

AN OVERVIEW OF INTUITIONISTIC FUZZY SUPRATOPOLOGICAL SPACES

Necla Turanli*

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Abstract

The purpose of this paper is to present some fundamental results on the so called “intuitionistic fuzzy supratopological spaces”, and then give some introductory results about several connectedness concepts and a notion of compactness in these spaces. The product of two intuitionistic fuzzy supratopological spaces is also considered and some results on the productivity of connectedness and compactness are presented.

Keywords: Intuitionistic fuzzy set, Intuitionistic fuzzy topology, Intuitionistic fuzzy supratopology, Intuitionistic fuzzy supratopological space, Fuzzy supracontinuity, Fuzzy super supraconnectedness, Fuzzy C_5 -supraconnectedness, Fuzzy strong supraconnectedness, Fuzzy C_s -supraconnectedness, Intuitionistic fuzzy supracompactness.

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1. Introduction

First we give the concept of “intuitionistic fuzzy set” defined by Atanassov as a generalization of the concept of fuzzy set given by Zadeh [12].

1.1. Definition. [1, 2] Let X be a nonempty set. An intuitionistic fuzzy set (IFS for short) on X is an object of the form

$$A = \{ \langle x, \mu_A(x), \gamma_A(x) \rangle : x \in X \}$$

where the fuzzy sets $\mu_A : X \rightarrow \mathbb{I}$ and $\gamma_A : X \rightarrow \mathbb{I}$, $\mathbb{I} = [0, 1]$, denote, respectively, the membership function and the nonmembership function of A , and satisfy $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$. An IFS $A = \{ \langle x, \mu_A(x), \gamma_A(x) \rangle : x \in X \}$ can be written in the form $A = \langle x, \mu_A, \gamma_A \rangle$, or simply $A = \langle \mu_A, \gamma_A \rangle$.

1.2. Definition. [2] Let $A = \{ \langle x, \mu_A(x), \gamma_A(x) \rangle : x \in X \}$ and $B = \{ \langle x, \mu_B(x), \gamma_B(x) \rangle : x \in X \}$ be IFS's on X . Then

- (a) $\bar{A} = \{ \langle x, \gamma_A(x), \mu_A(x) \rangle : x \in X \}$ (the complement of A);
- (b) $A \cap B = \{ \langle x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x) \rangle : x \in X \}$ (the meet of A and B);

*Hacettepe University, Department of Mathematics Education, 06532 Beytepe, Ankara, Turkey. E-mail : turanli@hacettepe.edu.tr

- (c) $A \cup B = \{ \langle x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x) \rangle : x \in X \}$ (the join of A and B);
- (d) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for each $x \in X$;
- (e) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$;
- (f) $\sqcup A = \{ \langle x, \mu_A(x), 1 - \mu_A(x) \rangle : x \in X \}$ and $\langle \rangle A = \{ \langle x, 1 - \gamma_A(x), \gamma_A(x) \rangle : x \in X \}$;
- (g) $\underline{1} = \{ \langle x, 1, 0 \rangle : x \in X \}$, $\underline{0} = \{ \langle x, 0, 1 \rangle : x \in X \}$

Now we shall give some fundamental definitions concerning intuitionistic fuzzy supratopological spaces:

1.3. Definition. [10] A family τ of IFS's on X is called an *intuitionistic fuzzy supratopology* (IFST for short) on X if $\underline{0} \in \tau$, $\underline{1} \in \tau$ and τ is closed under arbitrary suprema. Then we call the pair (X, τ) an *intuitionistic fuzzy supratopological space* (IFSTS for short). Each member of τ is called an *intuitionistic fuzzy supraopen set* and the complement of an intuitionistic fuzzy supraopen set is called an *intuitionistic fuzzy supraclosed set*. The *intuitionistic fuzzy supraclosure* of an IFS A is denoted by $s\text{-cl}(A)$. Here $s\text{-cl}(A)$ is the intersection of all intuitionistic fuzzy supraclosed sets containing A . The *intuitionistic fuzzy sinterior* of A will be denoted by $s\text{-int}(A)$. Here, $s\text{-int}(A)$ is the union of all intuitionistic fuzzy supraopen sets contained in A .

1.4. Example. Let $X = \{a, b\}$ and

$$A = \langle x, (0.3, 0.4), (0.4, 0.5) \rangle, B = \langle x, (0.4, 0.2), (0.5, 0.3) \rangle.$$

Then $\tau = \{ \underline{0}, \underline{1}, A, B, A \cup B \}$ is an IFST on X .

Now let $IF(X)$ denote the family of all intuitionistic fuzzy sets in X .

1.5. Proposition. *If β is an intuitionistic fuzzy supratopology on X , then*

$$\tau_\beta = \{ T \in IF(X) : (\forall A)(A \in \beta \implies T \cap A \in \beta) \}$$

is an intuitionistic fuzzy topology on X and $\tau_\beta \subseteq \beta$.

Proof. (1). $A \in \beta \implies \underline{0} \cap A = \underline{0} \in \beta$ and $\underline{1} \cap A = A \in \beta$. Hence $\underline{0}, \underline{1} \in \tau_\beta$.

(2). Let $\{A_i\} \subseteq \tau_\beta$ and $A \in \beta$. Since

$$\begin{aligned} \left(\bigcup A_i \right) \cap A &= \bigcup (A_i \cap A) \in \beta, \\ \left\langle \bigvee \mu_{A_i}, \bigwedge \gamma_{A_i} \right\rangle \cap \langle \mu_A, \gamma_A \rangle &= \left\langle \bigvee (\mu_{A_i} \wedge \mu_A), \bigwedge (\gamma_{A_i} \vee \gamma_A) \right\rangle \\ &= \bigvee \langle \mu_{A_i} \wedge \mu_A, \gamma_{A_i} \vee \gamma_A \rangle \\ &= \bigcup (A_i \cap A) \in \beta, \end{aligned}$$

we have $\bigcup A_i \in \tau_\beta$.

(3). Let $A_1, A_2, A \in \tau_\beta$. Then

$$\begin{aligned} (A_1 \cap A_2) \cap A &= \langle \mu_{A_1} \wedge \mu_{A_2}, \gamma_{A_1} \vee \gamma_{A_2} \rangle \cap \langle \mu_A, \gamma_A \rangle \\ &= \langle (\mu_{A_1} \wedge \mu_{A_2}) \wedge \mu_A, (\gamma_{A_1} \vee \gamma_{A_2}) \vee \gamma_A \rangle \\ &= \langle \mu_{A_1} \wedge (\mu_{A_2} \wedge \mu_A), \gamma_{A_1} \vee (\gamma_{A_2} \vee \gamma_A) \rangle \\ &= A_1 \cap (A_2 \cap A) \end{aligned}$$

Since $A_2 \cap A \in \beta$ and $A_1 \in \tau_\beta$, $A_1 \cap (A_2 \cap A) \in \beta$. Hence, $A_1 \cap A_2 \in \tau_\beta$. Thus τ_β is an intuitionistic fuzzy topology on X . If $T \in \tau_\beta$ then $T \in \beta$, since $\underline{1} \in \beta$ and $T \cap \underline{1} = T \in \beta$. \square

Let (X, τ) be an intuitionistic fuzzy supratopological space. Then $\tau_1 = \{\mu_A : A \in \tau\}$ is a fuzzy supratopological space on X in the sense of El-Monsef and Ramadan [8], and $\tau_2^c = \{\gamma_A : A \in \tau\}$ is the family of all fuzzy supraclosed sets of the fuzzy supratopological space $\tau_2 = \{1 - \gamma_A : A \in \tau\}$ on X . Furthermore, in (X, τ_1) and (X, τ_2) , the fuzzy sets μ_A and γ_A are fuzzy supraopen and fuzzy supraclosed sets, respectively. On the other hand, since τ_1 is a fuzzy supratopology on X ,

$$T_{\tau_1} = \{P \in \mathbb{I}^X : \mu_A \in \tau_1 \implies P \wedge \mu_A \in \tau_1\}$$

is a fuzzy topology on X , and $T_{\tau_1} \subseteq \tau_1$ (see Turanlı and Çoker [9]).

1.6. Definition. [6] Let (X, τ) be an IFSTS. The intuitionistic fuzzy set A is called

- (a) *intuitionistic fuzzy semiopen* in X iff there exists $B \in \tau$ such that $B \subseteq A \subseteq \text{s-cl}(B)$.
- (b) *intuitionistic fuzzy semiclosed* in X if there exists an intuitionistic fuzzy supraclosed set B in X such that $\text{s-int}(B) \subseteq A \subseteq B$.

Arbitrary unions of intuitionistic fuzzy semiopen sets are again intuitionistic fuzzy semiopen sets. But the intersection of two intuitionistic fuzzy semiopen sets is not necessarily an intuitionistic fuzzy semiopen set (cf. [6]). Therefore the family of all intuitionistic fuzzy semiopen sets in (X, τ) is again an intuitionistic fuzzy supratopology on X .

1.7. Definition. Let A be an IFS in an intuitionistic fuzzy supratopological space.

- (a) If $\text{s-int}(\text{s-cl}(A)) = A$, then we say that A is an *intuitionistic fuzzy regular supraopen set* in X ,
- (b) If $\text{s-cl}(\text{s-int}(A)) = A$, then we say that A is an *intuitionistic fuzzy regular supraclosed set* in X .

The union of two intuitionistic fuzzy regular supraopen sets need not be an intuitionistic fuzzy regular supraopen set; hence the family of all intuitionistic regular supraopen sets in an IFSTS is not an IFST on X , in general.

1.8. Definition. Let (X, τ) be an intuitionistic fuzzy supratopological space. An IFS $A \in IF(X)$ is called

- (a) *intuitionistic fuzzy semi-supraopen* iff $A \subseteq \text{s-cl}(\text{s-int}(A))$,
- (b) *intuitionistic fuzzy α -supraopen* iff $A \subseteq \text{s-int}(\text{s-cl}(\text{s-int}(A)))$,
- (d) *intuitionistic fuzzy β -supraopen* iff $A \subseteq \text{s-cl}(\text{s-int}(\text{s-cl}(A)))$,
- (d) *intuitionistic fuzzy presupraopen* iff $A \subseteq \text{s-int}(\text{s-cl}(A))$,

1.9. Example. Let

$$X = \{a, b\} \quad E = \langle x, (0.3, 0.3), (0.3, 0.3) \rangle, \quad F = \langle x, (0.4, 0.4), (0.4, 0.4) \rangle.$$

The family $\tau = \{\underline{0}, \underline{1}, E, F, E \cup F\}$ is an IFST on X . Here E and F are intuitionistic fuzzy regular supraopen sets.

1.10. Definition. [4] Let $A = \langle x, \mu_A, \gamma_A \rangle$ and $B = \langle x, \mu_B, \gamma_B \rangle$ be two intuitionistic fuzzy sets in X . A is said to be *quasi-coincident with B* (written $A q B$) iff there exists an element x in X such that $\mu_A(x) > \gamma_B(x)$ or $\gamma_A(x) < \mu_B(x)$.

1.11. Example. Let $X = \{a, b\}$ and take the IFS's

$$A = \langle x, (0.4, 0.4), (0.4, 0.3) \rangle \text{ and } B = \langle x, (0.3, 0.2), (0.5, 0.5) \rangle.$$

Since, for all x in X , $\mu_A(x) > \gamma_B(x)$ or $\gamma_A(x) < \mu_B(x)$, $A q B$ follows.

1.12. Proposition. [4] If A and B are intuitionistic fuzzy sets in X , then

- (a) $A \bar{q} B \iff A \subseteq \bar{B}$,
- (b) $A q B \iff A \not\subseteq \bar{B}$.

Here \bar{q} denotes the negation of the relation q .

1.13. Corollary. $A \not\subseteq \bar{B} \implies A \cap B \neq \emptyset$.

1.14. Definition. Let (X, τ) be an intuitionistic fuzzy supratopological space. The intuitionistic fuzzy sets A and B are said to be *weakly supraseparated*, if $\text{s-cl}(A) \subseteq \bar{B}$ and $\text{s-cl}(B) \subseteq \bar{A}$.

1.15. Definition. [5] Let X and Y be two nonempty sets and $f : X \rightarrow Y$ a function.

(a) If $B = \{\langle y, \mu_B(y), \gamma_B(y) \rangle : y \in Y\}$ is an IFS in Y , then the *preimage of B under f* , denoted $f^{-1}(B)$, is the IFS in X defined by

$$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x) \rangle : x \in X\}.$$

(b) If $A = \{\langle x, \lambda_A(x), \eta_A(x) \rangle : x \in X\}$ is an IFS in X , then the *image of A under f* , denoted by $f(A)$, is the IFS in Y defined by

$$f(A) = \{\langle y, f(\lambda_A)(y), (1 - f(1 - \eta_A))(y) \rangle : y \in Y\}.$$

For the sake simplicity, the symbol $f_-(\eta_A)$ is often used to denote $1 - f(1 - \eta_A)$ [5].

1.16. Definition. Let (X, τ) and (Y, δ) be IFSTS's, and $f : (X, \tau) \rightarrow (Y, \delta)$ a function. Then f is said to be *fuzzy supracontinuous* if the preimage of each IFS in δ is in τ .

2. Types of Fuzzy Supraconnectedness in IFSTS's

Here we shall give several types of fuzzy connectedness in IFSTS's as a generalization of Turanlı-Çoker [9, 11]:

2.1. Definition. An intuitionistic fuzzy supratopological space (X, τ) is said to be *fuzzy strongly supraconnected* if X has no non-zero intuitionistic fuzzy supraclosed sets A and B in X such that $\mu_A + \mu_B \leq 1$ and $\gamma_A + \gamma_B \geq 1$.

2.2. Note. Since $\mu_A + \gamma_A \leq 1$ and $\mu_B + \gamma_B \leq 1$ it follows that $\gamma_B + \gamma_B \geq 1 \implies \mu_A + \mu_B \leq 1$. Hence in Definition 2.1 the condition $\mu_A + \mu_B \leq 1$ is redundant.

2.3. Theorem. Let (X, τ) be an IFSTS. Then the following are equivalent:

- (1) (X, τ) is fuzzy strongly supraconnected.
- (2) There exist no intuitionistic fuzzy supraopen sets A, B in X such that $A \neq \underline{1} \neq B$ and $\mu_A + \mu_B \geq 1$.
- (3) There exists no $C \in \tau$ satisfying $C \neq \underline{1}$, $\frac{1}{2} \leq \mu_C$, and there exist no $A, B \in \tau$ with $A \neq \underline{1} \neq B$ and $A \cup B = \underline{1}$.

Proof. (1) \implies (2). Trivial in view of Note 2.2.

(2) \implies (3). The second condition is a special case of (2) and the first condition is obtained by considering $A = B = C$.

(3) \implies (1). If (X, τ) is not fuzzy strongly supraconnected then we have $A, B \in \tau$ with $A \neq \underline{1} \neq B$ and $\mu_A + \mu_B \geq 1$. Then $C = A \cup B \in \tau$, $C \neq \underline{1}$ and $2\mu_C \geq \mu_A + \mu_B \geq 1$, which contradicts (3). \square

2.4. Example. Let

$$X = \{a, b\}, \quad A = \langle x, (0.3, 0.4), (0.4, 0.5) \rangle, \quad B = \langle x, (0.4, 0.5), (0.5, 0.3) \rangle.$$

Then $\tau = \{\underline{0}, \underline{1}, A, B, A \cup B\}$ is an intuitionistic fuzzy supratopology on X . There exist no intuitionistic fuzzy supraopen sets A, B in X such that $\mu_A + \mu_B \geq 1$, $\gamma_A + \gamma_B \leq 1$. Therefore (X, τ) is fuzzy strongly supraconnected.

2.5. Theorem. *Let (X, τ) , (Y, δ) be intuitionistic fuzzy supratopological spaces and $f : (X, \tau) \rightarrow (Y, \delta)$ a surjective fuzzy supracontinuous function. If (X, τ) is fuzzy strongly supraconnected, then (Y, δ) is also fuzzy strongly supraconnected.*

Proof. Suppose that X is fuzzy strongly supraconnected but that there exists $A, B \in \delta$ satisfying $A \neq 1_Y \neq B$ and $\mu_A + \mu_B \geq 1$. Then for all $x \in X$ we have

$$\mu_{f^{-1}(A)}(x) + \mu_{f^{-1}(B)}(x) = f^{-1}(\mu_A)(x) + f^{-1}(\mu_B)(x) = \mu_A(f(x)) + \mu_B(f(x)) \geq 1,$$

which contradicts Theorem 2.3 since $f^{-1}(A), f^{-1}(B) \in \tau$ and clearly $f^{-1}(A) \neq 1_X \neq f^{-1}(B)$. \square

2.6. Definition. Let (X, τ) be an IFSTS. X is said to be *fuzzy super supraconnected*, if there exist no intuitionistic fuzzy regular supraopen set A in X such that $\underline{\sim} \neq A \neq \underline{\sim}$.

2.7. Theorem. (cf. [11]) *The following properties are equivalent:*

- (X, τ) is fuzzy super supraconnected.
- For each intuitionistic fuzzy supraopen set $A \neq \underline{\sim}$ in X , $s\text{-cl}(A) = \underline{\sim}$.
- For each intuitionistic fuzzy supraclosed set $A \neq \underline{\sim}$ in X , $s\text{-int}(A) = \underline{\sim}$.
- There exist no intuitionistic fuzzy supraopen sets A and B in X such that $A \neq \underline{\sim} \neq B$ and $A \subseteq \overline{B}$.
- There exists no intuitionistic fuzzy supraopen sets A and B in X such that $A \neq \underline{\sim} \neq B$ and $B = s\text{-cl}(A)$, $A = s\text{-cl}(B)$.
- There exists no intuitionistic fuzzy supraclosed sets A and B in X such that $A \neq \underline{\sim} \neq B$ and $B = s\text{-int}(A)$, $A = s\text{-int}(B)$.

2.8. Definition. Let (X, τ) be an IFSTS.

- X is called *fuzzy C_s -supradisconnected* if there exist two nonzero IFS's A and B in (X, τ) such that A and B are weakly separated and $\underline{\sim} = A \cup B$.
- X is called *fuzzy C_s -supraconnected* if (X, τ) is not fuzzy C_s -supradisconnected.

2.9. Definition. Let (X, τ) be an IFSTS.

- X is said to be *fuzzy C_5 -supradisconnected* if there exists an intuitionistic fuzzy supraopen and fuzzy supraclosed set A in X such that $\underline{\sim} \neq A \neq \underline{\sim}$.
- X is called *fuzzy C_5 -supraconnected* if X is not fuzzy C_5 -supradisconnected.
- X is said to be *fuzzy supradisconnected* if there exist intuitionistic fuzzy supraopen sets $A \neq \underline{\sim}$ and $B \neq \underline{\sim}$ such that $\underline{\sim} = A \cup B$ and $A \cap B = \underline{\sim}$.
- X is called *fuzzy supraconnected* if X is not fuzzy supradisconnected.

2.10. Theorem. *Let (X, τ) be an IFSTS.*

- If (X, τ) is fuzzy supradisconnected, then (X, τ) is fuzzy C_5 -supradisconnected.
- If (X, τ) is fuzzy C_5 -supraconnected, then (X, τ) is fuzzy supraconnected.

Proof. Straightforward. \square

2.11. Theorem. *Let $f : (X, \tau) \rightarrow (Y, \Phi)$ be a surjective fuzzy supracontinuous function. If X is fuzzy supraconnected, then Y is also fuzzy supraconnected.*

Proof. Immediate. \square

2.12. Theorem. *The following are equivalent:*

- (1) (X, τ) is fuzzy C_5 -supraconnected.
- (2) There exist no intuitionistic fuzzy supraopen sets A, B in X such that $A \neq \underline{0} \neq B$ and $A = \overline{B}$.
- (3) There exist no intuitionistic fuzzy sets A and B in X such that $A \neq \underline{0} \neq B$, $B = \overline{A}$, $B = \overline{s\text{-cl}(A)}$ and $A = \overline{s\text{-cl}(B)}$.

Proof. Straightforward. □

2.13. Remark. There is no relation between fuzzy strong supraconnectedness and fuzzy C_5 -supraconnectedness in IFSTS's. In other words, each of these properties does not imply the other.

2.14. Example. Let

$$X = \{a, b\}, \quad A = \langle x, (0.3, 0.5), (0.6, 0.5) \rangle, \quad B = \langle x, (0.6, 0.3), (0.3, 0.6) \rangle.$$

Then $\tau = \{\underline{0}, \underline{1}, A, B, A \cup B\}$ is an IFSTS on X , (X, τ) is not fuzzy C_5 -supraconnected, but is fuzzy strongly supraconnected.

2.15. Example. Let

$$X = \{a, b\}, \quad A = \langle x, (0.5, 0.6), (0.4, 0.3) \rangle, \quad B = \langle x, (0.5, 0.4), (0.2, 0.4) \rangle.$$

Then $\tau = \{\underline{0}, \underline{1}, A, B, A \cup B\}$ is an IFSTS on X , and for the intuitionistic fuzzy supraopen sets A, B in X we have $\mu_A + \mu_B \geq 1$, $\gamma_A + \gamma_B \leq 1$, whence (X, τ) is not fuzzy strongly supraconnected. However, it is fuzzy C_5 -supraconnected.

2.16. Note. Let (X, β) be an IFSTS, τ_β the intuitionistic fuzzy topology on X defined in Proposition 1.5. Since $\tau_\beta \subseteq \beta$, if (X, β) has one of the connectedness properties defined above so does (X, τ_β) . The converse is false in general. For example if β is the fuzzy C_5 -supradisconnected IFST of Example 2.15 then $\tau_\beta = \{\underline{1}, \underline{0}, A \cup B\}$, which is fuzzy C_5 -supraconnected.

3. Products of Intuitionistic Fuzzy Supratopological Spaces

Let (X, τ) and (Y, δ) be two intuitionistic fuzzy supratopological spaces, $A \in IF(X)$ and $B \in IF(Y)$. The product of A and B is defined by:

$$A \times B = \{ \langle (x, y), \mu_A(x) \wedge \mu_B(y), \gamma_A(x) \vee \gamma_B(y) \rangle : (x, y) \in X \times Y \}.$$

The product intuitionistic fuzzy supratopology on $X \times Y$ is defined as the coarsest intuitionistic fuzzy supratopology on $X \times Y$ making the projections $\pi_1 : X \times Y \rightarrow X$, $\pi_1(x, y) = x$ and $\pi_2 : X \times Y \rightarrow Y$, $\pi_2(x, y) = y$ fuzzy supracontinuous. Since for $G \in \tau$,

$$\pi_1^{-1}(\mu_G)(x, y) = \mu_G(\pi_1(x, y)) = \mu_G(x),$$

and likewise $\pi_2^{-1}(\gamma_G)(x, y) = \gamma_G(x)$ we see that

$$\pi_1^{-1}(G) = \langle (x, y), \mu_G(x), \gamma_G(x) \rangle = G \times \underline{1}_Y.$$

In just the same way, for $H \in \delta$,

$$\pi_2^{-1}(H) = \langle (x, y), \mu_H(y), \gamma_H(y) \rangle = \underline{1}_X \times H.$$

Hence $\{G \times \underline{1}_Y \mid G \in \tau\} \cup \{\underline{1}_X \times H \mid H \in \delta\}$ is a base for the product intuitionistic fuzzy supratopology β on $X \times Y$. However, since the families $\{G \times \underline{1}_Y \mid G \in \tau\}$ and

$\{1_{\underline{X}} \times H \mid H \in \delta\}$ are both closed under arbitrary unions because τ and δ are intuitionistic fuzzy supratologies, we obtain

$$\beta = \{(G \times 1_{\underline{Y}}) \cup (1_{\underline{X}} \times H) \mid G \in \tau, H \in \delta\}.$$

3.1. Example. Let $X = \{a, b\}$ and

$$A = \langle x, (0.3, 0.4), (0.4, 0.5) \rangle, B = \langle x, (0.4, 0.2), (0.5, 0.3) \rangle.$$

Now $\tau = \{0_{\underline{X}}, 1_{\underline{X}}, A, B, A \cup B\}$ is an intuitionistic fuzzy supratology on X . Secondly, let $Y = \{a, b\}$ and

$$C = \langle y, (0.3, 0.2), (0.6, 0.5) \rangle, D = \langle y, (0.4, 0.1), (0.5, 0.3) \rangle.$$

Then $\delta = \{0_{\underline{Y}}, 1_{\underline{Y}}, C, D, C \cup D\}$ is an intuitionistic fuzzy supratology on Y . The product of these spaces is as follows:

$$\begin{aligned} \beta = \{ & \langle (x, y), \begin{pmatrix} 0.3 & 0.3 \\ 0.4 & 0.4 \end{pmatrix}, \begin{pmatrix} 0.4 & 0.4 \\ 0.5 & 0.5 \end{pmatrix} \rangle, \langle (x, y), \begin{pmatrix} 0.4 & 0.4 \\ 0.2 & 0.2 \end{pmatrix}, \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.3 \end{pmatrix} \rangle, \\ & \langle (x, y), \begin{pmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{pmatrix}, \begin{pmatrix} 0.4 & 0.4 \\ 0.3 & 0.3 \end{pmatrix} \rangle, \langle (x, y), \begin{pmatrix} 0.3 & 0.2 \\ 0.3 & 0.2 \end{pmatrix}, \begin{pmatrix} 0.6 & 0.5 \\ 0.6 & 0.5 \end{pmatrix} \rangle, \\ & \langle (x, y), \begin{pmatrix} 0.4 & 0.1 \\ 0.4 & 0.1 \end{pmatrix}, \begin{pmatrix} 0.5 & 0.3 \\ 0.5 & 0.3 \end{pmatrix} \rangle, \langle (x, y), \begin{pmatrix} 0.4 & 0.2 \\ 0.4 & 0.2 \end{pmatrix}, \begin{pmatrix} 0.5 & 0.3 \\ 0.5 & 0.3 \end{pmatrix} \rangle, \\ & \langle (x, y), \begin{pmatrix} 0.4 & 0.3 \\ 0.4 & 0.4 \end{pmatrix}, \begin{pmatrix} 0.4 & 0.3 \\ 0.5 & 0.3 \end{pmatrix} \rangle, \langle (x, y), \begin{pmatrix} 0.4 & 0.4 \\ 0.3 & 0.2 \end{pmatrix}, \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.3 \end{pmatrix} \rangle, \\ & \langle (x, y), \begin{pmatrix} 0.4 & 0.4 \\ 0.4 & 0.2 \end{pmatrix}, \begin{pmatrix} 0.5 & 0.3 \\ 0.3 & 0.3 \end{pmatrix} \rangle, \langle (x, y), \begin{pmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{pmatrix}, \begin{pmatrix} 0.4 & 0.3 \\ 0.3 & 0.3 \end{pmatrix} \rangle, \\ & \langle (x, y), \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rangle, \langle (x, y), \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle \} \end{aligned}$$

In general a product of two IFSOS's is not necessarily supraopen. For example in the above example A and D are supraopen, but

$$A \times D = \langle (x, y), \begin{pmatrix} 0.3 & 0.1 \\ 0.4 & 0.1 \end{pmatrix}, \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.5 \end{pmatrix} \rangle \notin \beta$$

On the other hand the supraclosed sets with respect to $(X \times Y, \beta)$ are precisely the products $F \times K$, where F is supraclosed in (X, τ) and K supraclosed in (Y, δ) . To see this we need only note that if $G = \overline{F} \in \tau$ and $H = \overline{K} \in \delta$ then

$$\overline{(G \times 1_{\underline{Y}}) \cup (1_{\underline{X}} \times H)} = (\overline{G} \times 1_{\underline{Y}}) \cap (1_{\underline{X}} \times \overline{H}) = \overline{G} \times \overline{H} = F \times K.$$

3.2. Theorem. *Let $(X, \tau), (Y, \delta)$ be IFSTS's. Then the product IFSTS $(X \times Y, \beta)$ is fuzzy strongly supraconnected if and only if (X, τ) and (Y, δ) are fuzzy strongly supraconnected.*

Proof. \implies . This follows by Theorem 2.5 since the projections π_1 and π_2 are surjective fuzzy supracontinuous functions.

\impliedby . Let (X, τ) and (Y, δ) be fuzzy strongly supraconnected. Suppose that there exist $A, B \in \beta$ with $A \neq 1_{\underline{X}} \neq B$ and $A \cup B = 1_{\underline{X}}$. By the above comments we have $A = (G_1 \times 1_{\underline{Y}}) \cup (1_{\underline{X}} \times H_1)$ and $B = (G_2 \times 1_{\underline{Y}}) \cup (1_{\underline{X}} \times H_2)$ where $G_1, G_2 \in \tau$ and $H_1, H_2 \in \delta$. By hypothesis we have $G_1 \neq 1_{\underline{X}} \neq G_2, H_1 \neq 1_{\underline{Y}} \neq H_2$ and

$$\begin{aligned} 1_{\underline{X}} \times 1_{\underline{Y}} &= (G_1 \times 1_{\underline{Y}}) \cup (1_{\underline{X}} \times H_1) \cup (G_2 \times 1_{\underline{Y}}) \cup (1_{\underline{X}} \times H_2) \\ &= ((G_1 \cup G_2) \times 1_{\underline{Y}}) \cup (1_{\underline{X}} \times (H_1 \cup H_2)). \end{aligned}$$

It follows easily that $G_1 \cup G_2 = 1_X$ or $H_1 \cup H_2 = 1_Y$ (compare the proof of Theorem 4.5 below), which is a contradiction.

Now suppose we have $C \in \beta$ with $C \neq 1$ and $\frac{1}{2} \leq \mu_C$. Again, $C = (G \times 1_Y) \cup (1_X \times H)$ with $G \in \tau$, $H \in \delta$, so $G \neq 1_X$, $H \neq 1_Y$ and $\frac{1}{2} \leq \mu_G(x) \vee \mu_H(y)$ for all $x \in X$ and $y \in Y$. Since (X, τ) is fuzzy strongly supraconnected $\frac{1}{2} \not\leq \mu_G$ by Theorem 2.3, so there exists $x_0 \in X$ with $\mu_G(x_0) < \frac{1}{2}$. In the same way we have $y_0 \in Y$ with $\mu_H(y_0) < \frac{1}{2}$, which gives the contradiction $\mu_G(x_0) \vee \mu_H(y_0) < \frac{1}{2}$. Hence $(X \times Y, \beta)$ is fuzzy strongly supraconnected by Theorem 2.3. \square

If the product IFSTS $(X \times Y, \beta)$ is C_5 -supraconnected then clearly (X, τ) and (Y, δ) are also C_5 -supraconnected. However the converse does not hold in general, as shown by the following example.

3.3. Example. Let $X = \{a, b\}$, $Y = \{c\}$ and define $G = \langle x, (0.1, 0.2), (0.2, 0.2) \rangle$, $G_1 = \langle x, (0.1, 0.2), (0.2, 0.4) \rangle$ and $H = \langle y, 0.2, 0.4 \rangle$, $H_1 = \langle y, 0.2, 0.5 \rangle$. Then $\tau = \{0_X, 1_X, G, G_1, G \cup G_1\}$ is an IFST on X and $\delta = \{0_Y, 1_Y, H, H_1, H \cup H_1\}$ an IFST on Y . Clearly (X, τ) and (Y, δ) are C_5 -supraconnected. If we denote the product of τ and δ by β then

$$(G \times 1_Y) \cup (1_X \times H) = \langle (x, y), (0.2, 0.4), (0.2, 0.2) \rangle \in \beta.$$

On the other hand

$$\langle (x, y), (0.2, 0.4), (0.2, 0.2) \rangle = \overline{G_1} \times \overline{H_1}$$

is a supraclosed set in $(X \times Y, \beta)$ by the remarks above. This shows that $(X \times Y, \beta)$ is C_5 -supradisconnected.

4. Intuitionistic fuzzy supracompactness

First we present the basic concepts.

4.1. Definition. (cf. [5]) Let (X, τ) be an IFSTS.

- (a) If a family $\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle \mid i \in J\}$ of IFSOS's in X satisfies the condition

$$\bigcup \{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle \mid i \in J\} = 1,$$

then it is called a *fuzzy supraopen cover* of X . A finite subfamily of a fuzzy supraopen cover $\langle x, \mu_{G_i}, \gamma_{G_i} \rangle$, $i \in J$, of X , which is also a fuzzy supraopen cover of X , is called a *finite subcover* of $\langle x, \mu_{G_i}, \gamma_{G_i} \rangle$, $i \in J$.

- (b) A family $\{\langle x, \mu_{K_i}, \gamma_{K_i} \rangle \mid i \in J\}$ of IFSCS's in X satisfies the *finite intersection property* (FIP for short) if every finite subfamily $\{\langle x, \mu_{K_{i_k}}, \gamma_{K_{i_k}} \rangle \mid k = 1, 2, 3, \dots, n\}$ satisfies the condition

$$\bigcap_{k=1}^n \langle x, \mu_{K_{i_k}}, \gamma_{K_{i_k}} \rangle \neq 0.$$

4.2. Lemma. *The following are equivalent:*

- (1) $\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle \mid i \in J\}$ is a cover of X .
- (2) $\bigvee \{\mu_{G_i}(x) \mid i \in J\} = 1 \forall x \in X$.
- (3) $\bigwedge \{\gamma_{G_i}(x) \mid i \in J\} = 0 \forall x \in X$.

Proof. (1) \implies (2). Clear since $\bigcup\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle \mid i \in J\} = 1_X$ means $\bigvee\{\mu_{G_i} \mid i \in J\} = 1$ and $\bigwedge\{\gamma_{G_i} \mid i \in J\} = 0$, so in particular $\bigvee\{\mu_{G_i}(x) \mid i \in J\} = 1 \forall x \in X$.

(2) \implies (3). By the definition of IFS we have $\mu_{G_i}(x) + \gamma_{G_i}(x) \leq 1$ and so $\gamma_{G_i}(x) \leq 1 - \mu_{G_i}(x)$ for all $x \in X$. Hence

$$0 \leq \bigwedge\{\gamma_{G_i}(x) \mid i \in J\} \leq \bigwedge\{1 - \mu_{G_i}(x) \mid i \in J\} = 1 - \bigvee\{\mu_{G_i}(x) \mid i \in J\} = 0$$

by (2). Hence $\bigwedge\{\gamma_{G_i}(x) \mid i \in J\} = 0 \forall x \in X$.

(3) \implies (1). Using (3) it may be shown that $\bigvee\{\mu_{G_i}(x) \mid i \in J\} = 1 \forall x \in X$ by an argument similar to that used above. This, together with (3), gives $\bigcup\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle \mid i \in J\} = 1_X$, as required. \square

4.3. Definition. An IFSTS (X, τ) is called *fuzzy supracompact* if every fuzzy supraopen cover of X has a finite subcover.

Equivalently, (X, τ) is fuzzy supracompact if every family of IFSC's with the FIP has a non-empty meet. Clearly, Lemma 4.2 will be useful in establishing fuzzy supracompactness.

It is known that fuzzy supracompactness is preserved under a fuzzy supracontinuous surjection:

4.4. Theorem. *Let $(X, \tau), (Y, \phi)$ be IFSTS's and $f : X \rightarrow Y$ a fuzzy supracontinuous surjection. If (X, τ) is fuzzy supracompact, then so is (Y, ϕ) .*

Proof. See [5]. \square

4.5. Theorem. *Let $(X, \tau), (Y, \delta)$ be IFSTS's and $(X \times Y, \beta)$ their product. Then the product $(X \times Y, \beta)$ is fuzzy supracompact if and only if (X, τ) and (Y, δ) are fuzzy supracompact.*

Proof. \implies . If $(X \times Y, \beta)$ is fuzzy supracompact then by Theorem 4.4 so are (X, τ) and (Y, δ) , since the projections π_1 and π_2 are fuzzy supracontinuous surjections.

\Leftarrow . Let $(X, \tau), (Y, \delta)$ be fuzzy supracompact. Since $\beta = \{(G \times 1_Y) \cup (1_X \times H) \mid G \in \tau, H \in \delta\}$, a general fuzzy supraopen cover of $X \times Y$ has the form

$$C = \{(G_i \times 1_Y) \cup (1_X \times H_i) \mid i \in J\},$$

where $G_i = \langle x, \mu_{G_i}, \gamma_{G_i} \rangle \in \tau$ and $H_i = \langle y, \mu_{H_i}, \gamma_{H_i} \rangle \in \delta$. We claim that $\{G_i \mid i \in J\}$ is a cover of X or $\{H_i \mid i \in J\}$ a cover of Y . By Lemma 4.2 it is sufficient to prove $\bigvee\{\mu_{G_i}(x) \mid i \in J\} = 1$ or $\bigvee\{\mu_{H_i}(x) \mid i \in J\} = 1$. However, $(G_i \times 1_Y) \cup (1_X \times H_i) = \langle (x, y), \mu_{G_i}(x) \vee \mu_{H_i}(y), \gamma_{G_i}(x) \wedge \gamma_{H_i}(y) \rangle$, so $\bigvee\{\mu_{G_i}(x) \mid i \in J\} \vee \bigvee\{\mu_{H_i}(x) \mid i \in J\} = \bigvee\{\mu_{G_i}(x) \vee \mu_{H_i}(y) \mid i \in J\} = 1$, from which the stated result follows. Hence we have a finite subset J' of J for which $\bigvee\{\mu_{G_i}(x) \mid i \in J'\} = 1$ or $\bigvee\{\mu_{H_i}(x) \mid i \in J'\} = 1$, whence by Lemma 4.2 $\{(G_i \times 1_Y) \cup (1_X \times H_i) \mid i \in J'\}$ is a finite subcover of C . Hence $(X \times Y, \beta)$ is fuzzy supracompact. \square

4.6. Definition. (cf. [5]) Let (X, τ) be an IFSTS and A an IFS in X .

- (a) if a family $\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle : i \in J\}$ of IFSOS's in X satisfies the condition $A \subseteq \bigcup\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle : i \in J\}$, then it is called a *fuzzy supraopen cover* of A . A finite subfamily of the fuzzy supraopen cover $\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle : i \in J\}$ of A , which is also a fuzzy supraopen cover of A , is called a *finite subcover* of $\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle : i \in J\}$.

(b) An IFS $A = \langle x, \mu_A, \gamma_A \rangle$ in an IFSTS (X, τ) is called *fuzzy supracompact* if every fuzzy supraopen cover of A has a finite subcover.

4.7. Corollary. (cf.[5]) *An IFS $A = \langle x, \mu_A, \gamma_A \rangle$ in an IFSTS (X, τ) is fuzzy supracompact iff for each family $\delta = \{G_i : i \in J\}$, $G_i = \langle x, \mu_{G_i}, \gamma_{G_i} \rangle$, $i \in J$, of IFSOS's in X with the properties $\mu_A \subseteq \bigvee_{i \in J} \mu_{G_i}$ and $1 - \gamma_A \subseteq \bigvee_{i \in J} (1 - \gamma_{G_i})$, there exists a finite subfamily*

$\{G_{i_k} : k = 1, 2, \dots, n\}$ of δ such that $\mu_A \subseteq \bigvee_{k=1}^n \mu_{G_{i_k}}$ and $1 - \gamma_A \subseteq \bigvee_{k=1}^n (1 - \gamma_{G_{i_k}})$.

4.8. Corollary. *Let (X, τ) , (Y, ϕ) be IFSTS's and $f : X \rightarrow Y$ a fuzzy supracontinuous surjection. If A is fuzzy supracompact in (X, τ) then $f(A)$ is fuzzy supracompact in (Y, ϕ) .*

Proof. Let $B = \{G_i : i \in J\}$, where $G_i = \langle y, \mu_{G_i}, \gamma_{G_i} \rangle$, $i \in J$, be a fuzzy supraopen cover of $f(A)$. Then, by the definition of supracontinuity $A = \{f^{-1}(G_i) : i \in J\}$ is a fuzzy supraopen cover of A . Since A is fuzzy supracompact, there exists a finite subcover of A , that is G_{i_k} , $k = 1, 2, \dots, n$, such that $A \subseteq \bigcup_{k=1}^n f^{-1}(G_{i_k})$. Hence $f(A) \subseteq$

$f \bigcup_{k=1}^n f^{-1}(G_{i_k}) = \bigcup_{k=1}^n f(f^{-1}(G_{i_k})) \subseteq \bigcup_{k=1}^n G_{i_k}$. Therefore, $f(A)$ is fuzzy supracompact. \square

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