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Research Article

A Numerical Discussion for the European Put Option Model

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Abstract
The Black-Scholes equations have been increasingly popular over the last three decades since they provide more practical information for optional behaviours. Therefore, effective methods have been needed to analyse these models. This study will mainly focus on investigating the behaviour of the Black-Scholes equation for the European put option pricing model. To achieve this, numerical solutions of the Black-Scholes European option pricing model are produced by three combined methods. Spatial discretization of the Black-Scholes model is performed using a fourth-order finite difference (FD4) scheme that allows a highly accurate approximation of the solutions. For the time discretization, three numerical techniques are proposed: a strong-stability preserving Runge Kutta (SSPRK3), a fourth-order Runge Kutta (RK4) and a one-step method. The results produced by the combined methods have been compared with available literature and the exact solution. It has seen that the results with minimal computational effort are sufficiently accurate.

AMS (MOS) subject classifications. 91B99, 35Q91, 65M06

Keywords: Black-Scholes equation, Option pricing modelling, High-order finite difference, Temporal discretization

Avrupa Tipi Satış Opsiyonu Modeli için Nümerik bir Değerlendirme

Öz
Black-Scholes denklemleri opsiyon davranışlarında pratik bilgiler sağladığından son otuz yılda daha popüler hale gelmiştir. Bu nedenle, bu modelleri analiz etmek için etkili yöntemlere ihtiyaç duyulmaktadır. Bu çalışma temel olarak Avrupa tipi satış opsiyonu fiyatlama modeli için Black-Scholes denkleminin davranışını araştırmaya odaklanmıştır. Bunun için, Black-Scholes Avrupa tipi opsiyon fiyatlama modelinin sayısal çözümleri üç birleştirilmiş yöntem ile üretilmiştir. Black-Scholes modelinin uzaysal ayrıklaştırması, çözümlerin yüksek hassasiyetli yaklaşımlarına izin veren dördüncü mertebeden bir sonlu fark (FD4) şeması kullanılarak yapılmıştır. Zaman ayrıklaştırması için üç sayısal teknik kullanılmıştır: Kuvvetli kararlılık koruyan Runge-Kutta (SSPRK3), dördüncü mertebe Runge Kutta (RK4) ve tek adımlı bir yöntem. Birleştirilmiş yöntemlerle üretilen sonuçlar literatürde mevcut olan çözüm ve tam çözüm ile karşılaştırılmıştır. Sonuçların minimum hesaplama çabasıyla yeterince hassas olduğu görülmüştür.

Anahtar Kelimeler: Black-Scholes Denklemi, Opsiyon Fiyatlama Modellemesi, Yüksek Mertebe Sonlu Fark, Zaman Ayrıklaştırması

1. Introduction

Financial markets in the business world are growing rapidly and depend on many parameters. As part of the financial securities, options are used to assure assets for covering the risk in the stock price changes. There are many types of options but the most often used ones are the European options and American options. While the European options can only be exercised on the expiry date, the American options can be exercised on or before the expiry date.

Option pricing theory has made fast developments with the option pricing model proposed by Black and Scholes (1973) and

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previously by Merton (1973). This model has expressed the value of an option is equal to the value of a self-financing replicating portfolio comprising risk-less security and a risky stock. The model behaviour is represented by a partial differential equation of the form

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} + rSV_s - rV = 0, \quad (S,t) \in (0,\infty) \times [0,T) \quad (1.1)$$

with the boundary and terminal conditions depending on the type of option, where V is the price of the option with respect to the stock price S and time t. r and σ are referred to the interest rate and volatility, respectively.

Since the Black-Scholes model is very effective for pricing options, in the last few decades, various numerical techniques have been proposed to understanding behaviour of the valuation of an option. Under the influence of the Black-Scholes, to get rid of difficulties with payouts and potential bankruptcies, Cox and Ross (1976) published an article on the valuation of options based upon different jump and diffusion processes. Schwartz (1977) developed a numerical procedure for valuing options on the dividend-paying stock using a finite difference method. After that Courtadon (1982) used the same approach that is more accurate than the Schwartz approximation for the valuation of the option. Heston (1993) proposed a closed-form solution for the European call option with stochastic volatility. Wilmott et al. (1995) investigated the numerical solutions of the Black-Scholes equation extensively.

Although the Black-Scholes model is very effective for pricing options in a complete market without costs on transactions of risky and riskless securities, in the presence of transaction costs on trading in the riskless security or stock, it is no longer valid. To overcome this drawback, different models with transaction costs were proposed by Leland (1985), Boyle and Vorst (1992), Kusuoka (1995) and Barles and Soner (1998). After these studies, many authors took into account the models with transaction costs in their studies (Ankudinova and Ehrhardt, 2008; Company et al., 2008; Company et al., 2009; Lesmana and Wang, 2013; Mashayekhi and Fugger, 2015; Koleva et el., 2016).

Recently, many researchers have paid more attention to the methods using finite difference schemes to obtain more accurate solutions of the option pricing models. For instance, Tavella and Randall (2000) dealt with the finite difference solutions of the pricing equations of different types and investigated the stability analysis. Düring et al. (2003) extended the compact finite difference scheme of Rigal (1994) for a nonlinear Black-Scholes equation with transaction cost. Duffy (1976) investigated option pricing problems represented by a partial differential approach. Besides, to obtain high order accuracy in the solution of different option pricing models, some authors studied compact difference schemes (Zhao et al., 2007; Liao and Khaliq, 2009; Tangman et al., 2008; Düring et al., 2014; Jeong et al., 2018, Raol and Goura, 2020). Ankudinova and Ehrhardt (2008) focused on the numerical solution of several models defined by the nonlinear Black-Scholes equations for the European and American options with nonlinear volatilities. Company et al. (2008, 2009) put the Black-Scholes equations on their agenda by publishing two articles. They applied various difference scheme to the Barles and Soner model. Lesmana and Wang (2013) developed a numerical method based on an upwind finite difference scheme for the spatial discretization for a nonlinear European option pricing problem. Jeong et al. (2018) proposed a finite difference scheme for the solution of the Black-Scholes equation without boundary conditions. Rao et al. (Rao et al., 2018) considered numerical solutions of Black-Scholes equation governing four different option styles of European type with variable parameters by using high order difference approximation. Gulen et al. (2019) proposed a new approach based on a sixth-order finite difference scheme for the European put option problem with minimal computational effort. Besides, recently, some authors focused on construct to solutions of multidimensional Black-Scholes equations by using difference schemes (Heo et al., 2019, Kim et al., 2020, Yan et al., 2020).

As mentioned earlier, the option pricing problems are becoming increasingly important in the financial markets and academic community. However, the complexity and stochastic properties of these problems make it difficult to carry out the numerical solution of the problem. To overcome this difficulty, efficient approximation methods are needed to understand the behaviour of the option pricing problems. Although the above studies were successfully used to analyze option pricing problems, most of them lead to lower accuracy solutions. Besides, even the European put option pricing problem which we have considered has exact closed-form solutions, a more complex model does not have any closed-form. From all reasons above, to solve these problems computationally accurate approximate techniques are required. Therefore, in this paper, we aim to propose high-order efficient schemes whose order is superior to the literature to understand the behaviour of the European put option pricing model. For this purpose, we have examined the use of popular time discretization schemes, a third-order strong stability preserving Runge-Kutta (SSPRK3) scheme, a fourth-order Runge-Kutta (RK4) scheme and a one-step method are combined with a fourth-order finite difference scheme for the European put option problem. This choice provides a direct and accurate estimation of approximation error. While high-order approximations reduce the computational effort, the time integration techniques are important in accuracy and stability. Therefore, the chosen techniques could solve the model with high accuracy and minimal computational effort. Our computations have shown that the results from the current methods approximate the exact and the available solution in the literature very well. Furthermore, based on the literature review, this method has not been implemented for the problem represented by the Euroepan put option pricing and we believed that the proposed scheme can be adapted to analyze the other financial and real-world problems, especially nonlinear.

2. Pricing European Put Option under Black-Scholes Model

This paper considers a general case in which no transaction costs and restraint on transactions and adopts the PDE given in Eq.(1.1). For solving Eq.(1.1) uniquely, one final condition and two boundary conditions are required. These conditions are given according to the option types (put or call). The final and boundary conditions of the European put option problem is given as follows:

$$V(S,T) = \max\{K - S, 0\}, \quad 0 \le S < \infty \quad (2.1)$$

$$V(S,T) \sim 0, \quad S \longrightarrow \infty \quad (2.2)$$

$$V(0,T) = Ke^{r(T-t)}, \quad 0 \le t \le T \quad (2.3)$$

The exact solution of the European put option problem

$$V(S,t) = Ke^{-r(T-t)}N(-\gamma_2) - Se^{-\delta(T-t)}N(-\gamma_1) \quad (2.4)$$

with parameters

$$\gamma_1 = \frac{lnS - lnK + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$\gamma_2 = d_1 - \sigma\sqrt{T-t}$$

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}x^2} dx$$

where N, δ and K denote the standard normal cumulative probability distribution function, continuous dividend yield and exercise price, respectively (Leentvar, 2003).

3. Solution Methods

This section is dedicated to the numerical solutions of the European put option problem. In spatial discretization, a fourth-order finite difference method (FD4) is applied while the SSPRK3, the RK4 and the one-step time method are considered in temporal discretization.

3.1. Spatial Discretization

The spatial domain $[0, \infty)$ in which Eq.(1.1) is reconsidered with $[0, S_{max})$ where S_{max} is an artificial limit will be chosen large enough, approximately, which is larger than three or four times the exercise price. Then, the spatial domain $[0, S_{max})$ is discretized with uniformly spread N grids satisfying $0 = S_1 < S_2 < \cdots S_N = S_{max}$. The step size $dS = S_{i+1} - S_i$, $i = 0,1,\dots,N$ is equal to each other at any point *i*.

Spatial derivatives are computed by the FD4 scheme. The first derivative $\frac{\partial V}{\partial S}(S_i, t)$ is given by

$$\frac{\partial V}{\partial S}(S_i,t) = \frac{V(S_{i-2},t) - 8V(S_{i-1},t) + 8V(S_{i+1},t) - V(S_{i+2},t)}{12dS}. \quad (3.1)$$

Then, with the values $v_i(t)$ approximated to $V(S_i,t)$, Eq. (1.1) is transformed to the system of ordinary differential equations

$$\frac{\partial v(t)}{dt} = Av(t) + w \quad = 3,\dots,N-2 \quad (3.2)$$

where $v(t) = [v_1, v_2, \dots, v_N]^T, w$ is a column vector, with

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{12} & \frac{-8}{12} & \frac{8}{12} & \frac{-1}{12} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{-8}{12} & \frac{8}{12} & \frac{-1}{12} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12} & \frac{-8}{12} & \frac{8}{12} & \frac{-1}{12} & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \frac{1}{12} & \frac{-8}{12} & \frac{8}{12} & \frac{-1}{12} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{12} & \frac{-8}{12} & \frac{8}{12} & \frac{-1}{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For the points $i = 2$ and $i = N - 1$, the second-order approximation is applied.

The fourth order scheme (3.2) can be written in a more compact form as follows

$$\boldsymbol{v}' = \frac{1}{h}A\boldsymbol{v} \quad (3.3)$$

The second-order derivative terms are obtained by applying the first operator twice:

$$\boldsymbol{v}'' = \frac{1}{h^2}A\boldsymbol{v}' \quad (3.4)$$

3.2. Temporal Discretization

For the time discretization of Eq. (1.1), the SSPRK3, the RK4 and the one-step method have been considered. A class of high-order SSP time discretization technique was developed by Gottlieb et al. (2001) for solving hyperbolic conservation laws with stable spatial discretizations. The SSP methods guarantee the stability properties expected of the forward Euler method (Gottlieb et al., 2001).

The computational domain for the time consists of M points satisfying $0 = t^1 < t^2 < \cdots < t^M = T$. The uniform time step $dt = t^{n+1} - t^n$, $n = 1,2,\dots,M$ is equal to any point *n*. After applying the FD4 method, Eq. (3.2) is transformed into a set of ordinary differential equations in time as follows:

$$\frac{dv_i(t)}{dt} = Lv_i(t), \quad i = 3,\dots,N-2 \quad (3.5)$$

where L is the discretization form of

$$\mathcal{L} = -\frac{1}{2}\sigma^2 S^2 v_{SS} + rSv_s - rv. \quad (3.6)$$

Then, three different time discretization methods have been applied as explained in the following subsections.

3.2.1. The SSPRK3 Method

Eq.(3.5) is integrated in time with the consideration of the SSPRK3 scheme,

$$\frac{dv_i}{dt} = Lv_i. \quad (3.7)$$

The SSPRK3 scheme integrates the semi-discrete equation (3.7) from time to t_0 (step *k*) to $t_0 + dt$ (step *k + 1*) through the operations

$$v_i^{(1)} = v_i^n + dtLv_i^n$$

$$v_i^{(2)} = \frac{3}{4}v_i^n + \frac{1}{4}v_i^{(1)} + \frac{1}{4}dtLv^{(1)} \quad (3.8)$$

$$v_i^{n+1} = \frac{1}{3}v_i^n + \frac{2}{3}v^{(2)} + \frac{2}{3}dtLv^{(2)}$$

For the SSPRK3 method, the total variation (TV) of the numerical solution

$$TV(v) = \sum_i |v_{i+1} - v_i| \quad (3.9)$$

does not increase in time; i.e. the following so-called TVD property holds (Gottlieb et al., 2001):

$$TV(v^{k+1}) \le TV(v^k). \quad (3.10)$$

3.2.2. The RK4 Method

The RK4 scheme can be used to integrate the semi-discrete equation (3.5) through the operations

$$v_i^{(1)} = v_i^n + \frac{1}{2} dtLv_i^n$$

$$v_i^{(2)} = v_i^n + \frac{1}{2} dtLv_i^{(1)} \quad (3.11)$$

$$v_i^{(3)} = v_i^n + dtLv_i^{(2)}$$

$$v_i^{n+1} = v_i^n + dt\left[Lv_i^n + 2Lv_i^{(1)} + 2Lv_i^{(2)} + Lv_i^{(3)}\right].$$

3.2.3. The One-Step Method

The time discretization of Eq.(3.5) is obtained by the following one-step method (Ascher et al., 1995)

$$v^{n+1} = v^n + \theta_1 L_i^{n+1} + \theta_2 L_i^n + \theta_3 L_t^{n+1} + \theta_4 L_t^n \quad (3.12)$$

where, when $\theta_1 = \theta_2 = dt/2, \theta_3 = \theta_4 = 0$ the method is of order 2 known as Crank-Nicolson method and when $\theta_1 = \theta_2 = dt/2, \theta_3 = -\frac{dt^2}{12} \theta_4 = \frac{dt^2}{12}$, the method is of order 4 and L_t is the time derivation of $\mathcal{L}$. In this study, it is chosen the method is of order four.

4. Numerical Illustration

In this section, to illustrate performance of the proposed methods, numerical experiments on the European put option model have been performed. For the computations through the current schemes, computer codes have been produced in MATLAB 2018.

$T = 0.25, K = 10.0, r = 0.1, \sigma = 0.4, M = 2000, N = 200$ (Dura and Moşneagu, 2010) are used in all calculations. To compare accuracy of the proposed methods, the numerical values for the SSPRK3, the RK4, the one-step method, the literature (Dura and Moşneagu, 2010) and exact solution at time *t = 0* are presented in Table 1. It is seen that the SSPRK3 results and the RK4 results for the different stock prices are nearly the same as with the exact solution. The one-step method produces less accurate solutions than the other two. Due to non-smoothness of the final condition $max(K - S, 0)$, the option price $V(S,t)$ should not be very smooth near the strike price *K* and near *T*. As seen in Figure 1, for the selected parameters, one step method does not give the desired accurate solution for $7 < S < 20$, namely, near the strike price *K*. Besides, the qualitative behaviours of the European put option problem are plotted in Figure 2.

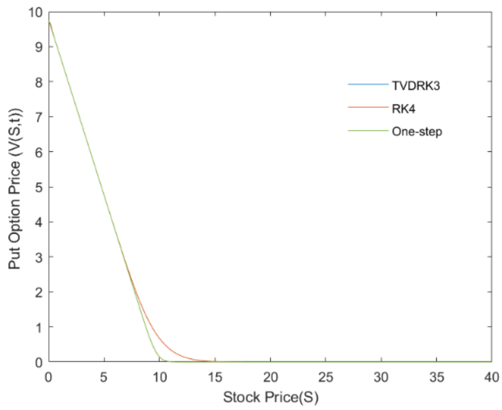

Figure 1. Solutions of the model by the methods SSPRK3, RK4, and the one-step at *t = 0.*

Since the proposed methods are explicit, the stability condition is given as follows:

$$dt \le \frac{dS^2}{2a}$$

where $a = \frac{1}{2}\sigma^2 S^2$ is the coefficient in front of the second derivative term in the Black-Scholes equation. In the present methods, this condition has been verified for each time and spatial steps.

To show accuracy of the methods, the convergence rates are calculated by

$$Rate(\|.\|_2) = \frac{\left\|V_{dS}^{dt} - V_{exact}(S_i, t^j)\right\|_2}{\left\|V_{dS/2}^{dt/2} - V_{exact}(S_i, t^j)\right\|_2} \quad (3.13)$$

where V_{dS}^{dt} represents the solution with spatial mesh size ds and time mesh size dt and $\|.\|_2$ is L^2-norm are given by

$$\left\|V_{dS}^{dt} - V_{exact}(S_i, t^j)\right\|_2 := \left(\sum_{1\le j\le M}\sum_{1\le i\le N}\left|V_i^j - V_{exact}(S_i, t^j)\right|^2 ds dt\right)^{1/2}$$

Table 1. Comparison of numerical solutions of the European put option model for various stock price values

S	SSPRK3	RK4	One-step Method	Dura and Moşneagu (2010)	Exact Solution
4.0	5.75309640	5.75309641	5.75309912	5.753102	5.753100
8.0	1.90209428	1.90209429	1.75331994	1.902102	1.902434
10.0	0.66888856	0.66888857	0.14301843	0.668360	0.669390
16.0	0.00532372	0.00532372	0.00	0.005419	0.005386
20.0	1.09130731e-04	1.09130661e-04	0.00	1.170806e-04	1.129336e-04

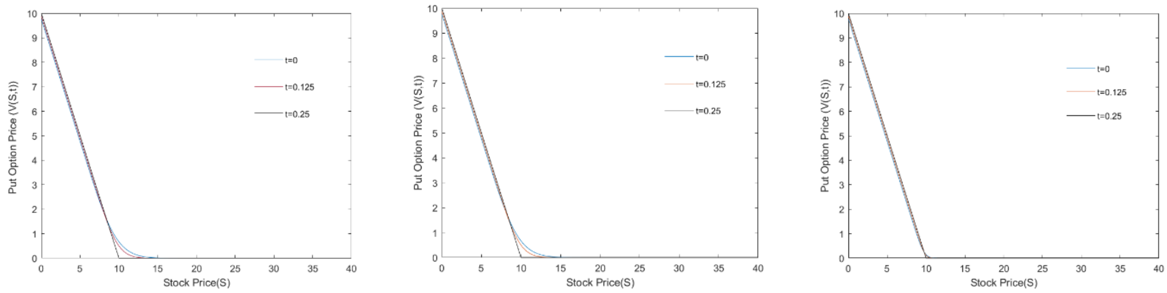

Figure 2. Solutions of the model by the SSPRK3, RK4 and one-step methods

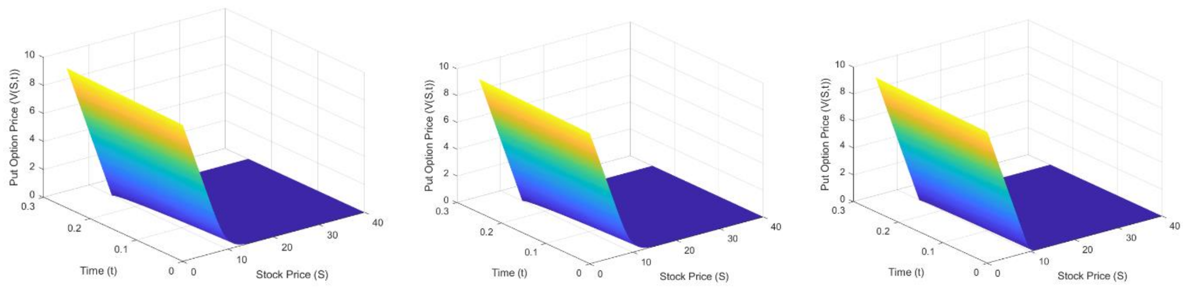

Figure 3. Solutions of the model by the SSPRK3, RK4 and one-step methods

Table 2. Convergence results for the model solved by the SSPRK3, RK4 and one-step methods

		SSPRK3		RK4		One-step method	
N	***M***	$\|.\|_{dS,2}$	Rate	$\|.\|_{dS,2}$	Rate	$\|.\|_{dS,2}$	Rate
50	500	0.00684723		0.00684725		0.24431677	
100	1000	0.00299818	2.28379550	0.00299817	2.28380979	0.25681997	0.95131531
200	2000	6.75679292e-4	4.43728265	6.75673287e-4	4.43730728	0.25630399	1.00201315
400	4000	1.45117522e-4	4.65608344	1.45128904e-4	4.65567690	0.23383970	1.09606705

It is quite clear from the results in Table 2 that the SSPRK3, the RK4 and one-step method are all convergent, but while the SSPRK3 and the RK4 has the fourth-order of accuracy, the one-step method has the first order of accuracy for space *x*. The non-smoothness of the payoff of the option in strike price $S=K$ can cause less accuracy in numerical computation. However, the one-step method has an advantage in computational time (CPU) than the other two methods at different *N* and *M* values, as seen in Table 3.

Table 3. CPU time (seconds)

N	*M*	SSPRK3	RK4	One-step metho
50	500	0.524095	1.760635	0.025198
100	1000	1.811687	6.288984	0.063711
200	2000	8.739596	31.350653	0.215877
400	4000	61.126498	211.443305	0.987350

5. Conclusion

This paper has proposed three combined methods for effectively solving the European put option pricing model. The FD4 scheme in space have been combined with the SSPRK3, RK4 and one-step method in time for solving the Black-Scholes equation for pricing the European put option. The convergence of the solutions has been measured by some error norms and it has been confirmed that the proposed methods are asymptotically convergent. Besides, the discussed computational procedures successfully worked to give very reliable and accurate solutions to the problem. The solutions obtained are compatible with available solution in the literature and exact solution and it has been seen that the present methods. This study is expected to provide a better understanding of the behavioural impacts of the economic models.

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