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Research Article

Ulam Stability in Real Inner-Product Spaces

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ABSTRACT. Roughly speaking an equation is called Ulam stable if near each approximate solution of the equation there exists an exact solution. In this paper, we prove that Cauchy-Schwarz equation, Orthogonality equation and Gram equation are Ulam stable.

Keywords: Ulam stability, inner-product space.

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1. INTRODUCTION

This paper is concerned with the Ulam stability of some classical equations arising in the context of inner-product spaces. For the general notion of Ulam stability see, e.q., [1]. Roughly speaking an equation is called Ulam stable if near every approximate solution there exists an exact solution; the precise meaning in each case presented in this paper is described in three theorems. Related results can be found in [2, 3, 4]. See also [5] for some inequalities in inner product spaces.

2. THE CAUCHY-SCHWARZ EQUATION

Let $(V, (\cdot|\cdot))$ be a real inner-product space. Consider the Cauchy-Schwarz equation, i.e.,

$$\|x\|^2\|y\|^2 - (x|y)^2 = 0. \tag{2.1}$$

The set of its solutions is

$$S = \{(x, y) \in V^2 : x, y \ \ are \ \ linearly \ \ dependent \ \ vectors\}. \tag{2.2}$$

Theorem 2.1. *Let $\varepsilon > 0$ and $(u, v) \in V^2$ an approximate solution of (2.1), i.e.,*

$$\|u\|^2\|v\|^2 - (u|v)^2 \leq \varepsilon. \tag{2.3}$$

Then there exists an exact solution $(x, y) \in S$ such that

$$\|u - x\|^2 + \|v - y\|^2 \leq \sqrt{\varepsilon}. \tag{2.4}$$

Proof. If u and v are linearly dependent, then it suffices to take $x = u, y = v$. So, let u and v be linearly independent. Then $u \neq 0$, $v \neq 0$; suppose that $\|v\| \leq \|u\|$ and let $w_t := u + tv, t \in \mathbb{R}$. Then $w_t \neq 0, t \in \mathbb{R}$; let $z_t := w_t/\|w_t\|$ and $W_t := Span\{z_t\}$. Let $x_t := pr_{W_t}u$ and $y_t := pr_{W_t}v$ be the orthogonal projections of u, respectively v, on W_t. Then x_t and y_t are linearly dependent

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vectors, i.e., $(x_t, y_t) \in V^2$ is a solution of (2.1). Moreover, $\|u-x_t\|^2+\|v-y_t\|^2 = \|u\|^2-(u|z_t)^2+\|v\|^2-(v|z_t)^2$. Let $a := (v|v), b := (u|v), c := (u|u)$. Then

$$ac - b^2 > 0; 0 < a \le c. \tag{2.5}$$

We have

$$\|u - x_t\|^2 + \|v - y_t\|^2 = c + a - (u|z_t)^2 - (v|z_t)^2 \tag{2.6}$$

and

$$s := s(t) := (u|z_t)^2 + (v|z_t)^2 = \frac{(u|w_t)^2 + (v|w_t)^2}{\|w_t\|^2} = \frac{(c + tb)^2 + (b + ta)^2}{\|w_t\|^2},$$

so that

$$s = \frac{(a^2 + b^2)t^2 + 2b(a + c)t + b^2 + c^2}{at^2 + 2bt + c}.$$

It follows that

$$(a^2 + b^2 - as)t^2 + 2b(a + c - s)t + b^2 + c^2 - cs = 0,$$

which entails

$$b^2(a + c - s)^2 - (a^2 + b^2 - as)(b^2 + c^2 - cs) \ge 0. \tag{2.7}$$

From (2.5) and (2.7), we deduce

$$s^2 - (a + c)s + ac - b^2 \le 0.$$

Let s_1 and s_2 be the roots of the corresponding equation, i.e.,

$$s_1 = \frac{a + c - \sqrt{(a - c)^2 + 4b^2}}{2}, s_2 = \frac{a + c + \sqrt{(a - c)^2 + 4b^2}}{2}.$$

Then $s_1 \le s(t) \le s_2$ for all $t \in \mathbb{R}$. By using (2.5), it is easy to prove that there exists $\tau \in \mathbb{R}$ such that $s(\tau) = s_2$. Now,

$$\|u - x_\tau\|^2 + \|v - x_\tau\|^2 = a + c - s(\tau) = a + c - s_2 = a + c - \frac{a + c + \sqrt{(a - c)^2 + 4b^2}}{2} =$$

$$= \frac{a + c - \sqrt{(a - c)^2 + 4b^2}}{2} \le \sqrt{ac - b^2} = \sqrt{\|u\|^2\|v\|^2 - (u|v)^2}.$$

Combined with (2.3), this gives (2.4) and the proof is finished.

□

3. THE ORTHOGONALITY EQUATION

Consider the orthogonality equation $(x|y) = 0$.

Theorem 3.2. *Let $\varepsilon > 0$ and $(u, v) \in V^2$ such that $\|u\| = \|v\| = 1$ and $|(u|v)| \le \varepsilon$. Then, there exists $(x, y) \in V^2$ such that $\|x\| = \|y\| = 1, (x|y) = 0$ and*

$$\|u - x\|^2 + \|v - y\|^2 \le (4 - 2\sqrt{2})\varepsilon. \tag{3.8}$$

Proof. (i) Let $(u|v) > 0$. Choose $w \in Span\{u, v\}, \|w\| = 1, (w|u) = 0$. Then $v = u\cos\alpha + w\sin\alpha$, for a suitable $\alpha \in [0, \frac{\Pi}{2})$. Define $x_t := u\cos t - w\sin t, y_t := u\sin t + w\cos t, t \in \mathbb{R}$. Then $\|x_t\| = 1, \|y_t\| = 1, (x_t|y_t) = 0$, and $\|u - x_t\|^2 + \|v - y_t\|^2 = \|(1-\cos t)u + w\sin t\|^2 + \|(\cos\alpha - \sin t)u + (\sin\alpha - \cos t)w\|^2 = (1-\cos t)^2 + \sin^2 t + (\cos\alpha - \sin t)^2 + (\sin\alpha - \cos t)^2 = 4 - 2((1+\sin\alpha)\cos t + \cos\alpha\sin t)$. Clearly $(1 + \sin\alpha)\cos t + \cos\alpha\sin t \le \sqrt{2 + 2\sin\alpha}, t \in \mathbb{R}$. Choose $\tau \in \mathbb{R}$ such that

$$(1 + \sin\alpha)\cos\tau + \cos\alpha\sin\tau = \sqrt{2 + 2\sin\alpha}.$$

Then

$$\|u - x_\tau\|^2 + \|v - y_\tau\|^2 = 4 - 2\sqrt{2}\sqrt{1 + \sin\alpha}. \tag{3.9}$$

Now consider the function $f(\alpha) = (4 - 2\sqrt{2})\cos\alpha - 4 + 2\sqrt{2}\sqrt{1+\sin\alpha}$, $\alpha \in [0, \frac{\Pi}{2}]$. It is easy to verify that $f(0) = f(\frac{\Pi}{2}) = 0$ and there exists $0 < \beta < \frac{\Pi}{2}$ such that f is increasing on $[0, \beta]$ and decreasing on $[\beta, \frac{\Pi}{2}]$. It follows that $f(\alpha) \geq 0, \alpha \in [0, \frac{\Pi}{2}]$; combined with (3.9), this yields

$$\|u - x_\tau\|^2 + \|v - y_\tau\|^2 \leq (4 - 2\sqrt{2})\cos\alpha. \tag{3.10}$$

On the other hand, $\cos\alpha = (u|v) \leq \varepsilon$, and so (3.8) is a consequence of (3.10).

(ii) If $(u|v) < 0$, it suffices to use the proof of (i) with v replaced by $-v$. Thus, the theorem is proved.

□

4. The Gram equation

Denote by $G(u_1, ..., u_m)$ the Gram determinant of the vectors $u_1, ..., u_m \in V$. Let $v_1, ..., v_n \in V$ be linearly independent vectors. Consider the equation

$$G(x, v_1, ..., v_n) = 0. \tag{4.11}$$

Theorem 4.3. *Let $\varepsilon > 0$ and $u \in V$ such that*

$$G(u, v_1, ..., v_n) \leq \varepsilon.$$

Then, there exists $x \in V$ which satisfy (4.11) and

$$\|u - x\| \leq \frac{1}{\sqrt{G(v_1, ..., v_n)}}\sqrt{\varepsilon}.$$

Proof. Let $W = Span\{v_1, ..., v_n\}$ and $x := pr_W u$. Then $x \in W$ and therefore it satisfies (4.11). Moreover,

$$\|u - x\| = \sqrt{\frac{G(u, v_1, ..., v_n)}{G(v_1, ..., v_n)}} \leq \frac{1}{\sqrt{G(v_1, ..., v_n)}}\sqrt{\varepsilon}.$$

□

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