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APPROXIMATION BY BERNSTEIN-CHLODOWSKY POLYNOMIALS

E. İbikli*

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Abstract

The weighted approximation of continuous functions by Bernstein-Chlodowsky polynomials and their generalizations are studied.

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1. Introduction

The classical Bernstein-Chlodowsky polynomials have the following form

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}b_n\right) C_n^k \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)^{n-k}, \tag{1.1}$$

where $0 \le x \le b_n$ and b_n is a sequence of positive numbers such that $\lim\limits_{n\to\infty} b_n = \infty$, $\lim\limits_{n\to\infty} \frac{b_n}{n} = 0$. These polynomials were introduced by Chlodowsky in 1932 as a generalization of Bernstein polynomials (1912) on an unbounded set. Although there have been many studies of Bernstein polynomials to the present date (see [1], [2], [6] and [7]), the Bernstein-Chlodowsky polynomials (1.1) have not been investigated well enough. The aim of this article is to investigate the problem of weighted approximations of continuous functions by Bernstein-Chlodowsky polynomials (1.1) (for a generalization of these polynomials see [4]).

*Ankara University, Faculty of Science, Department of Mathematics, Tandogan, Ankara, Turkey.

2. Main Results

Let $\phi(x)$ be a continuous and increasing function in $(-\infty,\infty)$ such that

$$\lim_{x\to\pm\infty}\phi(x)=\pm\infty$$

and

$$\rho(x)=1+\phi^2(x)\,.$$

Denote by C_ρ the space of all continuous functions f, satisfying the condition

$$|f(x)|\le M_f\rho(x)\qquad -\infty<x<\infty\,.$$

Obviously C_ρ is a linear normed space with the norm

$$\|f\|_\rho=\sup_{-\infty<x<\infty}\frac{|f(x)|}{\rho(x)}\,.$$

A Korovkin type theorem for linear positive operators L_n, acting from C_ρ to C_ρ, has been proved in [3], where the following results have been established.

2.1. Theorem. (See [3]) *There exists a sequence of positive linear operators L_n, acting from C_ρ to C_ρ, satisfying the conditions*

$$\lim_{n\to\infty}\|L_n(1,x)-1\|_\rho=0 \tag{2.1}$$

$$\lim_{n\to\infty}\|L_n(\phi,x)-\phi\|_\rho=0 \tag{2.2}$$

$$\lim_{n\to\infty}\|L_n(\phi^2,x)-\phi^2\|_\rho=0 \tag{2.3}$$

and there exists a function $f^*\in C_\rho$ for which

$$\overline{\lim_{n\to\infty}}\|L_nf^*-f^*\|_\rho>0.$$

2.2. Theorem. (See [3]) *The conditions* (2.1), (2.2), (2.3) *imply* $\lim_{n\to\infty}\|L_nf-f\|_\rho=0$ *for any function f belonging to the subset C_ρ^0 of C_ρ for which*

$$\lim_{|x|\to\infty}\frac{f(x)}{\rho(x)}$$

exists finitely.

Setting $\rho(x)=1+x^2$ and applying Theorem 2.2 to the operators

$$L_n(f,x)=\begin{cases}B_n(f,x) & \text{if } 0\le x\le b_n\\ f(x) & \text{if } x\notin[0,b_n]\end{cases}$$

we obtain,

2.3. Proposition. *The assertion*

$$\lim_{n\to\infty}\sup_{0\le x\le b_n}\frac{|L_n(f,x)-f(x)|}{1+x^2}=0 \tag{2.4}$$

holds for any function $f\in C_\rho^0$ with $\rho(x)=1+x^2$ $x\ge0$.

Note that conditions (2.1),(2.2) and (2.3) are fulfilled since

$$B_n(1,x)=1 \tag{2.5}$$

$$B_n(t,x)=x \tag{2.6}$$

$$B_n(t^2,x)=x^2+\frac{x(b_n-x)}{n} \tag{2.7}$$

and therefore

$$\sup_{0\le x\le b_n}\frac{|B_n(t^2,x)-x^2|}{1+x^2}=\frac{1}{n}\sup_{0\le x\le b_n}\frac{x(b_n-x)}{1+x^2}\le\frac{b_n}{n}.$$

In view of Theorem 2.1, the assertion (2.4) does not hold in general for an arbitrary function $f\in C_\rho$, $\rho(x)=1+x^2$. Moreover, the polynomials (1.1) are not able to approximate even the analytic function x^2 on the entire interval $[0,b_n]$ without weight, since (2.7) gives

$$\max_{0\le x\le b_n}[B_n(t^2,x)-x^2]=\frac{b_n^2}{4n}$$

which does not converge to zero for some sequences (b_n) as $n\to\infty$.

An affirmative solution of the problem of approximation of the function $f(x)=x^2$ on the unbounded interval may be obtained by considering the polynomials of Bernstein-Chlodowsky with b_n satisfying the condition $\frac{b_n^2}{n}\to 0$ as $n\to\infty$ in (1.1). That is,

$$\widetilde{B_n}(f,x)=\sum_{k=0}^{n}f\left(\frac{k}{n}b_n\right)C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\quad 0\le x\le b_n\,. \tag{2.8}$$

where $\lim\limits_{n\to\infty}\frac{b_n^2}{n}=0$. Then

$$\tilde{B_n}(t^2,x)=x^2+\frac{x(b_n-x)}{n},\quad 0\le x\le b_n, \tag{2.9}$$

and therefore

$$\max_{0\le x\le b_n}[\tilde{B_n}(t^2,x)-x^2]=\frac{b_n}{4n},$$

which tends to zero as $n\to\infty$.

We consider now the problem of the approximation of arbitrary continuous functions by the polynomials (2.8).

Firstly we shall consider a special case.

2.4. Lemma. *For any continuous function f vanishing on $[a,\infty)$, where $a>0$ is independent of n,*

$$\lim_{n\to\infty}\sup_{0\le x\le b_n}|\tilde{B_n}(f,x)-f(x)|=0.$$

Proof. Since by the given condition, f is bounded, say $|f(x)|\le M$, $0\le x\le a$, we can write for arbitrary small $\varepsilon>0$ the inequality

$$\left|f\left(\frac{k}{n}b_n\right)-f(x)\right|<\varepsilon+\frac{2M}{\delta^2}\left(\frac{k}{n}b_n-x\right)^2,$$

where $x\in[0,b_n]$ and $\delta=\delta(\varepsilon)$ are independent of n.

By the properties (2.5), (2.6) and (2.9)

$$\sum_{k=0}^{n}\left(\frac{k}{n}b_n-x\right)^2C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}=\frac{x(b_n-x)}{n}.$$

Therefore

$$\sup_{0\le x\le b_n}|\tilde{B_n}(f,x)-f(x)|=\varepsilon+\frac{2M}{\delta^2}\frac{b_n}{4n},$$

which completes the proof. □

2.5. Theorem. *Let f be a continuous function on the semiaxis $[0,\infty)$, for which*

$$\lim_{x\to\infty} f(x) = k_f < \infty.$$

Then

$$\lim_{n\to\infty} \sup_{0\le x\le b_n} |\widetilde{B_n}(f,x) - f(x)| = 0.$$

Proof. Obviously it is sufficient to prove this theorem in the case of $k_f = 0$. In this case, for any $\varepsilon > 0$ there exists a point x_0 such that

$$|f(x)| < \varepsilon, \quad x \ge x_0. \tag{2.10}$$

Consider the function g with properties: $g(x) = f(x)$ if $0 \le x \le x_0$, $g(x)$ is linear on $x_0 \le x \le x_0 + \frac{1}{2}$ and $g(x) = 0$ if $x \ge x_0 + \frac{1}{2}$.

Then

$$\sup_{0\le x\le b_n} |f(x) - g(x)| \le \sup_{x_0\le x\le x_0+\frac{1}{2}} |f(x) - g(x)| + \sup_{x\ge x_0+\frac{1}{2}} |f(x)|$$

and since

$$\max_{x_0\le x\le x_0+\frac{1}{2}} |g(x)| = |f(x_0)|$$

we have

$$\sup_{0\le x\le b_n} |f(x) - g(x)| \le 3\varepsilon$$

by the condition (2.10).

Now we obtain

$$\begin{aligned}\sup_{0\le x\le b_n} |\widetilde{B_n}(f,x) - f(x)| \le \sup_{0\le x\le b_n} \widetilde{B_n}(|f-g|,x) &+ \\ + \sup_{0\le x\le b_n} |\widetilde{B_n}(g,x) - g(x)| &+ \\ + \sup_{0\le x\le b_n} |f(x) - g(x)| & \\ \le 6\varepsilon + \sup_{0\le x\le b_n} |\widetilde{B_n}(g,x) - g(x)|&.\end{aligned}$$

where $g(x)$ vanishes in $x_0 + \frac{1}{2} \le x \le b_n$. By Lemma 2.4, we obtain the desired result. □

3. A Generalization

We now give a generalization of Bernstein-Chlodowsky polynomials, which can be used to approximate continuous functions on more general weighted spaces.

Let $\omega(x) \ge 1$ be any continuous function for $x \ge 0$. Let also

$$F_f(t) = f(t)\frac{1+t^2}{\omega(t)},$$

and consider the following generalization of the polynomials (1.1)

$$L_n(f,x) = \frac{\omega(x)}{1+x^2} \sum_{k=0}^{n} F_f\left(\frac{k}{n}b_n\right) C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}, \tag{3.1}$$

where $x \in [0, b_n]$ and b_n has the same property as in (1.1). In the case of $\omega(t) = 1 + t^2$ the operators (3.1) coincide with (1.1).

3.1. Theorem. *For a continuous function f satisfying the condition*

$$\lim_{x\to\infty}\frac{f(x)}{\omega(x)}=K_f<\infty,$$

the equality

$$\lim_{n\to\infty}\sup_{0\le x\le b_n}\frac{|L_n(f,x)-f(x)|}{\omega(x)}=0$$

holds.

Proof. Obviously

$$L_n(f,x)-f(x)=\frac{\omega(x)}{1+x^2}\left\{\sum_{k=0}^{n}F_f\left(\frac{kb_n}{n}\right)C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}-F_f(x)\right\},$$

and therefore

$$\sup_{0\le x\le b_n}\frac{|L_n(f,x)-f(x)|}{\omega(x)}=\sup_{0\le x\le b_n}\frac{|B_n(F_f,x)-F_f(x)|}{1+x^2}.$$

Also, $F_f(x)$ is a continuous function on $[0,\infty)$ satisfying $|F_f(x)|\le M_f(1+x^2)$, $x\ge 0$, since we have the inequality $|f(x)|\le M_f\omega(x)$ for f. Therefore, by Proposition 2.3 we obtain the desired result. □

Note that similar statements may also be obtained for the generalization of Bernstein-Chlodowsky polynomials considered in [5].

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