JAMAME Journal of Advanced Mathematics and Mathematics Education
2018, VOL. 01, NO. 1, 31-36

Galerkin Method for Numerical Solution of Advection-Diffusion Equation with constant coefficients

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Abstract In the present study, one-dimensional advection–diffusion equation with constant coefficients is solved using Galerkin Method. We give the generlized solution of this equation. Two examples are presented for the numerical solution of this equation and results are compared with exact solution.

Keywords: Advection-Diffusion Equation, Numerical Approximation, Galerkin Method.

1. Introduction and Formulation of the Problem

With respect to the importance of hydro-dynamic dispersion process studies in water quality management and pollution control particularly in aquifers, the dispersion has been referred to as a hydraulic mixing process by which the waste concentrations are attenuated while the waste pollutants are being transported downstream. The concentration distribution behavior with space and time is described by a partial differential equation of parabolic type known as advection–diffusion equation. This equation is equally important in soil physics, bio-physics, petroleum engineering and chemical engineering for describing similar processes [1]. There have been many researches related with these equations [2,3].

The mathematical expression of the one-dimensional advection-diffusion equation with the source term is as follows;

$$\psi_t - U\Delta\psi + D\psi_x = F(x,t), \quad x \in (0,l), t \in (0,T) \tag{1}$$

$$\psi(x,0) = \varphi(x), \tag{2}$$

$$\psi(x,t)\big|_{\Gamma} = 0, \quad t \in (0,T). \tag{3}$$

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Here $\psi(x,t)$ is concentration of substance, U is velocity of the flow and D is diffusion coefficient.

$\Delta = \frac{\partial^2}{\partial x^2}$ is the Laplace operator and Γ is the boundary of $(0,l)$. Also the initial displacement $\varphi(x)$ are given.

The space $L_2(0,l)$ is the set of all square integrable functions on the domain $(0,l)$. The Hilbert space called $H^{2,1}(0,l)$ is the set of functions taken from $L_2(0,l)$ to the spatial variables of second order and time variable of first order. The space $\overset{o}{H}{}^{2,1}(0,l)$ which is subspace of the space $H^{2,1}(0,l)$ consists of the functions vanishing at boundary of the domain $(0,l)$.

The paper is organized as follows: In section 2, we obtain the generalized solution for advection-diffusion equation. In section 3, we refer from the considered numerical method. In section 4, we give three examples and calculate their error norm on the space L_2.

Let us write the generalized solution of the considered problem (1)-(3) [4].

Definition 1. The generalized solution of the problem (1)-(3) is the function $\psi \in \overset{o}{H}{}^{2,1}(\Omega)$ and it satisfies the following integral equality;

$$\int_{\Omega} (-\psi_t + U\Delta\psi + D\psi_x)\mu dxdt = \int_{\Omega} F(x,t)\mu(x,t)dxdt \tag{4}$$

for all $\mu(x,t) \in L_2(\Omega)$ [5].

2. Numerical Solution: Galerkin Method

In this section, we give information about Galerkin method which is efficient to solve partial differential equations. Galerkin methods have been utilized to solve the problems encountered structural mechanics, dynamics, fluid flow, heat and mass transfer, acoustics, microwave theory, neutron transport, etc. The Galerkin method can be thought of as the calculus of variations performed backwards. That is, the Galerkin method seeks to find the weak form expressed in terms of integrals, and solve that, instead of solving the strong form of the problem as a differential equation.

Galerkin method is an influential numerical method for solving different types of partial differential equations [6,7,8]. The basic Galerkin methods with piecewise linear basis functions and quadratic basis functions have been compared in [9].

The main contribution of this study deal with numerical solution of advection-diffusion equation with constant coefficient.

By using Galerkin Method, the approximate solutions for the problem (1)-(3) are written as follows;

$$\psi^N(x,t)=\sum_{i=1}^{N} c_i^N(t)v_i(x)$$

where the coefficients $c_i^N(t)$ are the functions such that

$$c_i^N(t)=\langle \psi^N(x,t), v_i(x)\rangle_{L_2(0,l)}$$

for $i, j=1,2,...,N$ and the functions $v_i(x)$ are basis functions for which $\langle v_i(x), v_j(x)\rangle_{L_2(0,l)}=\delta_j^i$ is valid . Here δ_j^i is Kronecker delta. Let us write the equation (1) for the approximations $\psi^N(x,t)$ so that we obtain the coefficients $c_i^N(t)$

$$\sum_{i=1}^{N}\frac{dc_i^N(t)}{dt}v_i(x)-U\sum_{i=1}^{N}c_i^N(t)\Delta v_i(x)+D\sum_{i=1}^{N}c_i^N(t)v_i(x)=F(x,t) \quad (5)$$

and then integrate on the domain $(0,l)$ after multiplying both side of equality (5) with the function $v_j(x)$. Hence we obtain the equality such as,

$$\int_0^l\left[\sum_{i=1}^{N}\frac{dc_i^N(t)}{dt}v_i(x)-U\sum_{i=1}^{N}c_i^N(t)\Delta v_i(x)+D\sum_{i=1}^{N}c_i^N(t)v_i(x)\right]v_j(x)dx=\int_0^l F(x,t)v_j(x)dx \quad (6)$$

where $\langle v_i(x), v_j(x)\rangle_{L_2(0,l)}=0$ and $\langle \Delta v_i(x), v_j(x)\rangle_{L_2(0,l)}=0$ as $i\neq j$.

Let us we write this system in the matrix form of

$$\begin{aligned}&\frac{d}{dt}C^N(t)+V(t)C^N(t)=F(t)\\&C^N(0)=D,\end{aligned} \quad (7)$$

where $C^N(t)$ is the matrix of searched functions.

By calculating the coefficients $c_i^N(t)$, we obtain the approximate solution $\psi^N(t)$. The equation (21) states a system of first order ODE and the solution of this system is uniquely solvable.

3. Numerical Illustrations

In this section, we present test problems after giving theoretical information mentioned in the previous sections. Before giving illustrations, we refer from its properties and basis functions which we will use for solving examples. In this article, we take the following fundamental set as basis functions

$$\{v_i(x)\} = \left\{ \sqrt{\frac{2}{l}} \sin\left(\frac{\pi x}{l}\right), \ldots, \sqrt{\frac{2}{l}} \sin\left(\frac{i\pi x}{l}\right), \ldots, \sqrt{\frac{2}{l}} \sin\left(\frac{N\pi x}{l}\right) \right\}$$

for which the following statements are hold;

- $\langle v_i(x), v_j(x) \rangle_{L_2(0,l)} = \begin{Bmatrix} 0, & i \neq j \\ 1, & i = j \end{Bmatrix}$
- $\langle \Delta v_i(x), v_j(x) \rangle_{L_2(0,l)} = \begin{Bmatrix} 0, & i \neq j \\ \left(\frac{i\pi}{l}\right)^2, & i = j \end{Bmatrix}.$

Example 1. Let us consider problem (1)-(3) on the domain $(x,t) \in [0,2] \times [0,2]$ and $\varphi(x) = \sin(3\pi x), U = 3, D = 2$ and

$$F(x,t) = e^{-t}\left(27\pi^2 \sin(3\pi x) + 6\pi \cos(3\pi x) - \sin(3\pi x)\right)$$

In this case, by solving the system (7), numerical solution is obtained and here the exact solution is $\psi(x,t) = e^{-t} \sin(3\pi x)$.

The graphs of these solutions are given in Figure 1.

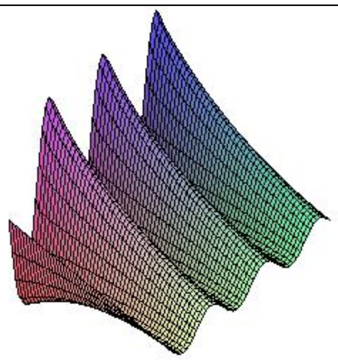
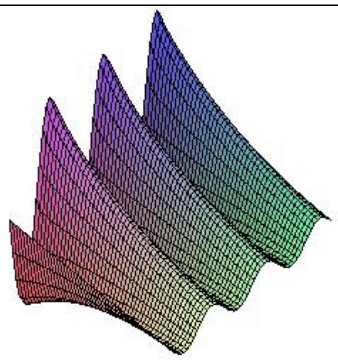
Approximate Solution	Exact Solution	Errors for $\lVert \psi^N - \psi \rVert_{L_2(0,l)}$	
		$t=0$	0
		$t=0.2$	0.008
		$t=0.4$	0.007
		$t=0.6$	0.005
		$t=0.8$	0.003
		$t=1$	0.002

Figure 1: The graph of the approximate solution and exact solution for $N = 20$.

Example 2. Let us analyze problem (1)-(3) on the domain $(x,t)\in[0,1]\times[0,2]$ and $U=8, D=1, \varphi(x)=0$

and

$$F(x,t)=4tx^3+x^4-102tx^2-2x^3+98tx+x^2-16t$$

In this case, the exact solution is

$$u(\mathrm{x},t)=x^2(x-1)^2t.$$

The graphs of these solutions are given in Figure 2.

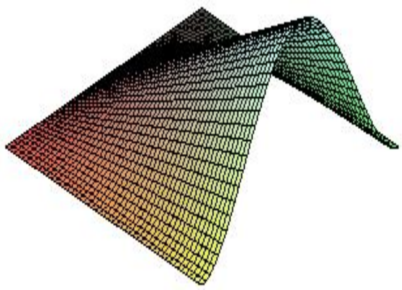
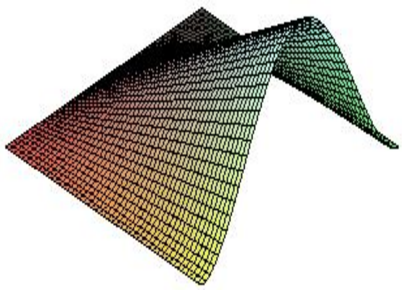
<table>
<tr><th rowspan="2">Approximate Solution</th><th rowspan="2">Exact Solution</th><th colspan="2">Errors for $\left\|u^N-u\right\|_{L_2(\phi)}$</th></tr>
<tr></tr>
<tr><td rowspan="6"></td><td rowspan="6"></td><td>$t=0$</td><td>0</td></tr>
<tr><td>$t=0.2$</td><td>0.73×10^{-8}</td></tr>
<tr><td>$t=0.4$</td><td>0.29×10^{-7}</td></tr>
<tr><td>$t=0.6$</td><td>0.67×10^{-7}</td></tr>
<tr><td>$t=8$</td><td>0.11×10^{-6}</td></tr>
<tr><td>$t=10$</td><td>0.18×10^{-6}</td></tr>
</table>

Figure 2: The graph of the approximate solution and exact solution for $N=20$. .

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