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A RELATED FIXED POINT THEOREM FOR TWO PAIRS OF MAPPINGS ON TWO COMPLETE METRIC SPACES

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Abstract

A new related fixed point theorem for two pairs of mappings on two complete metric spaces is obtained.

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1. Introduction

The following related fixed point theorem was proved in [1].

1.1. Theorem. *Let (X, d) and (Y, ρ) be complete metric spaces, let A, B be mappings of X into Y and let S, T be mappings of Y into X satisfying the inequalities*

$$d(SAx, TBx') \le c \max\{d(x, x'),\ d(x, SAx),\ d(x', TBx'),\ \rho(Ax, Bx')\},$$
$$\rho(BSy, ATy') \le c \max\{\rho(y, y'),\ \rho(y, BSy),\ \rho(y', ATy'),\ d(Sy, Ty')\}$$

for all x, x' in X and y, y' in Y, where $0 \le c < 1$. If one of the mappings A, B, S and T is continuous, then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

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2. Main Result

We now prove the following related fixed point theorem.

2.1. Theorem. *Let (X, d) and (Y, ρ) be complete metric spaces. Let A, B be mappings of X into Y and let S, T be mappings of Y into X satisfying the inequalities*

$$(1) \qquad d(SAx,\ TBx') \leq c\frac{f(x,x',y,y')}{h(x,x',y,y')},$$

$$(2) \qquad \rho(BSy,\ ATy') \leq c\frac{g(x,x',y,y')}{h(x,x',y,y')}$$

for all x, x' in X and y, y' in Y for which $h(x,x',y,y') \neq 0$, where

$$\begin{aligned} f(x,x',y,y') &= \max\{d(x,x')\rho(Ax,Bx'),\ d(x,x')d(Sy,Ty'),\\ &\qquad d(x,Ty')\rho(Ax,ATy'),\ d(x',Sy)\rho(Bx',BSy)\},\\ g(x,x',y,y') &= \max\{\rho(y,y')d(Sy,Ty'),\ \rho(y,y')\rho(Ax,Bx'),\\ &\qquad \rho(y,Bx')d(Sy,TBx'),\ \rho(y',Ax)d(Ty',SAx)\},\\ h(x,x',y,y') &= \max\{\rho(Ax,Bx'),\ d(SAx,TBx'),\ d(Sy,Ty'),\ \rho(BSy,ATy')\} \end{aligned}$$

and $0 \leq c < 1$. If one of the the mappings A, B, S and T is continuous, then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

Proof. Let $x = x_0$ be an arbitrary point in X. We define the sequences $\{x_n\}$ and $\{y_n\}$ in X and Y inductively by $y_{2n-1} = Ax_{2n-2}$, $x_{2n-1} = Sy_{2n-1}$, $y_{2n} = Bx_{2n-1}$ and $x_{2n} = Ty_{2n}$ for $n = 1, 2, \ldots$.

We will first of all suppose that for some n,

$$\begin{aligned} h(x_{2n},\ x_{2n-1},\ y_{2n-1},\ y_{2n}) &= \max\{\rho(Ax_{2n},\ Bx_{2n-1}),\ d(SAx_{2n}, TBx_{2n-1}),\\ &\qquad d(Sy_{2n-1},\ Ty_{2n}),\ \rho(BSy_{2n-1},\ ATy_{2n})\}\\ &= \max\{\rho(y_{2n+1},\ y_{2n}),\ d(x_{2n+1},\ x_{2n}),\\ &\qquad d(x_{2n-1},\ x_{2n}),\ \rho(y_{2n},\ y_{2n+1})\}\\ &= 0. \end{aligned}$$

Then putting $x_{2n-1} = x_{2n} = x_{2n+1} = z$ and $y_{2n} = y_{2n+1} = w$, we see that

$$SAz = TBz = z, \quad ATw = w, \quad Az = Bz = w, \quad Tw = z$$

from which it follows that

$$Sw = z, \quad BSw = w.$$

Similarly, $h(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n}) = 0$ for some n implies that there exist points z in X and w in Y such that

$$(3) \qquad SAz = TBz = z, \quad BSw = ATw = w, \quad Az = Bz = w, \quad Sw = Tw = z.$$

We will now suppose that $h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) \neq 0 \neq h(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n})$ for all n. Applying inequality (1), we get

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(SAx_{2n}, TBx_{2n-1})\\ &\leq c\frac{f(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})}{h(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})}\\ &= c\frac{\max\{d(x_{2n}, x_{2n-1})\rho(y_{2n+1}, y_{2n}), [d(x_{2n-1}, x_{2n})]^2\}}{\max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n+1}, x_{2n}), d(x_{2n-1}, x_{2n})\}} \end{aligned}$$

from which it follows that

(4) $$d(x_{2n}, x_{2n+1}) \le c \max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n}, y_{2n+1})\}.$$

It follows similarly on using inequality (1) again that

(5) $$d(x_{2n-1}, x_{2n}) \le c \max\{d(x_{2n-2}, x_{2n-1}), \rho(y_{2n-1}, y_{2n})\}.$$

Similarly, on using inequality (2), it follows that

(6) $$\rho(y_{2n}, y_{2n+1}) \le c \max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n-1}, y_{2n})\}$$

and

(7) $$\rho(y_{2n}, y_{2n-1}) \le c \max\{d(x_{2n-2}, x_{2n-1}), \rho(y_{2n-2}, y_{2n-1})\}.$$

It now follows from inequalities (4) and (6) that

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\le c \max\{d(x_{2n-1}, x_{2n}), cd(x_{2n-1}, x_{2n}), c\rho(y_{2n-1}, y_{2n)}\} \\ &\le c \max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n-1}, y_{2n})\}. \end{aligned}$$

It follows similarly from inequalities (5) and (7) that

$$d(x_{2n-1}, x_{2n}) \le c \max\{d(x_{2n-2}, x_{2n-1}), \rho(y_{2n-2}, y_{2n-1})\}$$

and so

(8) $$d(x_n, x_{n+1}) \le c^{n-1} \max\{d(x_1, x_2), \rho(y_1, y_2)\}.$$

Similarly,

(9) $$\rho(y_n, y_{n+1}) \le c^{n-1} \max\{d(x_1, x_2), \rho(y_1, y_2)\}.$$

Since $c < 1$, it follows from inequalities (8) and (9) respectively that $\{x_n\}$ is a Cauchy sequence in X with a limit z and $\{y_n\}$ is a Cauchy sequence in Y with a limit w.

Now suppose that A is continuous. Then

(10) $$w = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Ax_{2n} = Az$$

and

(11) $$\lim_{n\to\infty} f(z, x_{2n-1}, w, y_{2n}) = d(z, Sw)\rho(w, BSw),$$

(12) $$\lim_{n\to\infty} g(z, x_{2n-1}, w, y_{2n}) = \rho(w, Az)d(z, SAz),$$

(13) $$\lim_{n\to\infty} h(z, x_{2n-1}, w, y_{2n}) = \max\{d(Sw, z), \rho(BSw, w)\}.$$

If

(14) $$\max\{d(Sw, z), \rho(w, BSw)\} = 0,$$

then

(15) $$Sw = z, \qquad BSw = w, \qquad Bz = w.$$

If it were possible that

(16) $$\max\{d(Sw, z), \rho(w, BSw)\} \ne 0,$$

then we have on using inequality (1) and equations (11) and (13)

$$\begin{aligned} d(Sw, z) &= \lim_{n\to\infty} d(SAz, TBx_{2n-1}) \\ &\le \lim_{n\to\infty} c\frac{f(z, x_{2n-1}, w, y_{2n})}{h(z, x_{2n-1}, w, y_{2n})} \\ &= c\frac{d(z, Sw)\rho(w, BSw)}{\max\{d(Sw, z), \rho(w, BSw)\}} \\ &\le cd(Sw, z) \end{aligned}$$

and so $Sw = z$ since $c < 1$.

Further, using inequality (2) and equations (12) and (13), we have

$$\begin{aligned}\rho(BSw, w) &= \lim_{n\to\infty} \rho(BSw, ATy_{2n}) \\ &\le \lim_{n\to\infty} c\frac{g(z, x_{2n-1}, w, y_{2n})}{h(z, x_{2n-1}, w, y_{2n})} \\ &= 0,\end{aligned}$$

contradicting equation (16). Equations (14) and (15) must therefore hold.

Now suppose that $Tw \ne z$. Then

$$\lim_{n\to\infty} f(x_{2n}, z, w, w) = d(z, Tw)\rho(w, ATw), \tag{17}$$

$$\lim_{n\to\infty} h(x_{2n}, z, w, w) = \max\{d(z, Tw), \rho(w, ATw)\} \ne 0. \tag{18}$$

Then we have on using inequality (1) and equations (17) and (18)

$$\begin{aligned}d(z, Tw) &= \lim_{n\to\infty} d(SAx_{2n}, TBz) \\ &\le \lim_{n\to\infty} c\frac{f(x_{2n}, z, w, w)}{h(x_{2n}, z, w, w)} \\ &= c\frac{d(z, Tw)\rho(w, ATw)}{\max\{d(z, Tw), \rho(w, ATw)\}} \\ &\le cd(z, Tw),\end{aligned}$$

giving a contradiction, since $c < 1$. We must therefore have $Tw = z$ and equations (3) again follow.

By the symmetry, the same results again hold if one of the mappings B, S, T is continuous, instead of A.

To prove the uniqueness, suppose that TB and SA have a second distinct common fixed point z'. Then, using inequality (1), we have

$$\begin{aligned}d(z, z') &= d(SAz, TBz') \\ &\le c\frac{f(z, z', Az, Bz')}{h(z, z', Az, Bz')} \\ &= c\frac{\max\{d(z, z')\rho(w, Bz'), [d(z, z')]^2, d(z, z')\rho(w, Az')\}}{\max\{\rho(w, Bz'), d(z, z'), \rho(w, Az')\}} \\ &\le cd(z, z'),\end{aligned}$$

a contradiction since $c < 1$. The fixed point z must therefore be unique.

We can prove similarly that w is the unique common fixed point of BS and AT. This completes the proof of the theorem. □

2.2. Corollary. *Let A, B, S, T be selfmappings on the complete metric space (X, d) satisfying the inequalities*

$$d(SAx, TBy) \le c\frac{f(x, y)}{h(x, y)},$$

$$d(BSx, ATy) \le c\frac{g(x, y)}{h(x, y)}$$

for all x, y *in* X *for which* $h(x,y) \neq 0$*, where*

$$\begin{aligned} f(x,y) = \max\{&d(x,y)d(Ax,By), d(x,y)d(Sx,Ty), \\ &d(x,Ty)d(Ax,ATy), d(y,Sx)d(By,BSx)\}, \\ g(x,y) = \max\{&d(x,y)d(Sx,Ty), d(x,y)d(Ax,By), \\ &d(x,By)d(Sx,TBy), d(y,Ax)d(Ty,SAx)\}, \\ h(x,y) = \max\{&d(Ax,By), d(SAx,TBy), d(Sx,Ty), d(BSx,ATy)\} \end{aligned}$$

and $0 \leq c < 1$*. If one of the mappings* A, B, S *and* T *is continuous, then* SA *and* TB *have a unique common fixed point* z *and* BS *and* AT *have a unique common fixed point* w*. Further,* $Az = Bz = w$ *and* $Sw = Tw = z$*.*

References

[1] B. Fisher and P.P. Murthy, *Related fixed point theorems for two pairs of mappings on two metric spaces,* Kyungpook Math. J., **37**, 343-347, 1997.