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ON SOME DISTINGUISHED SUBSPACES AND RELATIONSHIP BETWEEN DUALS

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Abstract. In this paper, we observe some new spaces to obtain new β- and γ-type duality of a sequence space λ, related to the some sequence spaces. Before this we give some new distinguished subspaces of an FK space obtained by an operator of Aydın and Başar [2], *which is stronger than common C_1- Cesàro operator*. We also give some structural theorems and inclusions for these distinguished subspaces. Finally we prove some theorems related to the f-, a_s^r- and a_b^r- duality of a sequence space λ like Goes [14] and Buntinas [8]. These theorems are important to decade the duality of a sequence space in summability theory and topological sequence spaces theory.

1. Preliminaries, Background and Notation

The space of all complex- (or real-) valued sequences is given by $\omega = \mathbb{C}^{\mathbb{N}}$ (or $\mathbb{R}^{\mathbb{N}}$) and any linear subspace $\lambda \subset \omega$ is called a sequence space, (usually we assume that $\lambda \supset \phi$ where ϕ is the space of finitely non-zero sequences spanned by (δ^k), the sequence of kth position is 1 and all the others are 0).

Let $(b_n) \subset \lambda$ be a sequence in a normed sequence space $(\lambda, ||.||_\lambda)$. $\forall x \in \lambda$ if there exists a unique sequence $(\alpha_n) \subset \mathbb{K}$ such that $\lim_n \parallel x - \sum_{k=0}^{n} \alpha_k b_k \parallel = 0$ then (b_n) is called the Schauder base of λ. Zeller introduced a notion -*which is called AK property for a sequence space*- is the special case of $(b_n) = ((\delta^n))_{n\in\mathbb{N}}$ in this definition. It is clear example that the set $\{(\delta^k)\}_{k\in\mathbb{N}}$ is a Schauder base for the space c_0 and ℓ_p $(1 \leq p < \infty)$, which are the spaces of null sequences and p-absolutely convergent series, respectively.

A K space λ is a subspace of ω on which coordinate functionals $\pi_k(x) = x_k$ are continuous. A complete linear metric K space is an FK space. It is named BK if it is also normable.

Let each of $\lambda_i|_{i=1}^{n}$ be an FK space whose topologies are generated by paranorms $p^{(i)}$ $(i = 1, 2, ..., n)$, respectively. Then, $\lambda = \sum_i^n \lambda_i = \{\sum_i^n x^{(i)} : x^{(i)} \in \lambda_i\}$ is an FK space with the unrestricted inductive limit topology. The paranorm of λ is given by $q(z) = \inf\{\sum_i^n p^{(i)}(x^{(i)}) \big| x^{(i)} \in \lambda_i, z = \sum_i^n x^{(i)} \in X\}$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of FK spaces, each p_n the paranorm of λ_n and $(q_{nk})_k$ be the seminorms of λ_n. Then $\lambda = \bigcap_n^\infty \lambda_n$ is an FK space with paranorm $q = \sum_n \frac{p_n}{2^n(1+p_n)}$ and seminorms $(q_{nk})_k$, [10],[5]. These properties are also satisfied in BK spaces.

Coordinate-wise product of $x,\ y \in \omega$ is given by $xy = \big\{x_k y_k\big\}_{k\in\mathbb{N}}$. If $x \in \omega$ and $\lambda \subset \omega$ then $x\lambda = \big\{xy : y \in \lambda\big\}$. For sequence spaces $\lambda, \mu \subset \omega$, this product is

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given by $\lambda\mu = \left\{ \ xy \ : \ x \in \lambda, \ y \in \mu \right\}$. Smilarly, we notates the coordinate-wise sum of $x \in \lambda, \ y \in \mu, \ x+y = \left\{ x_k + y_k \right\}_{k\in\mathbb{N}}$ and $\lambda, \mu \subset \omega, \ \lambda + \mu = \left\{ x+y \middle| \ x \in \lambda, \ y \in \mu \right\}$.

The $\beta-$ and $\gamma-$ duality of the sequence space $\lambda \subset \omega$, also known as production λ^{cs} and λ^{bs} can be generalised as $\lambda^{\mu} = \left\{ \ y \in \omega | xy \in \mu, \forall x \in \lambda \right\}$ for all sequence spaces $\lambda, \mu \subset \omega$, where *cs* and *bs* are the spaces of convergent and bounded series, respectively [11]. We notates as $(\lambda^{\vartheta})^{\varsigma} = \lambda^{\vartheta\varsigma}$ for any $\vartheta-, \ \varsigma-$ duals. It is easy to see that both of these spaces are sequence spaces and there exists the inclusions $\phi \subset \lambda^{\beta} \subset \lambda^{\gamma}$. If $\lambda \subset \mu$ then $\mu^{\zeta} \subset \lambda^{\zeta}$, and for every sequence space λ we have $\lambda^{\zeta} = \lambda^{\zeta\zeta\zeta}$ and $\lambda \subset \lambda^{\zeta\zeta}$, where ζ is one of the duals $\beta-$ or $\gamma-$. If $\lambda^{\zeta\zeta} = \lambda$ then λ is called $\zeta-$ space [11]. A sequence space λ is *solid* iff $\{(y_k) \in \omega | \ \exists (x_k) \in \lambda, \forall k \in \mathbb{N} : |y_k| \leq |x_k|\} \subset \lambda$, that is, sequence space λ is *solid* provided that $yx \in \lambda$ whenever $y \in \ell_{\infty}$ and $x \in \lambda$, where ℓ_{∞} is the space of all bounded complex sequences.

Let $\lambda \supset \phi$ be a K space then f- (or sequential) dual of λ is given by $\lambda^f = \left\{ (f(\delta^k))_{k\in\mathbb{N}} | \text{ for some } f \in \lambda' \right\}$.

Let $\lambda \supset \phi$ be an FK space. n^{th}section of a sequence $x = (x_k) \in \lambda$ is given by $x^{[n]} = \sum_{k=1}^{n} x_k \delta^k$. A sequence x in this space; if the set $\{x^{[n]}\}_{n\in\mathbb{N}}$ is bounded in λ then has AB, if $x^{[n]} \to x($ or $x^{[n]} \to x$*(weakly)*) then has AK(or SAK), $\forall f \in \lambda'$, if the series $\sum_k x_k f(\delta^k)$ converges in λ then has FAK properties. One can give the SAK property as, $\forall f \in \lambda', \ f(x) = \sum_k x_k f(\delta^k)$. The spaces of the sequences which have these properties in λ, shown by $B_{\lambda}, S_{\lambda}, W_{\lambda}$ and F_{λ}, respectively. If $\lambda = B_{\lambda}$ (or F_{λ}, W_{λ} and S_{λ}) then we say λ is AB (or FAK, SAK and AK) space. $F_{\lambda}^{+} = \lambda^{f\beta}$ and $B_{\lambda}^{+} = \lambda^{f\gamma}$, we say $\Lambda_{\lambda} = \Lambda_{\lambda}^{+} \cap \lambda$, for $\Lambda = F, B$. It is true for every $\lambda \supset \phi$, $\phi \subset S_{\lambda} \subset W_{\lambda} \subset F_{\lambda} \subset B_{\lambda}$ and $W_{\lambda} \subset \overline{\phi}$, where $\overline{\phi}$ is closure of ϕ in λ [18]. If $\lambda = \overline{\phi}$ then λ is called AD space. Via Hahn-Banach theorem, $\lambda^f = \overline{\phi}^f$. Each λ which has AK then has AD, that is, $S_{\lambda} = \lambda \Rightarrow \lambda = \overline{\phi}$, if we want to see "$\Leftrightarrow$" in place of "$\Rightarrow$", the space $\lambda = \overline{\phi}$ must also be AB.

Let λ and μ be any sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, \ k \in \mathbb{N}$. Then, the matrix A defines a mapping from λ into μ, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = ((Ax)_n)$, the A-transform of x, exists and is in μ; where $(Ax)_n = \sum_k a_{nk} x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$, we shall denote the class of all such matrices. The *matrix domain* λ_A is given by $\lambda_A = \left\{ x = (x_k) \in w : Ax \in \lambda \right\}$, in sequel a *convergence domain* of an infinite matrix A is $c_A = \left\{ x = (x_k) \in w : Ax \in c \right\}$, which also an FK space, where c is the space of convergent sequences. A triangular (special name of triangle matrix) is $T = (t_{nk})$ such that, $t_{nk} = \left\{ \begin{array}{ccc} \neq 0 & , & k = n \\ 0 & , & k > n \end{array} \right\}$., $(n, k \in \mathbb{N})$. In this definition one can get diagonal matrix, although this is the general version. The transformation T is one to one and the inverse of T is again a triangular which has unique inverse S, that is, $x = T(S(x)) = S(T(x))$.

These spaces are BK spaces with the norm, $\parallel x \parallel_{a_{\infty}^r} = \parallel A^r x \parallel_{\infty}$, i.e.,

$$\parallel x \parallel_{a_{\infty}^r} = \sup_n \left| \frac{1}{n+1} \sum_{k=0}^{n} (1 + r^k) x_k \right|.$$

Now we shall define some spaces which may be used in $\beta-$ and γ- type duality of a sequence space. These are,

$$\begin{aligned} a_s^r &= \Big\{x=(x_j)\in w:\ \lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)x_j \ \text{ exists}\Big\} \\ &= \Big\{x=(x_j)\in w:\ \big(\sum_{j=0}^{k}x_j\big)_{k\in\mathbb{N}}\in a_c^r\Big\} \end{aligned}$$

and

$$\begin{aligned} a_b^r &= \Big\{x=(x_j)\in w:\ \sup_n\Big|\frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)x_j\Big|<\infty\Big\} \\ &= \Big\{x=(x_j)\in w:\ \big(\sum_{j=0}^{k}x_j\big)_{k\in\mathbb{N}}\in a_\infty^r\Big\} \end{aligned}$$

which are BK spaces with the norm, $\| x \|_{a_b^r}=\| x \|_{a_s^r}=\| A^r x \|_{bs}$, i.e.,

$$\| x \|_{a_b^r}=\sup_n\left|\frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)x_j\right|.$$

Parallelly to these spaces let us define a_s^r- and a_b^r- duality of a sequence space λ as following,

$$\begin{aligned} \lambda^{a_s^r} &= \Big\{(x_j)\in w:\ \lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)x_jy_j \ \text{ exists},\ \forall (y_j)\in\lambda\Big\} \\ &= \big\{x\in w: xy\in a_s^r,\ \forall y\in\lambda\big\} \end{aligned}$$

and

$$\begin{aligned} \lambda^{a_b^r} &= \Big\{(x_j)\in w:\ \sup_n\Big|\frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)x_jy_j\Big|<\infty,\ \forall (y_j)\in\lambda\Big\} \\ &= \big\{x\in w: xy\in a_b^r,\ \forall y\in\lambda\big\}, \end{aligned}$$

respectively. One can see that, $\phi\subset\lambda^{a_s^r}\subset\lambda^{a_b^r}$ and $\mu^\eta\subset\lambda^\eta$, when $\lambda\subset\mu$ and for every sequence space λ we have $\lambda^\eta=\lambda^{\eta\eta\eta}$ and $\lambda\subset\lambda^{\eta\eta}$, where η is one of the duals a_s^r or a_b^r. If $\lambda^{\eta\eta}=\lambda$ then λ is called $\eta-$ space.

Now, we introduce a new section for a sequence x which is thought with A^r method, then n^{th} section with A^r of x is

$$\begin{aligned} A^{r^n}\cdot x = x^{[n]_{A^r}} &= \frac{1}{n+1}\sum_{k=0}^{n}(1+r^k)P^kx \\ &= \frac{1}{n+1}\sum_{k=0}^{n}(1+r^k)x^{[k]} \end{aligned}$$

where, $(P^kx=\sum_{j=0}^{k}x_j\delta^j)$ and $A^{r^n}=\frac{1}{n+1}\sum_{k=0}^{n}(1+r^k)P^k$ is the A^{r^n} section operator. The set $\{A^{r^n}\cdot x\}=\{x^{[n]_{A^r}}\}$ of x is called set of A^r sections and shown by $A^r\cdot x$.

A sequence x in any K space $\lambda \supset \phi$ has A^rK property if $\frac{1}{n+1}\sum_{k=0}^{n}(1+r^k)x^{[k]} \to x$ in λ and we say λ is an A^rK- space if all elements of λ have this property. Similarly we can define the properties, SA^rK, FA^rK and A^rB. We shall use $\{x : X\}$ for the set of elements x possessing the property X. So,

$$\begin{aligned}
A^rS_\lambda &= \Big\{x \in \lambda \big| x = \lim_n \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)x_j\delta^j\Big\} \\
&= \Big\{x \in \lambda \big| A^rK\Big\}, \\
A^rW_\lambda &= \Big\{x \in \lambda \big| \frac{1}{n+1}\sum_{k=0}^{n}(1+r^k)x^{[k]} \rightharpoonup x \text{ in } \lambda\Big\}(\text{"}\rightharpoonup\text{" means } \textit{weakly}) \\
&= \Big\{x \in \lambda \big| f(x) = \lim_n \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)x_j f(\delta^j),\ \forall f \in \lambda'\Big\} \\
&= \Big\{x \in \lambda \big| SA^rK\Big\}, \\
A^rF^+_\lambda &= \Big\{x \in \omega \big| \Big(\frac{1}{n+1}\sum_{k=0}^{n}(1+r^k)x^{[k]}\Big)_{n\in\mathbb{N}} \text{ weakly Cauchy in } \lambda\Big\} \\
&= \Big\{x \in \omega \big| \big(x_n f(\delta^n)\big)_{n\in\mathbb{N}} \in a^r_s,\ \forall f \in \lambda'\Big\}, \\
A^rB^+_\lambda &= \Big\{x \in \omega \big| \Big(\frac{1}{n+1}\sum_{k=0}^{n}(1+r^k)x^{[k]}\Big)_{n\in\mathbb{N}} \text{ is bounded in } \lambda\Big\} \\
&= \Big\{x \in \omega \big| \big(x_n f(\delta^n)\big)_{n\in\mathbb{N}} \in a^r_b,\ \forall f \in \lambda'\Big\}.
\end{aligned}$$

One should keep in mind that, $A^rB_\lambda = A^rB^+_\lambda \cap \lambda$ and $A^rF_\lambda = A^rF^+_\lambda \cap \lambda$ which are the space of the sequences having A^rB and FA^rK properties, respectively. Now for example, if λ is an A^rB space (respectively A^rK space) then,

$$\sup_n \| x^{[n]_{A^r}} \|_\lambda < \infty, \text{ (respectively } \lim_n \| x^{[n]_{A^r}} - x \|_\lambda = 0).$$

2. Some General Properties of New Subspaces

We shall give some theorems related to these new distinguished subspaces of an FK space.

Theorem 2.1. *Let $\lambda \supset \phi$ be an FK space. Then following is true,*

$$\phi \subset A^rS_\lambda \subset A^rW_\lambda \subset A^rF_\lambda \subset A^rB_\lambda \subset \lambda \text{ and } \phi \subset A^rS_\lambda \subset A^rW_\lambda \subset \overline{\phi}.$$

Proof. We know for every $\lambda \supset \phi$,

$$\phi \subset S_\lambda \subset W_\lambda \subset F_\lambda \subset B_\lambda \subset \lambda.$$

So first inclusion can be omitted. Let us show $A^rW_\lambda \subset \overline{\phi}$. In ([18],3.0.1) by the Hahn - Banach theorem, $f = 0$ on ϕ that leads $f(x) = 0$ is also on A^rW_λ by the definition of A^rW_λ. So $x \in A^rW_\lambda$ then $x \in \overline{\phi}$. □

Theorem 2.2. *Distinguished subspaces of an FK space are monotone, i.e, let* $\Lambda = A^rS,\ A^rW,\ A^rF,\ A^rB,\ A^rF^+,\ A^rB^+$, *then* $\lambda \subset \mu \Rightarrow \Lambda_\lambda \subset \Lambda_\mu$.

Proof. We know inclusion map is continuous. So, if $\lambda \subset \mu$, then for every $x \in A^rS_\lambda$,

$$\frac{1}{n+1}\sum_{k=0}^{n}(1+r^k)x^{[k]} \to x$$

is same for A^rS_μ. Similarly, we can have same discussion for A^rW. Now, let us take $x \in A^rF^+$ (or A^rB^+). Then, for every $f \in \lambda'$, $\big(x_nf(\delta^n)\big) \in a_s^r$ (or a_b^r). And if $g \in \mu'$, then for $g|_\lambda \in \lambda'$ we can have $\big(x_ng(\delta^n)\big) \in a_s^r$ (or a_b^r) ([18], 4.2.4). Similarly, we can have same discussion for $A^rF,\ A^rB$. $\square$

Theorem 2.3. *Let* $\lambda_i|_{i=1}^m \supset \phi$ *be FK spaces with paranorms* $p^{(i)}$ $(i = 1, 2, ..., m)$ *and* $\lambda = \sum_i^m \lambda_i$. *If* $\Lambda = A^rS, A^rW, A^rF, A^rB$, *then* $\sum_i^m \Lambda_{\lambda_i} \subseteq \Lambda_\lambda$.

Proof. Let $\Lambda = A^rS$ and $x^{(i)} \in A^rS_{\lambda_i}$ $(i = 1, 2, ..., m)$. We have

$$p^{(1)}(x^{(1)^{[n]A^r}} - x^{(1)}) \to 0, ..., p^{(m)}(x^{(m)^{[n]A^r}} - x^{(m)}) \to 0,\ (n \to \infty),$$

i.e., $p^{(i)}(x^{(i)^{[n]A^r}} - x^{(i)}) \to 0\Big|_{i=1}^m$, then

$$\begin{aligned} q\big[(\sum_i^m x^{(i)^{[n]A^r}}) - (\sum_i^m x^{(i)})\big] &= q\big[\big(\lim_n \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)\sum_i^m x_j^{(i)}\delta^j\big) - (\sum_i^m x^{(i)})\big] \\ &= \inf_{x^{(i)}\in\lambda_i} \big\{p^{(1)}(x^{(1)^{[n]A^r}} - x^{(1)}) + ... + \\ &+ p^{(m)}(x^{(m)^{[n]A^r}} - x^{(m)})\big\}\ (i = 1, ..., m) \\ &\le p^{(1)}(x^{(1)^{[n]A^r}} - x^{(1)}) + ... + \\ &+ p^{(m)}(x^{(m)^{[n]A^r}} - x^{(m)}) \\ &\to 0. \end{aligned}$$

So, $\sum_i^m x_i \in A^rS_\lambda$.

Let $\Lambda = A^rW$, $x^{(i)} \in A^rW_{\lambda_i}$ $(i = 1, 2, ..., m)$ and $f \in \lambda'$. We have $f|_{\lambda_i} \in \lambda_i'$ $(i = 1, 2, ..., m)$. Since f is linear and continuous, then

$$\begin{aligned} f(\sum_i^m x^{(i)}) &= \sum_i^m f(x^{(i)}) \\ &= \lim_n \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)x_j^{(1)}f(\delta^j) + \\ &+ ... + \\ &+ \lim_n \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)x_j^{(m)}f(\delta^j) \\ &= \lim_n \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)(\sum_i^m x_j^{(i)})f(\delta^j). \end{aligned}$$

So, $\sum_i^m x^{(i)} \in A^rW_\lambda$. One can prove this theorem similarly for $\Lambda = A^rF, A^rB$. $\square$

Theorem 2.4. *Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of FK spaces and $\lambda = \bigcap_n \lambda_n$. If $\Lambda = A^rS, A^rW, A^rF, A^rB$ then $\Lambda_\lambda = \bigcap_n \Lambda_{\lambda_n}$.*

Proof. With Theorem 2.2, $\forall n \in \mathbb{N}$, $\Lambda_\lambda \subseteq \Lambda_{\lambda_n}$ and so for $\Lambda = A^rS, A^rW, A^rF, A^rB$ we have $\Lambda_\lambda \subseteq \bigcap_n \Lambda_{\lambda_n}$. Conversely, we shall show $\bigcap_n \Lambda_{\lambda_n} \subseteq \Lambda_\lambda$.

$\Lambda = A^rS$; Let $x \in \bigcap_n A^rS_{\lambda_n}$. Then, $\forall n, k \in \mathbb{N}$, $q_{nk}(x^{[n]_{A^r}} - x) \to 0$ and q_{nk} are also seminorms for λ and so $x^{[n]_{A^r}} \to x$ is satisfied in λ which implies $x \in A^rS_\lambda$.

$\Lambda = A^rW$; Let $x \in \bigcap_n A^rW_{\lambda_n}$ and $f \in \lambda'$. For each $i = 1, 2, ..., m$ there exists $f_i \in (\lambda_i)'$ such that $f = \sum_i^m f_i$ and $|f_i| \leq p_i$, ([16], 4.4 (problem 30), 11.3 (problem 26)). Since $f_i(x^{[n]_{A^r}}) \to f_i(x)$ for $i = 1, 2, ..., m$, $f(x^{[n]_{A^r}}) \to f(x)$ is also satisfied for λ. Hence $x \in A^rW_\lambda$.

$\Lambda = A^rF$; Let $x \in \bigcap_n A^rF_{\lambda_n}$ and $f \in \lambda'$. Therefore, there exists $a_i \in \lambda_i$ such that $f_i(x^{[n]_{A^r}}) \to a_i$ for $i = 1, 2, ..., m$. This also satisfied on λ, i.e., there exists a $b \in \lambda$ such that $f(x^{[n]_{A^r}}) \to b$. Hence, $x \in A^rF_\lambda$.

$\Lambda = A^rB$; Let $x \in \bigcap_n A^rB_{\lambda_n}$. Then, for any fixed j, k, there exists positives M_{jk} such that $q_{jk}(x^{[n]_{A^r}}) \leq M_{jk}$. Hence $x \in A^rB_\lambda$.

So proof has been completed. □

3. a_s^r- And a_b^r- Duality

In this section we will determine some relationship between the f-, a_s^r- and a_b^r-duality of a sequence space λ, with its distinguished subspaces.

Theorem 3.1. *Let $\lambda \supset \phi$ be an FK space. Then*

$$A^rB_\lambda^+ = \lambda^{fa_b^r} \quad and \quad A^rF_\lambda^+ = \lambda^{fa_s^r}.$$

Proof. We know that $z \in A^rB_\lambda^+$ iff $\left(z_n f(\delta^n)\right) \in a_b^r$. And for all $f \in \lambda'$ we have $f(\delta^n) \in \lambda^f$, from the definition of a_b^r we can have $z \in \lambda^{fa_b^r}$.

We can similarly show $A^rF_\lambda^+ = \lambda^{fa_s^r}$. So we omit it. □

Corollary 3.2. *Let $\lambda \supset \phi$ be an FK space. Then the distinguished subspaces $A^rB_\lambda^+$ and $A^rF_\lambda^+$ are a_b^r- and a_s^r- spaces, respectively.*

Theorem 3.3. *Let $\lambda \supset \phi$ be an FK space and $\overline{\phi}$ is the closure of ϕ in λ. If any μ FK space which has $\overline{\phi} \subset \mu \subset \lambda$ then $A^rB_\lambda^+ = A^rB_\mu^+$ and $A^rF_\lambda^+ = A^rF_\mu^+$.*

Proof. Since $(\overline{\phi})^f = \lambda^f$ for every K- space $\supset \phi$, we can have $(\overline{\phi})^f \subset \mu^f \subset \lambda^f = (\overline{\phi})^f$ and by apply a_b^r- dual to the every side, we have desired result with Theorem 3.1. □

Theorem 3.4. *Let $\lambda \supset \phi$ be an FK space. Then, λ is an A^rB (resp. FA^rK) space iff $\lambda^f \subset \lambda^{a_b^r}$ (resp. $\lambda^{a_s^r}$).*

Proof. $\{\Rightarrow\}$: One should keep in mind Theorem 3.1 and the properties of a_b^r-(resp. a_s^r-) duality of a sequence space with hypothesis of the theorem. So we have

$$\lambda^f \subset \lambda^{fa_b^r a_b^r} \subset \lambda^{a_b^r} \text{ (resp. } \lambda^f \subset \lambda^{fa_s^r a_s^r} \subset \lambda^{a_s^r})$$

from $\lambda \subset A^rB_\lambda^+ = \lambda^{fa_b^r}$ (resp. $\lambda \subset A^rF_\lambda^+ = \lambda^{fa_s^r}$).

$\{\Leftarrow\}$: Suppose that $\lambda^f \subset \lambda^{a_b^r}$ (resp. $\lambda^f \subset \lambda^{a_s^r}$). From properties of a_b^r- (*resp.* a_s^r-) duality, we have

$$\lambda \subset \lambda^{a_b^r a_b^r} \subset \lambda^{fa_b^r} = A^rB_\lambda^+ \text{ (resp. } \lambda \subset \lambda^{a_s^r a_s^r} \subset \lambda^{fa_s^r} = A^rF_\lambda^+)$$

by applying a_b^r- (*resp.* a_s^r-) duality to both side of inclusion. Hence, λ is an A^rB (resp. FA^rK) space. □

One can prove the necessity of this theorem by different way. That is, let $\lambda \supset \phi$ be an FA^rK space and $y \in \lambda^f$. Since λ is an FA^rK space, for all $x \in \lambda$ and for all $f \in \lambda'$, $\lim_n \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)x_j f(\delta^j)$ exists so we have

$$\lim_n \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)x_j y_j$$

by taking $f(\delta_j)=y_j$. Therefore $y \in \lambda^{a_s^r}$.

Corollary 3.5. *Let $\lambda \supset \phi$ be a $BK-A^rB$ space, then $\lambda^{a_s^r}$ is closed in λ^f, since $\lambda^{a_s^r}$ is closed in $\lambda^{a_b^r}$.*

Theorem 3.6. *For an FK space $\lambda \supset \phi$, the following are true.*
(i) If λ is an AD space, then $\lambda^{a_s^r}=\lambda^{a_b^r}$.
(ii) $\lambda^\beta \subset \lambda^\sigma \subset \lambda^{a_s^r} \subset \lambda^{a_b^r} \subset \lambda^f$, where λ^σ is given in [13] *as,* $\lambda^\sigma = \left\{(x_j)\in w : \lim_n \frac{1}{n+1}\sum_{k=0}^{n}\sum_{j=0}^{k} x_j y_j \;\; \textit{exists},\ \forall y \in \lambda\right\}$.

Proof. *(i)* Let us take $y \in \lambda^{a_b^r}$ and for all $x \in \lambda$,

$$f_n(x)=\frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=j}^{n}(1+r^k)x_j y_j$$

then $\{f_n\}$ is point-wise bounded and so equicontinuous, [18].

Now, for $m \le n$,

$$\lim_n f_n(\delta^m)=y_m$$

and so $\phi \subset \{x : \lim_n f_n(x) \text{ exists }\}$. Therefore, via Convergence lemma, ([18], 1.0.5, 7.0.3) $\{x : \lim_n f_n(x) \text{ exists }\}$ is closed subspace of λ. Since λ is an AD space,

$$\lambda = \{x : \lim_n f_n(x) \text{ exists }\} = \overline{\phi}$$

and then for all $x \in \lambda$ $\lim_n f_n(x)$ exists. So $y \in \lambda^{a_s^r}$.

Anymore, $\lambda^{a_s^r}=\lambda^{a_b^r}$ since for all x, if $x \in \lambda^{a_s^r}$ then $x \in \lambda^{a_b^r}$.

(ii) From hypothesis $\overline{\phi} \subset \lambda$. Since $\overline{\phi}$ is an A^rK space so is an AD and FA^rK space, therefore

$$\lambda^{a_b^r} \subset (\overline{\phi})^{a_b^r} = (\overline{\phi})^{a_s^r} = (\overline{\phi})^f = \lambda^f$$

by ([18], 7.2.4), *(i)* and Teorem 3.4.
So proof has been completed. □

We can have following result by reading Theorem 3.4 and 3.6

Corollary 3.7. *Let $\lambda \supset \phi$ be an FK space. Then, λ is an A^rB (resp. FA^rK) space iff $\lambda^f=\lambda^{a_b^r}$ (resp. $\lambda^f=\lambda^{a_s^r}$).*

Theorem 3.8. *Let $\lambda \supset \phi$ be an FK space and $\overline{\phi} \subset A^rB_\lambda$. Then, $\overline{\phi}$ is an A^rK space and the*

$$A^rS_\lambda = A^rW_\lambda = \overline{\phi}.$$

Proof. Let λ be an A^rB space and define $f_n : \lambda \to \lambda$ by

$$f_n(x) = x - \frac{1}{n+1}\sum_{k=0}^{n}(1+r^k)x^{[k]}.$$

Then $\{f_n\}$ is point-wise bounded and so equicontinuous by ([18], 7.0.2). Since $f_n \to 0$ on ϕ then also $f_n \to 0$ on $\overline{\phi}$ by ([18], 7.0.3). So $\overline{\phi} \subset A^rS_\lambda$ and therefore

$$A^rS_\lambda = A^rW_\lambda = \overline{\phi}$$

by also keeping in mind Theorem 2.1. □

Lemma 3.9. *$\lambda \supset \phi$ be an FK space such that $\overline{\phi}$ has FA^rK. Then*

$$A^rF_\lambda^+ = \overline{\phi}^{\,a_s^r a_s^r}.$$

Proof. We know from Theorem 3.1 $A^rF_\lambda^+ = \lambda^{fa_s^r}$. And we know $\lambda^f = (\overline{\phi})^f$ by ([18], 7.2.4). Now, with a_s^r- duality of both side then we have $\lambda^{fa_s^r} = (\overline{\phi})^{fa_s^r}$ by Corollary 3.7. □

Corollary 3.10. *Let $\lambda \supset \phi$ be an FK space. Then λ has FA^rK iff $\overline{\phi}$ has A^rK and $\lambda \subset \overline{\phi}^{\,a_s^r a_s^r}$.*

Theorem 3.11. *$\lambda \supset \phi$ be an FK space. Then the following statements are equivalent.*

(i) *λ is an FA^rK space ,* ***(ii)*** *$\lambda \subset A^rF_\lambda^{\ a_s^r a_s^r}$,* ***(iii)*** *$\lambda \subset A^rW_\lambda^{\ a_s^r a_s^r}$,*
(iv) *$\lambda \subset A^rS_\lambda^{\ a_s^r a_s^r}$,* ***(v)*** *$\lambda^{a_s^r} = A^rF_\lambda^{a_s^r} = A^rW_\lambda^{\ a_s^r} = A^rS_\lambda^{\ a_s^r}$.*

Proof. It is spontaneously seen that $(iv) \Rightarrow (iii) \Rightarrow (ii)$ holds by definition of each space.
$(ii) \Rightarrow (i)$: Suppose that $\lambda \subset A^rF_\lambda^{\ a_s^r a_s^r}$. Then,

$$\lambda^f \subset \lambda^{fa_s^r a_s^r} = A^rF_\lambda^{+a_s^r} \subset A^rF_\lambda^{a_s^r} \subset \lambda^{a_s^r}$$

by applying f- duality to every side. Hence, we have desired result by Theorem 3.4.
$(i) \Rightarrow (iv)$: Suppose that λ is an FA^rK space, then $\overline{\phi} = A^rS_\lambda$ by Corollary 3.10.
$(iv) \Rightarrow (v)$: For every $\lambda \supset \phi$,

$$A^rS_\lambda \subset A^rW_\lambda \subset A^rF_\lambda \subset \lambda$$

holds by Theorem 2.1. We have

$$\lambda^{a_s^r} \subset A^rF_\lambda^{a_s^r} \subset A^rW_\lambda^{a_s^r} \subset A^rS_\lambda^{a_s^r}$$

by applying a_s^r- duality to every side. Finally,

$$\lambda \subset A^rS_\lambda^{a_s^r a_s^r}$$

by hypothesis, so we have $A^rS_\lambda^{a_s^r} \subset \lambda^{a_s^r}$ by applying a_s^r- duality to everyside.
$(v) \Rightarrow (iv)$: Suppose that (v). One can easily have

$$A^rS_\lambda^{a_s^r a_s^r} = \lambda^{a_s^r a_s^r} \supset \lambda$$

by applying a_s^r duality.

So proof has been completed. □

Theorem 3.12. *$\lambda \supset \phi$ be an FK space. Then the following statements are equivalent.*

$$\textbf{(i) } \lambda \text{ is an } SA^rK, \quad \textbf{(ii) } \lambda \text{ is an } A^rK, \quad \textbf{(iii) } \lambda^{a_s^r} \widehat{=} \lambda', \ (f \to f(\delta^k))$$

Proof. Clearly $(ii) \Rightarrow (i)$. $(i) \Rightarrow (ii)$: Since λ has SA^rK, it has also A^rB and from Theorem 2.1 we have $A^rW_\lambda \subset \overline{\phi}$ so it has to have AD. So with Theorem 3.8, λ has A^rK.
$(ii) \Rightarrow (iii)$: Since λ is an A^rK space, $A^rS_\lambda = \lambda$ is an AD space and so $\lambda^f = \lambda'$ ([5], (7.2.11)). Also $\lambda^f = \lambda^{a_s^r}$ by Corollary 3.7.
$(iii) \Rightarrow (i)$: Suppose that (iii) holds. Then, there is $u \in \lambda^{a_s^r}$ such that

$$f(x) = \lim_n \frac{1}{n+1} \sum_{j=0}^{n} \sum_{k=j}^{n} (1+r^k) u_j x_j$$

for all $f \in \lambda'$ and $x \in \lambda$. So $f(\delta^j) = u_j$ by Theorem 3.6, hence $x \in A^rW_\lambda$. □

Theorem 3.13. *Let $\lambda \supset \phi$ be an FK space. Then the following statements are equivalent.*

$$\textbf{(i) } A^rW_\lambda \text{ is closed in } \lambda, \quad \textbf{(ii) } \overline{\phi} \subset A^rB_\lambda, \quad \textbf{(iii) } \overline{\phi} \subset A^rF_\lambda,$$
$$\textbf{(iv) } \overline{\phi} = A^rW_\lambda, \quad \textbf{(v) } \overline{\phi} = A^rS_\lambda, \quad \textbf{(vi) } A^rS_\lambda \text{ is closed in } \lambda.$$

Proof. $(iv) \Rightarrow (i)$ and $(v) \Rightarrow (vi)$ are clear. $(v) \Rightarrow (iv)$, $(iv) \Rightarrow (iii)$, $(v) \Rightarrow (ii)$ and $(iii) \Rightarrow (ii)$ are by Theorem 2.1. Since $\overline{\phi}$ is an A^rK space we have $\overline{\phi} \subset A^rS_\lambda$ and so $(ii) \Rightarrow (v)$. In the other hand, $(i) \Rightarrow (iv)$ and $(vi) \Rightarrow (v)$ from

$$\phi \subset A^rS_\lambda \subset A^rW_\lambda \subset \overline{\phi}.$$

So proof has been completed. □

References

[1] B. Altay and F. Başar, *Certain topological properties and duals of the matrix domain of a triangle matrix in a sequence space*, J. Math. Anal. Appl. **336**(1)(2007), 632–645.

[2] C. Aydın and F. Başar, *On the new sequence spaces which include the spaces c_0 and c*, Hokkaido Math. J. **33**(1)(2004), 1–16.

[3] F. Başar, *A note on the triangle limitation methods*, Fırat Üniv. Fen Müh. Bil. Dergisi, **5** (1) (1993), 113-117.

[4] F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, Istanbul, 2012.

[5] J. Boos, *Classical and Modern Methods in Summability*, Oxford University Press. New York, Oxford, 2000.

[6] M. Buntinas, *Convergent and bounded Cesàro sections in FK-spaces*, Math. Zeitschr., 121 (1971), 191-200.

[7] M. Buntinas, *On sectionally dense summability fields*, Math. Zeitschr., 132 (1973), 141-149.

[8] M. Buntinas, *On Toeplitz sections in sequence spaces*, Math. Proc. Camb. Phil. Soc., 78 (1975), 451-460.

[9] İ. Dağadur, *On Some subspaces of an FK space*, Mathematical Communications, 7 (2002), 15-20.

[10] R. Devos, *Combinations of distinguished subsets and conullity*, Math. Zeitschr., 192 (1986), 447-451.

[11] D. J. H. Garling, *The β- and γ-duality of sequence spaces*, Proc. Camb. Phil. Soc., 63 (Jan. 1967), 963-981.

[12] D. J. H. Garling, *On topological sequence spaces*, Proc. Camb. Phil. Soc., 63 (1967), 997-1019.

[13] G. Goes and S. Goes, *Sequences of bounded variation and sequences of fourier coefficients. I*, Math. Zeitschr.,**118**(1970), 93-102.
[14] G. Goes, *Summan von FK-räumen funktionale abschnittskonvergenz und umkehrsatz*, Tohoku. Math. J., 26(1974), 487-504.
[15] E. Malkowsky, *Recent results in the theory of matrix transformations in sequence spaces*, Mat. Vesnik **49**(1997), 187–196.
[16] A. Wilansky, *Functional Analysis*, Blaisdell Press, 1964.
[17] A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw Hill, New York, 1978.
[18] A. Wilansky, *Summability Through Functional Analysis*, North-Holland, Amsterdam, 1984.
[19] K. Zeller, *Allgemeine eigenschaften von limitierungsverfahren*, Math. Zeitschr., 53 (1951), 463-487.

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