions of (1.1)

Theorem 2.3. Every solution of (1.1) is bounded.

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of (1.1). Let $M = max\{x_{n-1}, x_{n-2}\}$. From (1.1)

$$x_{n+1}=\frac{\alpha x_n x_{n-1}+\beta x_n x_{n-2}}{\gamma x_{n-1}+\theta x_{n-2}} \le \frac{\alpha x_n M+\beta x_n M}{\gamma M+\theta M} < \frac{(\gamma+\theta)x_n M}{\gamma M+\theta M},$$

which implies that $x_{n+1} < x_n$ *for* $n \ge 0$. *Then*

$$\lim_{n\to\infty} x_n = \bar{x}.$$

Then, the proof is complete.

3. Computational examples

In this section, I perform computational examples to illustrate the validity of main results. In order to better express the numerical samples, a graph of the solutions was obtained by using matlab. These graphs are drawn with different parameters and different starting conditions.

- In Fig.1, The equilibrium $\bar{x}_1 = 0$ of (1.1) is shown to be global asymptotically stable under the initial conditions $x_{-2} = 3.456, x_{-1} = 7.879, x_0 = -6.841$ and the parameters $\alpha = 3, \beta = 4, \gamma = 5, \theta = 6$ that meet the condition $\alpha+\beta < \gamma+\theta$.

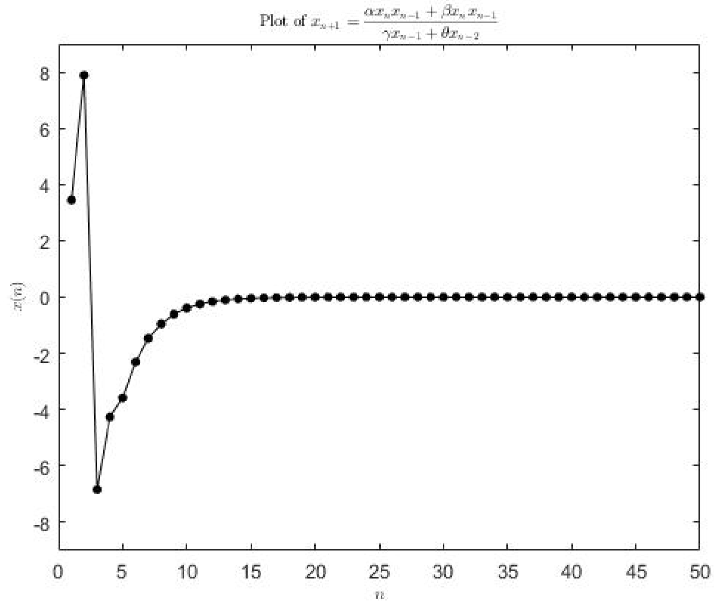

Figure 1. *Stability of the solutions of (1.1) under the condition* $\alpha+\beta<\gamma+\theta$.

- In Fig.2, The equilibrium $\overline{x}_2 = 15 \in \mathbb{R}$ of (1.1) is shown to be global asymptotically stable under the initial conditions $x_{-2} = -4.159, x_{-1} = 4.751, x_0 = -8.874$ and the parameters $\alpha = 3, \beta = 5, \gamma = 6, \theta = 2$ that meet the conditions $\alpha+\beta=\gamma+\theta$.

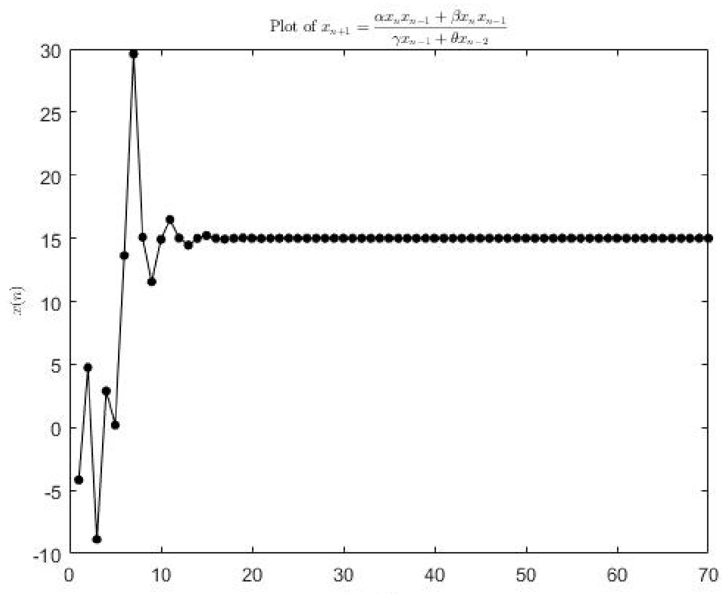

Figure 2. *Behavior of (1.1) under the condition* $\alpha+\beta=\gamma+\theta$.

- In Fig.3, (1.1) is shown to be not global asymptotically stable under the initial conditions $x_{-2} = -2.074, x_{-1} = 7.358, x_0 = -3.189$ and the parameters $\alpha = 3, \beta = 5, \gamma = 4, \theta = 3$ that meet the condition $\alpha+\beta>\gamma+\theta$.

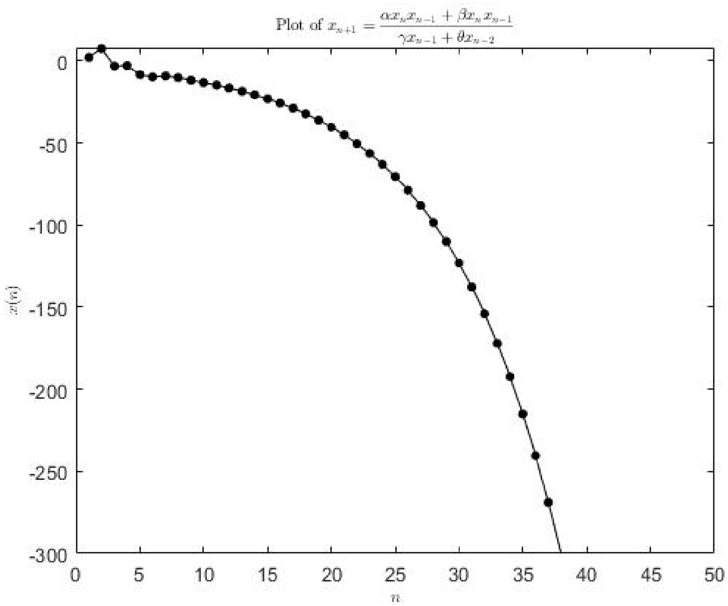

Figure 3. *Unboundness solutions of (1.1) under the condition* $\alpha + \beta > \gamma + \theta$.

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