Hacettepe Journal of Mathematics and Statistics
Volume 32 (2003), 13 – 15

SOME CHARACTERIZATIONS OF LIFTING MODULES IN TERMS OF PRERADICALS

Nil Orhan*

Received 27. 05. 2003 : Accepted 24. 10. 2003

Abstract

Some results on the direct sum of two certain lifting modules are given in terms of left preradicals.

Keywords: Lifting modules, left preradicals.

2000 AMS Classification: 16 D 99, 16 L 99

1. Preliminaries

Throughout this paper, all rings will have identities and all modules will be unital right modules. Let M be a module. Any small submodule K of M is denoted by $K \ll M$. The socle of M is denoted by $\mathrm{Soc}(M)$, the Jacobson radical of M by $\mathrm{Rad}(M)$.

Let M be a module. M is called *lifting* if for every submodule N of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2 \ll M_2$. By [3, 41.12], M is lifting if and only if M is amply supplemented and every supplement submodule of M is a direct summand of M. Recall that any submodule N of M is called a *supplement* of any submodule K of M if $M = N + K$ and $N \cap K \ll N$ (in this case, N is also called a *supplement* submodule of M) and the module M is called *amply supplemented* if for any submodules A and B of M with $M = A + B$, there exists a supplement X of A with $X \subseteq B$. M is called *supplemented* if every submodule of M has a supplement in M. It is well-known that a direct sum of lifting modules need not be a lifting module in general (see [1]). In this note we characterize the direct sum of two lifting modules in terms of left preradicals.

We begin by explaining left preradicals. A functor r from the category of right R–modules to itself is called a *left preradical* if it has the following two properties

(i) $r(M)$ is a submodule of M for every right R–module M,
(ii) $f(r(M)) \subseteq r(M')$ for every homomorphism $f : M \longrightarrow M'$ between right R–modules M and M'.

*Department of Mathematics, Hacettepe University, 06532 Beytepe, Ankara, Turkey
E-mail: `nilorhan@hacettepe.edu.tr`

It is clear that the socle and the Jacobson radical are left preradicals.

2. Lifting Modules in terms of Left Preradicals

2.1. Proposition. *Let R be a ring, let r be a left preradical in the category of right R–modules and M a lifting module. Then M has a decomposition $M = M_1 \oplus M_2$ with M_1 and M_2 lifting modules, $r(M_1) = M_1$ and $r(M_2) \ll M_2$. In the case that $r(M_2) = 0$, M_2 is M_1–projective.*

Proof. Since $r(M)$ is a submodule of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq r(M)$ and $r(M) \cap M_2 \ll M_2$. Now, $r(M) = r(M_1) \oplus r(M_2)$ implies that $r(M) \cap M_2 = r(M_2) \oplus (M_2 \cap r(M_1)) = r(M_2) \ll M_2$. Also, $r(M) \cap M_1 = M_1 = r(M_1) \oplus (M_1 \cap r(M_2)) = r(M_1)$. By [2, Lemma 4.7], M_1 and M_2 are lifting. Hence the first part is proved.

For the second part, let N be a submodule of M with $M = N + M_1$. Since M is amply supplemented, there exists a submodule N' of M such that $M = N' + M_1$, $N' \cap M_1 \ll N'$ and $N' \subseteq N$. Note that N' is a direct summand of M. Assume $M = N' \oplus K'$ for some submodule K' of M. Now, $r(M) = r(M_1) = M_1 = r(N') \oplus r(K')$ and $r(N') = N' \cap M_1$. Therefore, $N' \cap M_1$ is a direct summand of M. On the other hand, $r(N') \ll M$. Thus $r(N') = 0$, namely $M = N' \oplus M_1$. Hence M_2 is M_1–projective by [3, 41.14]. □

2.2. Corollary. *Let M be a lifting module. Then $M = M_1 \oplus M_2$ is a direct sum of lifting modules M_1 and M_2 such that $\mathrm{Rad}(M_1) = M_1$ and $\mathrm{Rad}(M_2) \ll M_2$. In the case that $\mathrm{Rad}(M_2) = 0$, M_2 is M_1–projective.*

2.3. Corollary. *Let M be a lifting module. Then $M = M_1 \oplus M_2$ is a direct sum of lifting modules M_1 and M_2 such that $\mathrm{Soc}(M_1) = M_1$ and $\mathrm{Soc}(M_2) \ll M_2$. In the case that $\mathrm{Soc}(M_2) = 0$, M_2 is M_1–projective.*

Let M_1 and M_2 be modules. Then M_1 is *small* M_2–projective if every homomorphism $f : M_1 \longrightarrow M_2/A$, where A is a submodule of M_2 and $\mathrm{Im} f \ll M_2/A$, can be lifted to a homomorphism $\varphi : M_1 \longrightarrow M_2$ (see [1]).

A module M is said to have the *finite exchange property* if, for every finite index set I, whenever $M \oplus N = \oplus_{i \in I} A_i$ for modules N and A_i, $i \in I$, then $M \oplus N = M \oplus (\oplus_{i \in I} B_i)$ for submodules B_i of A_i, $i \in I$ (see [2]).

2.4. Theorem. *Let R be a ring, let r be a left preradical in the category of right R–modules, M_1 a module with the finite exchange property satisfying $r(M_1) = M_1$, and M_2 a module with $r(M_2) = 0$. Then $M = M_1 \oplus M_2$ is a lifting module if and only if M is amply supplemented, M_1 and M_2 are lifting, M_1 is small M_2–projective and M_2 is M_1–projective.*

Proof. The necessity follows from [1, Proposition 3.3] and Proposition 2.1. The sufficiency follows from [1, Theorem 2.8]. □

2.5. Corollary. *Let M_1 be a module with the finite exchange property satisfying $\mathrm{Rad}(M_1) = M_1$, and M_2 a module with $\mathrm{Rad}(M_2) = 0$. Then $M = M_1 \oplus M_2$ is a lifting module if and only if M is amply supplemented, M_1 and M_2 are lifting, M_1 is small M_2–projective and M_2 is M_1–projective.*

2.6. Corollary. *Let M_1 be a module with the finite exchange property satisfying $\mathrm{Soc}(M_1) = M_1$, and M_2 a module with $\mathrm{Soc}(M_2) = 0$. Then $M = M_1 \oplus M_2$ is a lifting module if and only if M is amply supplemented, M_1 and M_2 are lifting, M_1 is small M_2–projective and M_2 is M_1–projective.*

Now, let us consider another preradical Z^*. Note that for a right R–module M, $Z^*(M) = \{m \in M : mR \ll E(mR)\}$, where $E(mR)$ is the injective hull of mR. Then we have the following result.

2.7. Corollary. *Let M_1 be a module with finite exchange property and $Z^*(M_1) = M_1$, M_2 be a module with $Z^*(M_2) = 0$. Then $M = M_1 \oplus M_2$ is a lifting module if and only if M is amply supplemented, M_1 and M_2 are lifting, M_1 is small M_2–projective and M_2 is M_1–projective.*

Finally, we give the following fact:

2.8. Theorem. *Let R be a ring, let r be a left preradical in the category of right R–modules and let $M = M_1 \oplus M_2$ have the finite exchange property and satisfy $r(M_1) = 0$. If M is lifting, then M_1 is $r(M_2) = r(M)$-projective.*

To prove the above theorem we need the following lemma.

2.9. Lemma. *Let $M = M_1 \oplus M_2$ be a supplemented module. Then the following are equivalent.*

(i) *M_2 is small M_1-projective.*

(ii) *M_2 is X-projective, for every small submodule X of M_1.*

Proof. $(i) \Rightarrow (ii)$: Let $A \leq X \ll M_1$ and $f : M_2 \to X/A$ be a homomorphism. Clearly, $Imf \ll M_1/A$. By assumption, there exists a homomorphism $g : M_2 \to M_1$ such that $\pi g = if$, where $i : X/A \to M_1/A$ is the inclusion map and $\pi : M_1 \to M_1/A$ is the natural epimorphism. Since $g(M_2) \leq X$, f can be lifted to g.

$(ii) \Rightarrow (i)$: Let $A \leq M_1$ and $\alpha : M_2 \to M_1/A$ be a homomorphism such that $Im\alpha = T/A \ll M_1/A$. Since M is supplemented, A has a supplement B in M_1. Let $B/(A \cap B) = (T \cap B)/(A \cap B) + L/(A \cap B)$ for any submodule $L/(A \cap B)$ of $B/(A \cap B)$. Then $M_1 = A + B = A + (T \cap B) + L = T + L$. Since $T/A \ll M_1/A$, $M_1/A = (L + A)/A$ and hence $M_1 = L + A$. By the minimality of B in M_1, $L = B$. Therefore $(T \cap B)/(A \cap B) \ll B/(A \cap B)$. Now $T \cap B \ll B$. Let $X = T \cap B$. Since $T = A + X$, we define the epimorphism $\varphi : X \to T/A$ such that $\varphi(x) = x + A$, where $x \in X$. So there exists a homomorphism $\beta : M_2 \to X$ such that $\varphi\beta = \alpha$. Hence α can be lifted to the homomorphism $i\beta$, where $i : X \to M_1$ is the inclusion map. □

Proof of Theorem 2.8:

By [1, Proposition 3.3], M_1 is small M_2-projective. By Proposition 2.1, M_2 has a decomposition $M_2 = M_{21} \oplus M_{22}$ with M_{21} and M_{22} lifting modules, $r(M_{21}) = M_{21}$ and $r(M_{22}) \ll M_{22}$. By Lemma 2.9, M_1 is $r(M_{22})$-projective. Also by Theorem 2.4, M_1 is M_{21}-projective. Thus M_1 is $r(M_{22}) \oplus M_{21} = r(M_{22}) \oplus r(M_{21}) = r(M_2) = r(M)$-projective. □

References

[1] Keskin, D. *On lifting modules*, Comm. Algebra, **28** (7), 3427–3440, 2000.

[2] Mohamed, S. H. and Müller, B. J. *Continuous and discrete modules*, London Math. Soc. LNS **147** (Cambridge Univ. Press, Cambridge 1990).

[3] Wisbauer, R. *Foundations of module and ring theory*, (Gordon and Breach, Philadelphia 1991).