Advances in the Theory of Nonlinear Analysis and its Applications **5** (2021) No. 3, 421–432.
https://doi.org/10.31197/atnaa.758962
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

Some fixed point results in ordered b-metric space with an auxiliary function

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Abstract

The purpose of this paper is to establish some fixed point results for a class of generalized (ϕ, ψ)-weak contraction mapping in complete partially ordered b-metric space. This mapping necessarily have a unique fixed point under ordered relation in the space. Also, the results for common fixed point and coincidence point of the self mappings are presented. These results generalize and extend an existing results in the literature. Some illustrations are given at the end to support the results.

Keywords: Ordered b-metric space; Generalized (ϕ, ψ)-weak contraction; Common fixed point; Coincidence point.
2010 MSC: Subject Classification 46T99, 41A50; 54H25.

1. Introduction

In generalized metric spaces, the fixed points of mappings satisfying certain contractive conditions are of great importance in acquiring the unique solution of many problems in pure and applied mathematics. First, Ran and Reuings [33] have extended the result in this direction, discussed the existence of fixed points for certain mappings in ordered metric space and also presented applications to matrix equations. Afterwords, the result of [33] has been extended by Nieto et al. [28, 29] involving nondecreasing mappings and used their results in obtaining the unique solutions of ordinary differential equations. At the same time, the

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Received June 27, 2020; Accepted: May 31, 2021; Online: June 2, 2021.

results regarded to generalized contractions in ordered metric spaces were studied by Agarwal et al. [1] and, O'Regan et al. [31]. Later the theory of coupled fixed points for certain maps was first introduced by Bhaskar and Lakshmikantham [6] and then applied the results to a periodic boundary value problem in acquiring the unique solution. Thereafter, the concept of coupled coincidence, common fixed point results was first initiated by Lakshmikantham and Ćirić [24], which were the extensions of Bhaskar and Lakshmikantham [6] involving monotone property of a function in ordered metric space. More work relevant to coupled fixed point results under different contractive conditions in various spaces can be found from [10, 11, 17, 22, 38, 39, 40, 42].

b-metric space is one of many generalizations to an usual metric, which was first initiated by Bakhtin [4] in his work and then extensively used by Czerwik in [13, 14]. Thereafter, lot of improvements have been done in finding fixed points for single and multi-valued operator in a b-metric space, the readers may refer to [2, 3, 5, 12, 16, 20, 26, 27, 32, 34, 35, 36, 37, 41, 43, 44].

In this paper, we have introduced a classes of (ϕ, ψ)-weak contractions to obtain a unique fixed point for a self mapping and a common fixed point, a coincidence point for two self mappings in complete partially ordered b-metric spaces. These results generalize and extended the results of [8, 30] and several results from [15, 19, 21].

2. Preliminaries

Definition 2.1. *[14, 41] A mapping $d : P \times P \to [0, +\infty)$, where P is a non-empty set is said to be a b-metric, if it satisfies the properties given below for any $\nu, \xi, \mu \in P$ and for some $s \geq 1$,*

(a). $d(\nu, \xi) = 0$ *if and if* $\nu = \xi$,

(b). $d(\nu, \xi) = d(\xi, \nu)$,

(c). $d(\nu, \xi) \leq s\,(d(\nu, \mu) + d(\mu, \xi))$.

And then (P, d, s) is known as a b-metric space.

Definition 2.2. *[14, 41] Let (P, d, s) be a b-metric space. Then*

(1). a sequence $\{\nu_n\}$ is said to converge to ν if $\lim_{n\to+\infty} d(\nu_n, \nu) = 0$ and written as $\lim_{n\to+\infty} \nu_n = \nu$.

(2). $\{\nu_n\}$ is said to be a Cauchy sequence in P, if $\lim_{n,m\to+\infty} d(\nu_n, \nu_m) = 0$.

(3). (P, d, s) is said to be complete if every Cauchy sequence in P is convergent.

Definition 2.3. *[41] A metric d on P together with a partially ordered relation $\leq$ is called a partially ordered b-metric space. It is denoted by $(P, d, \leq)$.*

Definition 2.4. *[41] If the metric d is complete then $(P, d, \leq)$ is called complete partially ordered b-metric space.*

Definition 2.5. *[39] Let $(P, \leq)$ be a partially ordered set. A mapping $S : P \to P$ is said to be monotone nondecreasing, if $S(\nu) \leq S(\xi)$ for all $\nu, \xi \in P$ with $\nu \leq \xi$.*

Definition 2.6. *[40] A point $\nu \in A$, where $A \neq \emptyset \subseteq P$ is called a coincidence (or common fixed) point for two self-mappings f and S, if $f\nu = S\nu$ $(f\nu = S\nu = \nu)$.*

Definition 2.7. *[40] Two self-maps f and S defined over a subset A of P are called commuting, if $fS\nu = Sf\nu$, for all $\nu \in A$.*

Definition 2.8. *[40] Two self-mappings f and S defined over $A \subseteq P$ are called compatible, if any sequence $\{\nu_n\}$ with $\lim_{n\to+\infty} f\nu_n = \lim_{n\to+\infty} S\nu_n = \mu$, for some $\mu \in A$ then $\lim_{n\to+\infty} d(Sf\nu_n, fS\nu_n) = 0$.*

Definition 2.9. *[40] A pair of self-maps (f, S) on $A \subseteq P$ is called weakly compatible, if $Sf\nu = fS\nu$, when $S\nu = f\nu$ for some $\nu \in A$.*

Definition 2.10. *[38] Let f and S be two self-mappings over $(P, \leq)$. Then S is called monotone f-nondecreasing, if*

$$f\nu \leq f\xi \Rightarrow S\nu \leq S\xi, \text{ for any } \nu, \xi \in P.$$

Definition 2.11. *[38] If very two elements of a nonempty subset A of P are comparable then A is called well ordered set.*

The concept of acquiring fixed points in metric space using control functions was initiated by Khan et al. [23].

Definition 2.12. *[23] A self-map ϕ defined on $[0, +\infty)$ is said to be an altering distance function, if ϕ is continuous and monotone increasing with $\phi(t) = 0$ if and if $t = 0$.*

Note 2.13. *(1). Let us denote the set of all altering distance functions on $[0, +\infty)$ by Φ.*

(2). Similarly, Ψ denoted by the set of all lower semi-continuous functions on $[0, +\infty)$ with $\psi(t) = 0$ if and if $t = 0$.

Lemma 2.14. *[18] Let P be a non-empty set and $f : P \to P$ be a mapping. Then there exists a subset E of P such that $fE = fP$ and $f : E \to P$ is one-to-one.*

In 1975, Dass and Gupta [15] proved the following fixed point result in a complete metric space.

Theorem 2.15. *[15] Suppose (P, d) is a complete metric space. Let $S : P \to P$ be a mapping such that there exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ satisfying*

$$d(S\nu, S\xi) \leq \alpha \frac{d(\xi, S\xi)\,[1 + d(\nu, S\nu)]}{1 + d(\nu, \xi)} + \beta d(\nu, \xi), \tag{1}$$

for any distinct $\nu, \xi \in P$. Then S has a unique fixed point in P.

The generalization of above result in partially ordered metric space was obtained by Cabrera et al. [7] in 2013. Later Chandok et al. [9] generalized the result of [7] using control functions in the same space.

Again, Theorem 2.15 was generalized by Jaggi [21] in 1977 and proved the following:

Theorem 2.16. *[21] Suppose (P, d) is a complete metric space. A self mapping S on P such that*

$$d(S\nu, S\xi) \leq \alpha \frac{d(\xi, S\xi)\ d(\nu, S\nu)}{d(\nu, \xi)} + \beta d(\nu, \xi), \tag{2}$$

for all $\nu, \xi \in P$ with $\nu \neq \xi$, where $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Then S has a unique fixed point in P.

This result was again proved by Harjani et al. [19] in complete metric space endowed with partial order relation. Later the result of [19] was generalized by Luong et al. [25] involving altering distance functions which satisfies a weak contractive condition of rational type auxiliary functions in ordered metric space. Thereafter, the result [25] was generalized and extended by Chandok et al. [8] in 2013 and obtained coupled fixed point, common fixed point results for weak contractive mapping in partially ordered metric space. These results were again generalized by Nguyen T. Hieu et al. [30] in partially order b-metric space by involving altering distance functions.

3. Main Results

To begin this section with the following theorem.

Theorem 3.1. *Let* $(P, d, s, \leq)$ *be a complete partially ordered b-metric space with parameter* $s > 1$. *Let* $S : P \to P$ *be a continuous, nondecreasing mapping with regards to* $\leq$ *such that there exists* $\nu_0 \in P$ *with* $\nu_0 \leq S\nu_0$. *Suppose that*

$$\phi(sd(S\nu, S\xi)) \leq \phi(M(\nu, \xi)) - \psi(M(\nu, \xi)), \tag{3}$$

where $\phi \in \Phi, \psi \in \Psi$, *for any* $\nu, \xi \in P$ *with* $\nu \leq \xi$ *and*

$$M(\nu, \xi) = \max\{\frac{d(\nu, S\nu)\ d(\xi, S\xi)}{1 + d(\nu, \xi)}, \frac{d(\nu, S\xi) + d(\xi, S\nu)}{2s}, d(\nu, S\nu), d(\xi, S\xi), d(\nu, \xi)\}. \tag{4}$$

Then S *has a fixed point in* P.

Proof. For some $\nu_0 \in P$ such that $S\nu_0 = \nu_0$, then the proof is finished. Assume that $\nu_0 < S\nu_0$, then construct a sequence $\{\nu_n\} \subset P$ by $\nu_{n+1} = S\nu_n$, for $n \geq 0$. But S is nondecreasing then we obtain the following expression by mathematical induction

$$\nu_0 < S\nu_0 = \nu_1 \leq S\nu_1 = \nu_2 \leq ... \leq S\nu_{n-1} = \nu_n \leq S\nu_n = \nu_{n+1} \leq ... . \tag{5}$$

If for some $n_0 \in \mathbb{N}$ such that $\nu_{n_0} = \nu_{n_0+1}$ then from (5), ν_{n_0} is a fixed point of S and we have nothing to prove. Suppose that $\nu_n \neq \nu_{n+1}$, i.e., $d(\nu_n, \nu_{n+1}) > 0$, for all $n \geq 1$. Since $\nu_n > \nu_{n-1}$, for any $n \geq 1$ then from (3), we have

$$\begin{aligned} \phi(d(\nu_n, \nu_{n+1})) = \phi(d(S\nu_{n-1}, S\nu_n)) &\leq \phi(sd(S\nu_{n-1}, S\nu_n)) \\ &\leq \phi(M(\nu_{n-1}, \nu_n)) - \psi(M(\nu_{n-1}, \nu_n)), \end{aligned} \tag{6}$$

where

$$\begin{aligned} M(\nu_{n-1}, \nu_n) &= \max\{\frac{d(\nu_{n-1}, S\nu_{n-1})\ d(\nu_n, S\nu_n)}{1 + d(\nu_{n-1}, \nu_n)}, \frac{d(\nu_{n-1}, S\nu_n) + d(\nu_n, S\nu_{n-1})}{2s}, \\ &\qquad d(\nu_{n-1}, S\nu_{n-1}), d(\nu_n, S\nu_n), d(\nu_{n-1}, \nu_n)\}. \\ &\leq \max\{d(\nu_n, \nu_{n+1}), \frac{d(\nu_{n+1}, \nu_n) + d(\nu_n, \nu_{n-1})}{2}, d(\nu_{n-1}, \nu_n)\} \\ &= \max\{d(\nu_n, \nu_{n+1}), d(\nu_{n-1}, \nu_n)\}, \end{aligned} \tag{7}$$

which implies that

$$\phi(d(\nu_n, \nu_{n+1})) \leq \phi(\max\{d(\nu_n, \nu_{n+1}), d(\nu_{n-1}, \nu_n)\}) - \psi(\max\{d(\nu_n, \nu_{n+1}), d(\nu_{n-1}, \nu_n)\}). \tag{8}$$

If $\max\{d(\nu_n, \nu_{n+1}), d(\nu_{n-1}, \nu_n)\} = d(\nu_n, \nu_{n+1})$ for some $n \geq 1$, then from (8), we get

$$\phi(d(\nu_n, \nu_{n+1})) \leq \phi(d(\nu_n, \nu_{n+1})) - \psi(d(\nu_n, \nu_{n+1})) < \phi(d(\nu_n, \nu_{n+1})), \tag{9}$$

which is a contradiction under (9). Thus, $\max\{d(\nu_n, \nu_{n+1}), d(\nu_{n-1}, \nu_n)\} = d(\nu_{n-1}, \nu_n)$ for $n \geq 1$ and we have from (8) again,

$$\phi(d(\nu_n, \nu_{n+1})) \leq \phi(d(\nu_n, \nu_{n-1})) - \psi(d(\nu_n, \nu_{n-1})) < \phi(d(\nu_n, \nu_{n-1})). \tag{10}$$

Thus, the sequence $\{d(\nu_n, \nu_{n-1})\}$ for $n \geq 1$ is monotone non-increasing and bounded below. As a result we have

$$\lim_{n \to +\infty} d(\nu_n, \nu_{n-1}) = \rho \geq 0. \tag{11}$$

Now, taking the upper limit on both sides of (10), we obtain

$$\phi(\rho) \leq \phi(\rho) - \lim_{n \to +\infty} \inf \psi(d(\nu_n, \nu_{n-1})) \leq \phi(\rho) - \psi(\rho) < \phi(\rho), \tag{12}$$

which is a contradiction under (12). Thus, $\rho = 0$. Hence, $d(\nu_n, \nu_{n-1}) \to 0$ as $n \to +\infty$.

Next, we prove that $\{\nu_n\}$ is a Cauchy sequence in P. Assume contrary that $\{\nu_n\}$ is not a Cauchy sequence. Then for some $\epsilon > 0$, we can get two subsequences $\{\nu_{m_j}\}$ and $\{\nu_{n_j}\}$ of $\{\nu_n\}$, where n_j is the smallest index such that

$$n_j > m_j > j, \quad d(\nu_{m_j}, \nu_{n_j}) \geq \epsilon \tag{13}$$

and

$$d(\nu_{m_j}, \nu_{n_j-1}) < \epsilon. \tag{14}$$

Applying the triangular inequality in (13), we get

$$\begin{aligned} \epsilon \leq d(\nu_{m_j}, \nu_{n_j}) &\leq sd(\nu_{m_j}, \nu_{n_j-1}) + sd(\nu_{n_j-1}, \nu_{n_j}) \\ &\leq s^2 d(\nu_{m_j}, \nu_{m_j-1}) + s^2 d(\nu_{m_j-1}, \nu_{n_j-1}) + sd(\nu_{n_j-1}, \nu_{n_j}). \end{aligned} \tag{15}$$

Similarly, we have

$$d(\nu_{m_j-1}, \nu_{n_j-1}) \leq sd(\nu_{m_j-1}, \nu_{m_j}) + sd(\nu_{m_j}, \nu_{n_j-1}) \leq sd(\nu_{m_j-1}, \nu_{m_j}) + s\epsilon. \tag{16}$$

Letting $j \to +\infty$ in equations (15) and (16) and combining together we obtain the following inequality

$$\frac{\epsilon}{s^2} \leq \lim_{j \to +\infty} \sup d(\nu_{m_j-1}, \nu_{n_j-1}) \leq s\epsilon. \tag{17}$$

Again using the triangular inequality, one can obtain the following inequalities

$$\frac{\epsilon}{s^2} \leq \lim_{j \to +\infty} \inf d(\nu_{m_j-1}, \nu_{n_j-1}) \leq s\epsilon, \tag{18}$$

and

$$\frac{\epsilon}{s} \leq \lim_{j \to +\infty} \sup d(\nu_{m_j-1}, \nu_{n_j}) \leq s\epsilon^2. \tag{19}$$

Let

$$\begin{aligned} M(\nu_{m_j-1}, \nu_{n_j-1}) = \max\{ &\frac{d(\nu_{m_j-1}, S\nu_{m_j-1}) \ d(\nu_{n_j-1}, S\nu_{n_j-1})}{1 + d(\nu_{m_j-1}, \nu_{n_j-1})}, \\ &\frac{d(\nu_{m_j-1}, S\nu_{n_j-1}) + d(\nu_{n_j-1}, S\nu_{m_j-1})}{2s}, d(\nu_{m_j-1}, S\nu_{m_j-1}), \\ &d(\nu_{n_j-1}, S\nu_{n_j-1}), d(\nu_{m_j-1}, \nu_{n_j-1})\} \\ = \max\{ &\frac{d(\nu_{m_j-1}, \nu_{m_j}) \ d(\nu_{n_j-1}, \nu_{n_j})}{1 + d(\nu_{m_j-1}, \nu_{n_j-1})}, \frac{d(\nu_{m_j-1}, \nu_{n_j}) + d(\nu_{n_j-1}, \nu_{m_j})}{2s}, \\ &d(\nu_{m_j-1}, \nu_{m_j}), d(\nu_{n_j-1}, \nu_{n_j}), d(\nu_{m_j-1}, \nu_{n_j-1})\}. \end{aligned} \tag{20}$$

From (20), we obtain the following inequalities

$$\frac{\epsilon}{s^2} \leq \lim_{j \to +\infty} \sup M(\nu_{m_j-1}, \nu_{n_j-1}) \leq s\epsilon \tag{21}$$

and

$$\frac{\epsilon}{s^2} \leq \lim_{j \to +\infty} \inf M(\nu_{m_j-1}, \nu_{n_j-1}) \leq s\epsilon. \tag{22}$$

Form (5), we have $\nu_{m_j-1} < \nu_{n_j-1}$, then

$$\phi(sd(\nu_{m_j}, \nu_{n_j})) = \phi(sd(S\nu_{m_j-1}, S\nu_{n_j-1})) \leq \phi(M(\nu_{m_j-1}, \nu_{n_j-1})) - \psi(M(\nu_{m_j-1}, \nu_{n_j-1})). \tag{23}$$

Now, letting $j \to +\infty$ in (23) and using equations (21) and (22), we obtain that

$$\begin{aligned}\phi(s\epsilon) &\leq \phi(s \lim_{j\to+\infty} d(\nu_{m_j}, \nu_{n_j})) \\ &\leq \phi(\lim_{j\to+\infty} \sup M(\nu_{m_j-1}, \nu_{n_j-1})) - \lim_{j\to+\infty} \inf \psi(M(\nu_{m_j-1}, \nu_{n_j-1})) \\ &\leq \phi(s\epsilon) - \psi(\lim_{j\to+\infty} \inf M(\nu_{m_j-1}, \nu_{n_j-1})) \\ &< \phi(s\epsilon),\end{aligned} \tag{24}$$

this is a contradiction under (24). Hence, $\{\nu_n\}$ is a Cauchy sequence and converges for some $\mu \in P$ as P is complete. Also, the continuity of S implies that

$$S\mu = S(\lim_{n\to+\infty} \nu_n) = \lim_{n\to+\infty} S\nu_n = \lim_{n\to+\infty} \nu_{n+1} = \mu. \tag{25}$$

Therefore, μ is a fixed point of S in P. □

By weakening the continuity property of a map S in Theorem 3.1, we have the following result.

Theorem 3.2. *In Theorem 3.1, if P has a property that, the sequence $\{\nu_n\}$ is a nondecreasing such that $\nu_n \to v$ implies that $\nu_n \leq v$, for all $n \in \mathbb{N}$, i.e., $v = \sup \nu_n$ then a non continuous mapping S has a fixed point in P.*

Proof. From Theorem 3.1, we take the same sequence $\{\nu_n\}$ in P such that $\nu_0 \leq \nu_1 \leq \nu_2 \leq \nu_3 \leq ... \leq \nu_n \leq \nu_{n+1} \leq ...$, i.e., $\{\nu_n\}$ is a nondecreasing Cauchy sequence and converges to v in P. Therefore from the hypotheses, we have $\nu_n \leq v$ for all $n \in \mathbb{N}$, which implies that $v = \sup \nu_n$.

Next, we prove that v is a fixed point of S in P, that is $Sv = v$. Suppose $Sv \neq v$, that is $d(Sv, v) \neq 0$. Let

$$\begin{aligned}M(\nu_n, v) &= \max\{\frac{d(\nu_n, S\nu_n)\ d(v, Sv)}{1 + d(\nu_n, v)}, \frac{d(\nu_n, Sv) + d(v, S\nu_n)}{2s}, d(\nu_n, S\nu_n), d(v, Sv), d(\nu_n, v)\} \\ &= \max\{\frac{d(\nu_n, \nu_{n+1})\ d(v, Sv)}{1 + d(\nu_n, v)}, \frac{d(\nu_n, Sv) + d(v, \nu_{n+1})}{2s}, d(\nu_n, \nu_{n+1}), d(v, Sv), d(\nu_n, v)\}.\end{aligned} \tag{26}$$

Letting $n \to +\infty$ and from $\lim_{n\to+\infty} \nu_n = v$, we get

$$\lim_{n\to+\infty} M(\nu_n, v) = \max\{0, \frac{d(v, Sv)}{2s}, 0, d(v, Sv), 0\} = d(v, Sv). \tag{27}$$

We know that $\nu_n \leq v$, for all n then from contraction condition (3), we get

$$\phi(d(\nu_{n+1}, Sv)) = \phi(d(S\nu_n, Sv) \leq \phi(sd(S\nu_n, Sv) \leq \phi(M(\nu_n, v)) - \psi(M(\nu_n, v)). \tag{28}$$

Letting $n \to +\infty$ and using equation (27), we get

$$\phi(d(v, Sv)) \leq \phi(d(v, Sv)) - \psi(d(v, Sv)) < \phi(d(v, Sv)), \tag{29}$$

which is a contraction under (29). Thus, $Sv = v$, that is S has a fixed point v in P. □

The uniqueness of an existing fixed point in Theorem 3.1 and Theorem 3.2 can get, if P has the following property:

For any ν, $\xi \in P$, there exists $w \in P$ such that $w \leq \nu$ and $w \leq \xi$.

Theorem 3.3. *If P satisfies the above mentioned condition in Theorem 3.1 (or Theorem 3.2) then S has a unique fixed point.*

Proof. From Theorem 3.1 (or Theorem 3.2), we conclude that S has a nonempty set of fixed points. Suppose that ν^* and ξ^* be two fixed points of S then, we claim that $\nu^* = \xi^*$. Suppose that $\nu^* \neq \xi^*$, then from the hypotheses we have

$$\phi(d(S\nu^*, S\xi^*)) \leq \phi(sd(S\nu^*, S\xi^*)) \leq \phi(M(\nu^*, \xi^*)) - \psi(M(\nu^*, \xi^*)), \tag{30}$$

where

$$\begin{aligned} M(\nu^*, \xi^*) &= \max\{\frac{d(\nu^*, S\nu^*)\ d(\xi^*, S\xi^*)}{1 + d(\nu^*, \xi^*)}, \frac{d(\nu^*, S\xi^*) + d(\xi^*, S\nu^*)}{2s}, d(\nu^*, S\nu^*), \\ &\quad d(\xi^*, S\xi^*), d(\nu^*, \xi^*)\} \\ &= \max\{\frac{d(\nu^*, \nu^*)\ d(\xi^*, \xi^*)}{1 + d(\nu^*, \xi^*)}, \frac{d(\nu^*, \xi^*) + d(\xi^*, \nu^*)}{2s}, d(\nu^*, \nu^*), d(\xi^*, \xi^*), d(\nu^*, \xi^*)\} \\ &= \max\{0, \frac{d(\nu^*, \xi^*)}{s}, 0, 0, d(\nu^*, \xi^*)\} \\ &= d(\nu^*, \xi^*). \end{aligned} \tag{31}$$

From equation (30), we have

$$\phi(d(\nu^*, \xi^*)) = \phi(d(S\nu^*, S\xi^*)) \leq \phi(d(\nu^*, \xi^*)) - \psi(d(\nu^*, \xi^*)) < \phi(d(\nu^*, \xi^*)), \tag{32}$$

which is a contradiction under (32). Hence, $\nu^* = \xi^*$. □

Now, we have the results below, which are the generalizations of Theorems 2.1 & 2.2 of [30] and the Corollaries 2.1 & 2.2 of [8] in a b-metric space.

Corollary 3.4. *Let $(P, d, s, \leq)$ be a partially ordered b-metric space with a parameter $s > 1$. Suppose $S, f : P \to P$ are two continuous mappings such that*

(C_1). *for some $\psi \in \Psi$ and $\phi \in \Phi$ with*

$$\phi(sd(S\nu, S\xi)) \leq \phi(M_f(\nu, \xi)) - \psi(M_f(\nu, \xi)), \tag{33}$$

for any ν, $\xi \in P$ such that $f\nu \leq f\xi$ and

$$\begin{aligned} M_f(\nu, \xi) = \max\{&\frac{d(f\nu, S\nu)\ d(f\xi, S\xi)}{1 + d(f\nu, f\xi)}, \frac{d(f\nu, S\xi) + d(f\xi, S\nu)}{2s}, \\ &d(f\nu, S\nu), d(f\xi, S\xi), d(f\nu, f\xi)\}, \end{aligned} \tag{34}$$

(C_2). *$SP \subset fP$ and fP is a complete subspace of P,*

(C_3). *S is a monotone f-non decreasing mapping,*

(C_4). *S and f are compatible.*

If for some $\nu_0 \in P$ such that $f\nu_0 \leq S\nu_0$, then S and f have a coincidence point in P.

Proof. By using lemma 2.14, we obtain a complete subspace fE of P, where $E \subset P$ and f is one-to-one self mapping on P. By Corollary 2.1 of [30], we have a sequence $\{f\nu_n\} \subset fE$ for some $\nu_0 \in E$ with $f\nu_{n+1} = S\nu_n = g(f\nu_n)$, for $n \geq 0$, where g is a self-mapping on fE with $g(f\nu) = S\nu$, $\nu \in E$. Therefore, from the hypotheses we have

$$\phi(s\ d(g(f\nu), g(f\xi))) \leq \phi(M_f(\nu, \xi)) - \psi(M_f(\nu, \xi)), \tag{35}$$

for all ν, $\xi \in P$ with $f\nu \leq f\xi$ and,

$$M_f(\nu,\xi) = \max\{\frac{d(f\nu, g(f\nu))\; d(f\xi, g(f\xi))}{1 + d(f\nu, f\xi)}, \frac{d(f\nu, g(f\xi)) + d(f\xi, g(f\nu))}{2s} \\ d(f\nu, g(f\nu)), d(f\xi, g(f\xi)), d(f\nu, f\xi)\}. \tag{36}$$

The similar argument from Theorem 3.1, we have a Cauchy sequence $\{f\nu_n\}$, which converges for some $v \in fE$. Thus the compatibility of S and f, we have

$$\lim_{n\to+\infty} d(f(S\nu_n), S(f\nu_n)) = 0. \tag{37}$$

Further, the triangular inequality of a b-metric we have

$$d(Sv, fv) \leq sd(Sv, S(f\nu_n)) + s^2 d(S(f\nu_n), f(S\nu_n)) + s^2 d(f(S\nu_n), fv). \tag{38}$$

Therefore, we arrive at $d(Sv, fv) = 0$ as $n \to +\infty$ in (38). Hence, v is a coincidence point of S and f in P. □

Replace the condition, weakly compatible instead of $(C4)$ in Corollary 3.4, we obtaining the following result.

Corollary 3.5. *If P has the property in Corollary 3.4 instead of the compatibility of S, f that, for any nondecreasing sequence $\{f\nu_n\} \subset P$ such that $\lim_{n\to+\infty} f\nu_n = f\nu$ implies that $f\nu_n \leq f\nu$ for $n \in \mathbb{N}$, that is $f\nu = \sup f\nu_n$, then S and f have a common fixed point in P, if for some coincidence point μ of S, f with $f\mu \leq f(f\mu)$. Furthermore, the set of common fixed of S, f is well ordered if and only if S and f have one and only one common fixed point.*

Proof. It is obvious from Corollary 3.4 and Theorem 3.2 that S and f have a coincidence point in P, as $f\mu = g(f\mu) = S\mu$ for some μ in P.

Next, assume that a pair of self mappings (S, f) is weakly compatible and let ϑ in P is such that $\vartheta = S\mu = f\mu$. Then $S\vartheta = S(f\mu) = f(S\mu) = f\vartheta$. Hence,

$$\begin{aligned} M(\mu,\vartheta) &= \max\{\frac{d(f\mu, S\mu)\; d(f\vartheta, S\vartheta)}{1 + d(f\mu, f\vartheta)}, \frac{d(f\mu, S\vartheta) + d(f\vartheta, S\mu)}{2s}, d(f\mu, S\mu), \\ &\qquad d(f\vartheta, S\vartheta), d(f\mu, f\vartheta)\} \\ &= \max\{0, \frac{d(S\mu, S\vartheta)}{s}, 0, 0, d(S\mu, S\vartheta)\} \\ &= d(S\mu, S\vartheta), \end{aligned} \tag{39}$$

and thus from contraction condition, we have

$$\phi(d(S\mu, S\vartheta)) \leq \phi(M(\mu,\vartheta)) - \psi(M(\mu,\vartheta)) \leq \phi(d(S\mu, S\vartheta)) - \psi(d(S\mu, S\vartheta)). \tag{40}$$

Hence, we get $d(S\mu, S\vartheta) = 0$ by the property of ψ. Therefore, $S\vartheta = f\vartheta = \vartheta$.

Eventually, by following Theorem 3.3, we deduce that S and f have one and only one common fixed point if and only if the set of common fixed points of S and f is well ordered. □

We illustrate the usefulness of the obtained results in different cases such as continuity and discontinuity of a metric d in a space P as follows.

Example 3.6. *Define a metric $d : P \to P$ as below and $\leq$ is an usual order in P, where $P = \{1, 2, 3, 4, 5\}$*

$$\begin{aligned} d(\nu,\xi) &= d(\xi,\nu) = 0,\ \ if\ \nu,\xi = 1,2,3,4,5\ and\ \nu = \xi, \\ d(\nu,\xi) &= d(\xi,\nu) = 1,\ \ if\ \nu,\xi = 1,2,3,4\ and\ \nu \neq \xi, \\ d(\nu,\xi) &= d(\xi,\nu) = 6,\ \ if\ \nu = 1,2,3\ and\ \xi = 5, \\ d(\nu,\xi) &= d(\xi,\nu) = 12,\ \ if\ \nu = 4\ and\ \xi = 5. \end{aligned}$$

Define a mapping $S : P \to P$ by $S1 = S2 = S3 = S4 = 1, S5 = 3$ and let $\phi(t) = \frac{t}{2}$, $\psi(t) = \frac{t}{3}$ for $t \in [0, +\infty)$. Then S has a fixed point in P.

Proof. It is apparent that, $(P, d, s, \leq)$ is a complete partially ordered b-metric space for $s = 2$. Consider the possible cases for ν, ξ in P.
Case 1. Suppose $\nu, \xi \in \{1, 2, 3, 4\}$ and $\nu < \xi$ then

$$\phi(2d(S\nu, S\xi)) = 0 \leq \phi(M(\nu, \xi)) - \psi(M(\nu, \xi)).$$

Case 2. Suppose that $\nu \in \{1, 2, 3, 4\}$ and $\xi = 5$, then $d(S\nu, S\xi) = d(1, 3) = 1$, $M(4, 5) = 12$ and $M(\nu, 5) = 6$, for $\nu \in \{1, 2, 3\}$. Therefore, we have the following inequality,

$$\phi(2d(S\nu, S\xi)) \leq \frac{M(\nu,\xi)}{6} = \phi(M(\nu, \xi)) - \psi(M(\nu, \xi)).$$

Thus, the condition (3) of Theorem 3.1 and Theorem 3.2 holds. Furthermore, the remaining assumptions in Theorem 3.1 and Theorem 3.2 are fulfilled. Hence, S has a fixed point in P as Theorem 3.1 and Theorem 3.2 is appropriate to S, ϕ, ψ and $(P, d, s, \leq)$. □

Example 3.7. *A metric $d : P \to P$, where $P = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\frac{1}{n}, ...\}$ with usual order $\leq$ is defined as follows*

$$d(\nu,\xi) = \begin{cases} 0 & ,\ if\ \nu = \xi \\ 1 & ,\ if\ \nu \neq \xi \in \{0,1\} \\ |\nu - \xi| & ,\ if\ \nu,\xi \in \{0, \frac{1}{2n}, \frac{1}{2m} : n \neq m \geq 1\} \\ 5 & ,\ otherwise. \end{cases}$$

A mapping $S : P \to P$ is such that $S0 = 0, S\frac{1}{n} = \frac{1}{9n}$ for all $n \geq 1$ and let $\phi(t) = t$, $\psi(t) = \frac{3t}{4}$ for $t \in [0, +\infty)$. Then S has a fixed point in P.

Proof. It is obvious that for $s = \frac{9}{2}$, $(P, d, s, \leq)$ is a complete partially ordered b-metric space and also by definition, d is discontinuous b-metric space. Now, for $\nu, \xi \in P$ with $\nu < \xi$, then consider the following possible cases:
Case 1. If $\nu = 0$ and $\xi = \frac{1}{n}$, $n \geq 1$, then $d(S\nu, S\xi) = d(0, \frac{1}{9n}) = \frac{1}{9n}$ and $M(\nu, 1) = 5$ and, $M(\nu, \xi) = \frac{1}{n}$ for $\xi = \frac{1}{n}$, $n > 1$. Therefore, we have

$$\phi\left(\frac{9}{2}d(S\nu, S\xi)\right) \leq \frac{M(\nu,\xi)}{2} = \phi(M(\nu, \xi)) - \psi(M(\nu, \xi)).$$

Case 2. If $\nu = \frac{1}{m}$ and $\xi = \frac{1}{n}$ with $m > n \geq 1$, then

$$d(S\nu, S\xi) = d(\frac{1}{9m}, \frac{1}{9n}) \text{ and } M(\nu,\xi) \geq \frac{1}{n} - \frac{1}{m} \text{ or } M(\nu,\xi) = 5.$$

Therefore,

$$\phi\left(\frac{9}{2}d(S\nu, S\xi)\right) \leq \frac{M(\nu,\xi)}{2} = \phi(M(\nu, \xi)) - \psi(M(\nu, \xi)).$$

Hence, the condition (3) of Theorem 3.1 and the remaining assumptions are all satisfied. Thus, S has a fixed point in P. □

Example 3.8. *Let $P = C[0,1]$ be the set of all continuous functions. Let us define a b-metric d on P by*

$$d(\theta_1, \theta_2) = \sup_{t\in[0,1]} \{|\theta_1(t) - \theta_2(t)|^2\}$$

for all $\theta_1, \theta_2 \in P$ with partial order $\leq$ defined by $\theta_1 \leq \theta_2$ if $0 \leq \theta_1(t) \leq \theta_2(t) \leq 1$, for all $t \in [0,1]$. Let $S : P \to P$ be a mapping defined by $S\theta = \frac{\theta}{5}, \theta \in P$ and the two altering distance functions as $\phi(t) = t$, $\psi(t) = \frac{t}{3}$, for any $t \in [0, +\infty]$. Then S has a unique fixed point in P.

Proof. It is clear that $(P, d, s, \leq)$ is a complete partially ordered b-metric space with parameter $s = 2$ and fulfill all conditions of Theorem 3.1 and Theorem 3.2. Furthermore, for any $\theta_1, \theta_2 \in P$, the function $\min(\theta_1, \theta_2)(t) = \min\{\theta_1(t), \theta_2(t)\}$ is also continuous and all the conditions of Theorem 3.3 are satisfied. Hence, S has a unique fixed point $\theta = 0 \in P$. □

Corollary 3.9. *Let $(P, d, s, \leq)$ be a complete partially ordered b-metric space with parameter $s > 1$. Let $S : P \to P$ be a continuous, nondecreasing mapping with regards to $\leq$. If there exists $k \in [0, 1)$ and for any $\nu, \xi \in P$ with $\nu \leq \xi$ such that*

$$d(S\nu, S\xi) \leq \frac{k}{s} \max\{\frac{d(\nu, S\nu)\ d(\xi, S\xi)}{1 + d(\nu, \xi)}, \frac{d(\nu, S\xi) + d(\xi, S\nu)}{2s}, d(\nu, S\nu), d(\xi, S\xi), d(\nu, \xi)\}, \tag{41}$$

then S has a fixed point in P, if there exists $\nu_0 \in P$ with $\nu_0 \leq S\nu_0$.

Proof. Set $\phi(t) = t$ and $\psi(t) = (1 - k)t$, for all $t \in (0, +\infty)$ in Theorem 3.1. □

Corollary 3.10. *In Corollary 3.9, if P has a property that, the sequence $\{\nu_n\}$ is a nondecreasing such that $\nu_n \to v$, implies that $\nu_n \leq v$, for all $n \in \mathbb{N}$, i.e., $v = \sup \nu_n$ then a non continuous mapping S has a fixed point in P.*

Proof. The proof follows from Theorem 3.2. □

Corollary 3.11. *In addition to the hypotheses of Corollary 3.9 (or Corollary 3.10), suppose that for every $v, \xi \in P$, there exists $w \in P$ such that $w \leq v$ and $w \leq \xi$, then one can obtains the uniqueness of a fixed point of the mapping S in P.*

Proof. The proof follows from Theorem 3.3. □

Acknowledgements

The authors do thankful to the editor and anonymous referees for their valuable suggestions and comments which improved the quality and presentation of this paper.

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