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Some Results on Especial Diophantine Sets with Size-3

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Abstract

Purpose of this paper is to determine some regular non-extendible $D(n)$ triples for some fixed integer n. Besides, paper includes a number of algebraic properties for such diophantine sets with size three.

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1.Introduction and Preliminaries

There are a lot of significant and attracted results in the literature related with Diophantine sets and equations. One of them was started by a Greek mathematician Diophantus of Alexandria in the 3rd century. Mathematicians have been interested in Diophantine sets for a long time before due to unsolved problems in the literature.

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For this paper, we use some basic notions such as quadratic reciprocity and residues ([3, 7, 13-16,22]), legendre symbol ([5, 12]) and Diophantine sets with their regularity ([6,8-10, 17-21]) as well as significant books ([1, 2, 4, 6, 10, 16]) from algebraic and elemantary number theories. We obtain regular non-extendibility of some *D*(*n*) Diophantine triples where n is 31 or -31.Additionally, we demontrate that some types of elements can not be in *D*(∓ 31).

Definition 1.1. ($[\mathbf{6}, \mathbf{8}, \mathbf{9}]$) A Diophantine *m*-tuple with the property *D*(*n*) (it sometimes representatives as $\boldsymbol{P_n}$ with *m*-tuples) for *n* an integer is an *m*-tuple of different positive integers $\{\boldsymbol{\beta_1}, \dots, \boldsymbol{\beta_n}\}$ such that $\boldsymbol{\beta_i \beta_j + n}$ is always a square of an integer for every distinct i, j.

As a special case, If *n* =3 then it is called by *D*(*n*) - Diophantine triple.

Definition 1.2. ($[\mathbf{8}]$) If $\boldsymbol{D(n)}$- triple $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ satisfies the following condition

$$\boldsymbol{(w - v - u)^2 = 4(u.v + n)} \qquad \mathbf{(1.1)}$$

then $\{u, v, w\}$ is called Regular Diophantine Triple.

Definition 1.3. ($[13, 15]$)Let q be an odd prime and u be an integer such that $gcd(u, q) = 1$. The quadratic residue symbol $\left(\frac{\boldsymbol{u}}{\boldsymbol{q}}\right)$ is defined to be 1 or -1 according as the congruence $x^2 \equiv u$ (mod q) is solvable or not.

Also, Quadratic Reciprocity law was formulated by Euler although Legendre discovered it independently of Euler in 1785. We can see some results on this law as follows:

Theorem 1.1. ($[12]$) (Quadratic Reciprocity Law) Let p, q be different odd primes. Then,

(i) If $p \equiv 1(mod\ 4)$ or $q \equiv 1(mod\ 4)$, then p is a square (mod q) if and only if q is a square (mod p).
(ii) If $p \equiv 3(mod\ 4)$ and $q \equiv 1(mod\ 4)$, then p is a square (mod q) if and only if q is not a square (mod p).

Theorem 1.2. ($[2, 12, 14, 22]$) (First Supplement to the Quadratic Reciprocity Law) Let q be an odd prime. Then, -1 is a square (mod q) necessary and sufficient condition $q \equiv 1(mod\ 4)$ holds.

Theorem 1.3. ([12,14]) (Second Supplement to the Quadratic Reciprocity Law) Let p be an odd prime. Then 2 is a square mod q if and only if $q \equiv 1, 7\ (mod\ 4)$.

Definition 1.4. ([5, 7]) The symbol $\left(\frac{\alpha}{q}\right)$ is called Legendre Symbol, if $\left(\frac{\alpha}{q}\right)$ equals to 1. It means that α is a quadratic residue.

$$\left(\frac{\alpha}{q}\right) = \begin{cases} 1 & \text{if } \alpha \text{ is a quadratic residue modulo } q \\ -1 & \text{if } \alpha \text{ is a non} - \text{quadratic residue modulo } q \end{cases} \tag{1.2}$$

Proposition 1.1. ([5, 7]) Let q be an odd prime. Then following properties are satisfied.

(a) $m \equiv n\ (mod\ q)$ implies $\left(\frac{m}{q}\right) = \left(\frac{n}{q}\right)$

(b) The Legendre Symbol is multiplicative: $\left(\frac{m}{q}\right).\left(\frac{n}{q}\right) = \left(\frac{m.n}{q}\right)$ where m, n are integers and coprime to q prime.

2. Theorems And Results

Theorem 2.1. $P_{+31} = \{2, 9, 25\}$ is regular but non-extendible Diophantine triple.

Proof. For regularity, we consider the condition (1.1) of Definition 1.2. So, it is seen that $P_{+31} = \{2, 9, 25\}$ is a regular triple. We suppose that $\{2, 9, 25\}$ can extendible to Diophantine quadruple for any positive integer ϑ and $\{2,9,25\ \vartheta\}$ is a P_{+31}quadruple. Then, there are a_1, a_2, a_3 integers such that,

$$2\vartheta + 31 = {a_1}^2 \tag{2.1}$$

$$9\vartheta + 31 = {a_2}^2 \tag{2.2}$$

$$25\vartheta + 31 = {a_3}^2 \tag{2.3}$$

Eliminating ϑ between (2.2) and (2.3), we obtain following equation:

$$25{a_2}^2 - 9{a_3}^2 = 496 \tag{2.4}$$

Applying factorization method on the (2.4), we get a table as follows:

Table 2.1. Solutions of $25{a_2}^2 - 9{a_3}^2 = 496$

Solutions	1.Class of Solutions	2.Class of Solutions
(a_2, a_3)	$(\mp 25, \mp 41)$	$(\mp 7, \mp 9)$

Considering (2.1) and (2.2), we obtain

$$9{a_1}^2 - 2{a_2}^2 = 217 \tag{2.5}$$

Considering above solutions, we have ${a_2}^2 = 625$, ${a_2}^2 = 49$. If we put these values into the (2.5) we have ${a_1}^2 = 163$, ${a_1}^2 = 35$. This is a contradiction since a_1 isn't an integer.

So, there is no such ϑ positive integer and $P_{+31} = \{2, 9, 25\}$ can be non-extended to P_{+31} Diophantine quadruple.

Theorem 2.2. A set $P_{+31} = \{3,46,75\}$ is both regular and non-extendible to the P_{+31} Diophantine quadruple.

Proof. $P_{+31} = \{3,46,75\}$ satisfies (1.1) regularity condition of Definition 1.2. So, it is regular. Supposing that $P_{+31} = \{3,46,75, \omega\}$ be a P_{+31} Diophantine quadruple for positive integer ω. There are b_1, b_2, b_3 integers such that

$$3\omega + 31 = {b_1}^2 \tag{2.6}$$

$$46\omega + 31 = {b_2}^2 \tag{2.7}$$

$$75\omega + 31 = {b_3}^2 \tag{2.8}$$

hold. Dropping ω from (2.6) and (2.8), we have

$$25{b_1}^2 - {b_3}^2 = 744 \tag{2.9}$$

If we use factorization method into the (2.9), we obtain solutions in a following table:

Table 2.2. Solutions of $25{b_1}^2 - {b_3}^2 = 744$

Solutions	1.Class of Solutions	2.Class of Solutions
(b_1, b_3)	$(\mp 19, \mp 91)$	$(\mp 13, \mp 59)$

Dropping ω from (2.6) and (2.7), then we have

$$-3{b_2}^2 + 46{b_1}^2 = 1333 \tag{2.10}$$

From Table 2.2, we have ${b_1}^2 = 361$ or ${b_1}^2 = 169$. If we substitute them into the (2.10), we obtain ${b_2}^2 = 5091$ or ${b_2}^2 = 2147$ respectively. It is a contradiction and b_2 isn't integer solution for (2.10).

Thus, there is not positive integer ω and $P_{+31} = \{3,46,75\}$ can not be extended to P_{+31} Diophantine quadruple.

Theorem 2.3. $P_{+31} = \{3,75,110\}$ is a regular triple but can not nonextendible to P_{+31} Diophantine triple.

Proof. First of all, let's show that $P_{+31} = \{3,75,110\}$ is a regular triple. If we use regularity condition for $P_{+31} = \{3,75,110\}$, it is easily seen that the set holds condition (1.1). That is why, set is regular.

Similarly, let us assume that $P_{+31} = \{3,75,110,\alpha\}$ be a Diophantine quadruple for positive integer α. So, we get $c_1, c_2, c_3 \in \mathbb{Z}$ such that following equations are satisfied.

$$3\alpha + 31 = {c_1}^2 \tag{2.11}$$

$$75\alpha + 31 = {c_2}^2 \tag{2.12}$$

$$110\alpha + 31 = {c_3}^2 \tag{2.13}$$

If we reduce α from (2.11) and (2.12), we obtain an equation as same as (2.9) for (c_1, c_2). So, we have the solutions as same as Table 2.2 for (c_1, c_2). Dropping α from (2.11) and (2.13), then we obtain

$$110{c_1}^2 - 3{c_3}^2 = 3317 \tag{2.14}$$

Substituting ${c_1}^2 = 361$ or ${c_1}^2 = 169$ into the (2.14), ${c_3}^2 = 12131$, or ${c_3}^2 = 5091$ are got. It is seen that c_3 is not integer solution for (2.14). Thus, it is a contradiction.

Therefore, there is no positive integer α and also $P_{+31} = \{3,75,110\}$ can not be extended to P_{+31} Diophantine quadruple.

Theorem 2.4. A set $P_{+31} = \{9,25,66\}$ regular diophantine triple and nonextendible to P_{+31} quadruple.

Proof. $P_{+31} = \{9,25,66\}$ holds (1.1) condition in the Definition 1.2 . That is why it is regular. Assume that $P_{+31} = \{9,25,66,g\}$ is Diophantine quadruple for $g \in \mathbb{Z}^+$. Definition 1.1 implies that

$$9g + 31 = {d_1}^2 \tag{2.15}$$

$$25g + 31 = {d_2}^2 \tag{2.16}$$

$$66g + 31 = {d_3}^2 \tag{2.17}$$

for $d_1, d_2, d_3 \in \mathbb{Z}$. Simplification of (2.15) and (2.16), we have;

$$25{d_1}^2 - 9{d_2}^2 = 496 \tag{2.18}$$

This equation is similar to (2.4) for (d_1, d_2). From Table 2.1, we obtain ${d_1}^2 = 625$, ${d_1}^2 = 49$. From (2.15) and (2.17), we obtain

$$22{d_1}^2 - 3{d_3}^2 = 589 \tag{2.19}$$

Substituting ${d_1}^2 = 625$, ${d_1}^2 = 49$ into the (2.19), we have ${d_3}^2 = 4387$ and ${d_3}^2 = 163$, respectively. This is a contradiction since d_3 is not integer solution of (2.19).

Hence, $P_{+31} = \{9,25,66\}$ can not extendible to P_{+31} Diophantine quadruple.

Theorem 2.5. Both $P_{-31} = \{2,100,128\}$ and $P_{-31} = \{2,128,160\}$ are regular Diophantine triple and also non-extendible.

Proof. We can see that both $P_{-31} = \{2,100,128\}$ and $P_{-31} = \{2,128,160\}$ are regular Diophantine triples from (1.1) condition.

Suggesting that $\{2,100,128,\hbar\}$ is a P_{-31} Diophantine quadruple for positive integer $\hbar$. Then,, there are $e_1, e_2, e_3 \epsilon \mathbb{Z}$ such that

$$2\,\hbar - 31 = {e_1}^2 \tag{2.20}$$

$$100\,\hbar - 31 = {e_2}^2 \tag{2.21}$$

$$128\,\hbar - 31 = {e_3}^2 \tag{2.22}$$

Simplifying $\hbar$ between (2.20) and (2.22), we get

$$-64{e_1}^2 + {e_3}^2 = 1953 \tag{2.23}$$

and similarly from (2.20) and (2.21)

$$-50{e_1}^2 + {e_2}^2 = 1519 \tag{2.24}$$

By factorizing (2.23), we obtain following table for solutions.

Table 2.3. Solutions of $-64{e_1}^2 + {e_3}^2 = 1953$

Solutions	1.Class of Solutions	2.Class of Solutions	3.Class of Solutions	4.Class of Solutions
(e_3, e_1)	$(\mp 977, \mp 122)$	$(\mp 143, \mp 17)$	$(\mp 113, \mp 13)$	$(\mp 47, \mp 2)$

By substituting ${e_1}^2 = 14884$, ${e_1}^2 = 289$, ${e_1}^2 = 169$, ${e_1}^2 = 4$ into the (2.24), we get ${e_2}^2 = 745719$ ${e_2}^2 = 15969$, ${e_2}^2 = 8450$, ${e_2}^2 = 1719$ respectively. It is seen that it is a contradiction since $e_2 \notin \mathbb{Z}$.

As a consequence, $P_{-31} = \{2,100,128\}$ can not be extended to P_{-31} Diophantine quadruple.

Let $P_{-31} = \{2, 128, 160, \mathcal{L}\}$ be a Diophantine quadruple for $\mathcal{L} \in \mathbb{Z}^+$. From Definition 1.1, we have

$$2\mathcal{L} - 31 = {f_1}^2 \quad (2.25)$$

$$128\mathcal{L} - 31 = {f_2}^2 \quad (2.26)$$

$$160\mathcal{L} - 31 = {f_3}^2 \quad (2.27)$$

for $f_1, f_2, f_3 \in \mathbb{Z}$. Dropping $\mathcal{L}$ from (2.25) and (2.26), we have an equation like (2.19). Hence, Table2.3 can be used for (f_2, f_1) instead of (e_3, e_1). From (2.25) and (2.27), we also have

$$-80{f_1}^2 + {f_3}^2 = 2449 \quad (2.28)$$

Putting ${f_1}^2 = 14884$, ${f_1}^2 = 289$, ${f_1}^2 = 169$, ${f_1}^2 = 4$ into the (2.28), we have ${f_3}^2 = 1193169$, ${f_3}^2 = 25569$, ${f_3}^2 = 15969$, ${f_3}^2 = 2749$. It is a contradiction because f_3 is not an integer solution of (2.28).

Therefore, $P_{-31} = \{2, 128, 160\}$ is nonextendable to P_{-31} Diophantine triple.

Theorem 2.6. A set $P_{-31} = \{4, 64, 98\}$ is not only regular but also nonextendible Diophantine triple.

Proof. Regularity of $P_{-31} = \{4, 64, 98\}$ can be easily seen from (1.1). In the same way, supposing that $P_{-31} = \{4, 64, 98, \mathcal{M}\}$ be a Diophantine quadruple for positive integer $\mathcal{M}$. We get $g_1, g_2, g_3 \in \mathbb{Z}$ such that

$$4\mathcal{M} - 31 = {g_1}^2 \quad (2.29)$$

$$64\mathcal{M} - 31 = {g_2}^2 \quad (2.30)$$

$$98\mathcal{M} - 31 = {g_3}^2 \quad (2.31)$$

Dropping $\mathcal{M}$ from (2.29) and (2.30), we get an equation as follows:

$${g_2}^2 - 16{g_1}^2 = 465 \quad (2.32)$$

and if we eliminate $\mathcal{M}$ from (2.29) and (2.31), then

$$2{g_3}^2 - 49{g_1}^2 = 1457 \quad (2.33)$$

is obtained. Table 2.4 is got from (2.32) as follows:

Table 2.4. Solutions of $g_2{}^2 - 16g_1{}^2 = 465$

Solutions	1.Class of Solutions	2.Class of Solutions	3.Class of Solutions	4.Class of Solutions
(g_2, g_1)	$(\mp 233, \mp 58)$	$(\mp 79, \mp 19)$	$(\mp 49, \mp 11)$	$(\mp 23, \mp 2)$

Substituting $g_1{}^2 = 3364$, $g_1{}^2 = 361$, $g_1{}^2 = 121$, $g_1{}^2 = 4$, into the (2.33), $g_3{}^2 = \frac{166293}{2}$, $g_3{}^2 = 9573$, $g_3{}^2 = 3693$, $g_3{}^2 = \frac{1653}{2}$ are got. it is a contradiction since g_3 is not integer solution for (2.33)

So, $P_{-31} = \{4, 64, 98\}$ can not be extended to P_{-31} Diophantine quadruple.

Theorem 2.7. Following conditions are satisfied for $P_{\pm 31}$ sets.

(a) There isn't any set P_{+31} includes any multiplication of 4,7, 13, or 19.
(b) There is no set P_{-31} involves any multiplication of 3,11, 13, or 17.

Proof. **(a) (i)** Supposing that a is an element of set P_{+31}. If $4m \in P_{+31}$ for $m \in Z$,then

$$4ma + 31 = \mathrm{X}^2 \tag{2.34}$$

satisfy for some integer X. Applying (mod 4) on the (2.34), we obtain

$$\mathrm{X}^2 \equiv 3 \ (mod\ 4) \tag{2.35}$$

Since $\mathrm{X} \in Z$, then X is even or odd integer. So, (2.35) can not has a solution. This is a contradiction. Hence, $4m \notin P_{+31}$ for $m \in Z$.

(ii) Similarly, assuming that $b \in P_{+31}$ and $7n \in P_{+31}$ for $(n \in Z)$. So,

$$7nb + 31 = \Psi^2 \tag{2.36}$$

holds for integer Ψ. By (mod 7), we obtain

$$\Psi^2 \equiv 3 \ (mod\ 7) \tag{2.37}$$

From Theorem 1.1 and Definition 1.4,

$$\left(\frac{3}{7}\right)\left(\frac{7}{3}\right) = (-1)^{\frac{3-1}{2} \cdot \frac{7-1}{2}} = -1 \tag{2.38}$$

is hold.. Using Proposition 1.1, we have $\left(\frac{7}{3}\right) = \left(\frac{1}{3}\right) = +1$ and substituting it into the (2. 38) then $\left(\frac{3}{7}\right) = -1$ is obtained . This implies that equivalent (2.38) isn't solvable. So, $7n \notin P_{+31}$ for $n \in Z$.

(iii) In the same vein, if $c \in P_{+31}$ and $13k \in P_{+31}$ for $(k \in Z)$, then

$$13kc + 31 = \Omega^2 \tag{2.39}$$

holds for integer Ω. Applying (mod 13) on (2.39), we have

$$\Omega^2 \equiv 5 \ (mod\ 13) \tag{2.40}$$

Using Theorem 1.1 and Definition 1.4, then

$$\left(\frac{5}{13}\right)\left(\frac{13}{5}\right) = (-1)^{\frac{5-1}{2}\cdot\frac{13-1}{2}} = +1 \tag{2.41}$$

is satisfied. By substituting $\left(\frac{13}{5}\right) = \left(\frac{3}{5}\right) = -1$ into the (2.41) then we get $\left(\frac{5}{13}\right) = -1$. This is a contradiction since (2.40) can not sovable. Therefore, $13k \notin P_{+31}$ for $k \in Z$.

(iv) Assume that if $d \in P_{+31}$ and $19l \in P_{+31}$. Then,

$$19ld + 31 = \mathrm{T}^2 \tag{2.39}$$

holds for integer T. Applying (mod 19) on (2.39), we get

$$\mathrm{T}^2 \equiv 12 \ (mod\ 19) \tag{2.40}$$

From Legendre symbol's properties and Theorem 1.3, we have

$$\left(\frac{12}{19}\right) = \left(\frac{4}{19}\right).\left(\frac{3}{19}\right) = \left(\frac{2}{19}\right)\left(\frac{2}{19}\right)\left(\frac{3}{19}\right) \tag{2.41}$$

Using Definition 1.4, we obtain $\left(\frac{3}{19}\right) = -1$. It requires that $\left(\frac{12}{19}\right) = -1$ and it is a contradiction So, $19l \notin P_{+31}$ for $l \in Z$.

(b) Along the same line, assume that any multiply of 3,11, 13, or 17 are in the P_{-31}.

Briefly and respectively, we get

$$\mathrm{A}^2 \equiv 2 \ (mod\ 3) \tag{2.42}$$

$$\mathrm{B}^2 \equiv 2 \ (mod\ 11) \tag{2.43}$$

$$\mathrm{C}^2 \equiv 8 \ (mod\ 13) \tag{2.44}$$

$$\mathrm{D}^2 \equiv 3 \ (mod\ 17) \tag{2.45}$$

From Theorem 1.3, we calculate $\left(\frac{2}{3}\right) = -1$, $\left(\frac{2}{11}\right) = -1$ and $\left(\frac{2}{13}\right) = -1$. This is a contradiction since (2.42), (2.43) and (2.44) are unsolvable. Besides, from Theorem 1.1 and Definition 1.4, we get $\left(\frac{3}{17}\right) = -1$ which means that (2.45) is not solvable. So, any multiplication of 3,11, 13, or 17 are not in the P_{-31}.

Remark 2.8. One may find different regular non- extendible triples P_{+31} or P_{-31} and also extend the Theorem 2.7 using our method.

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