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The Directional Curves of Spacelike and Timelike Frenet Curves in E_1^3

Semra Yurttançıkmaz[1], Sezai Kızıltuğ[2], Ali Çakmak[3]

[1]Department of Mathematics, Faculty of Sciences,
Ataturk University, Erzurum, Turkey
semrakaya@atauni.edu.tr
[2]Department of Mathematics, Faculty of Arts and Sciences,
Erzincan Binali Yıldırım University, Erzincan, Turkey
skiziltug@erzincan.edu.tr
[3]Department of Mathematics, Faculty of Arts and Sciences,
Bitlis Eren University, Bitlis, Turkey
acakmak@beu.edu.tr

Abstract: In this paper, we define some special curves by using spacelike and timelike curves in three dimensional Minkowski space. Also, we give some new characterizations and results for these curves.

Keywords: Associated curves, osculating directional curves, osculating donor curves.

1. Introduction

The Lorentz geometry or more specifically the Minkowski geometry is used to explain a lot of events in mathematical physics. For example; the gravitation of a single star, perihelion procession, bending of light and black holes. At the same time, Einstein's relativity theory gains meaning with this geometry [1].

On the other hand, the theory of curves is still one of the most important interesting topics in differential geometry and it is being studied by many mathematicians until now [2, 3, 4, 5, 6, 7, 8, 9, 10]. The most fascinating and important subject of the theory of curves is to obtain the characterizations for a regular curve or a curve pair such as helix, slant helix, plane curve, spherical curve, etc. The definitions of aforementioned curves can be given by considering Frenet planes. The curve for which the position vector always lies on its rectifying, osculating or normal planes are called rectifying curve, osculating curve or normal curve, respectively [11].

In addition to some special curves, frequently examine in terms of differential geometry the curve pairs which there exits some relationships between their Frenet vectors or curvatures. Lately, a new curve pair in the Minkowski space has been defined by Choi, Kim and Ali [12]. They have represented the notion of the principal (binormal)-directional curve and the principal (binormal)-donor curve of the Frenet curve in the Minkowski space.

Here, we introduce the notion of osculating directional curve and osculating donor curve of the spacelike and timelike Frenet curve in the Minkowski 3-space E_1^3 and give some characterizations and results for these curves. Moreover, we show that osculating-direction curve and its osculating donor curve form a Mannheim pair.

2. Preliminaries

The Minkowski 3-space E_1^3 is the real vector space $\mathbb{R}^3$ provided with the standart flat metric given by

$$\langle\, ,\rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. An arbitrary vector $v = (v_1, v_2, v_3)$ in E_1^3 can have one of three Lorentzian causal characters; it can be spacelike if $\langle v, v\rangle > 0$ or $\vec{v} = 0$, timelike if $\langle v, v\rangle < 0$ and null (lightlike) if $\langle v, v\rangle = 0$and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\alpha: I \to E_1^3$ can locally be spacelike, timelike or null, if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike) [13, 14]. We say that a timelike vector is future pointing or past pointing if the first component of the vector is positive or negative, respectively. For any vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in E_1^3, in the meaning of Lorentz vector product of $\vec{u}$ and $\vec{v}$ is defined by

$$\vec{u} \times \vec{v} = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2, u_1v_3 - u_3v_1, u_2v_1 - u_1v_2),$$

where $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$ and $e_1 \times e_2 = -e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = -e_2$.

Denote by $\{T, N, B\}$ the moving Frenet frame along the curve $\alpha(s)$ in E_1^3. In this case, the Frenet formulas are given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \varepsilon_B\kappa & 0 & \tau \\ 0 & \varepsilon_T\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $\varepsilon_x = \langle X, X\rangle$, κ is the curvature function and $\tau$is the torsion function. A spacelike curve $\alpha(s)$ with timelike principal normal N (timelike binormal vector B) is called a spacelike curve of type 1(type 2). Then we have $\tau(s) = \varepsilon_N\varepsilon_B\langle B', N\rangle$ and the Frenet vectors satisfy the relation

$$B = \varepsilon_T\varepsilon_N T \times N,\ N = \varepsilon_B\varepsilon_T B \times T, T = \varepsilon_N\varepsilon_B N \times B, \varepsilon_B = -\varepsilon_T\varepsilon_N.$$

Theorem 2.1. Let α be a unit speed non-lightlike curve in the Minkowski 3-space. Then α is a slant helix if and only if either one the next two functions

$$\frac{\kappa^2}{(\tau^2 \pm \kappa^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$$

or

$$\frac{\kappa^2}{(\kappa^2 - \tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$$

is constant everywhere $\tau^2 - \kappa^2$ does not vanish [14].

Now, we define some associated curves of a curve α in E_1^3 defined on an open interval $I \subset \mathbb{R}$. For a non-lightlike Frenet curve $\alpha: I \to E_1^3$, consider a vector field X given by

$$X(s) = u(s)T(s) + v(s)N(s) + w(s)B(s) \tag{1}$$

where u, v and w are arbitrary differentiable functions of s which is the arc length parameter of α. Let

$$\varepsilon_T u^2(s) + \varepsilon_N v^2(s) + \varepsilon_B w^2(s) = \pm 1 \tag{2}$$

holds. In [15], Choi and Kim introduced the definition of X -direction curve and X -donor curve in E^3. We consider this definition in E_1^3 as follows.

Definition 2.1. *Let α be a non-lightlike Frenet curve in Minkowski 3-space $E_1^3$and X be a unit vector field satisfying the equations (1) and (2). The integral curve $\gamma: I \to E_1^3$ of X is called an X-direction curve of α. The curve α whose X - direction curve is γ is called the X-donor curve of γ in E_1^3.*

Definition 2.2. *An integral curve of principal normal vector $N(s)$ (resp. binormal vector $B(s)$) of α in (2) is called the principal-direction curve (resp. binormal-direction curve) of α in E_1^3.*

Remark 2.1. A principal-direction (resp. the binormal-direction) curve is an integral curve of $X(s)$ with $u(s) = w(s) = 0, v(s) = 1$ (resp. $u(s) = v(s) = 0, w(s) = 1$) for all s in (1) [15].

3. Osculating direction and osculating donor curves of non-lightlike curves in E_1^3

Önder and Kızıltuğ introduced the definition of osculating-direction and osculating-donor curves in Euclidean space E^3 [16].

In the following, we will define these concepts in Minkowski 3-space E_1^3.

Definition 3.1. *Let α be a non-lightlike Frenet curve in E_1^3 and X be a unit vector field lying on the osculating plane of α and defined by*

$$X(s) = u(s)T(s) + v(s)N(s), u(s) \neq 0, v(s) \neq 0, \quad (3)$$

and satisfying that the vectors $X'(s)$ and $B(s)$ are linearly dependent. The integral curve $\gamma: I \to E_1^3$ of $X(s)$ is called an osculating-direction curve of α. The curve α whose osculating-direction curve is γ is called the osculating-donor curve in E_1^3.

Since $X(s)$ is a unit vector and $\gamma: I \to E_1^3$ is an integral curve of $X(s)$, without loss of generality we can take s as the arc length parameter of γ and we can give the following characterizations in the view of these information.

Theorem 3.1. Let α be a non-lightlike Frenet curve in E_1^3, $X(s) = u(s)T(s) + v(s)N(s)$ be a vector valued function and the curve $\gamma: I \to E_1^3$ be an integral curve of X. Then, γ is an osculating-direction curve of α if and only if

i) $u = sinh(-\int \kappa(s)ds), v = cosh(-\int \kappa(s)ds),$

if α is a timelike curve and γ is a spacelike curve type 2 or α is a spacelike curve of type 1 and γ is a timelike curve,

ii) $u = cosh(-\int \kappa(s)ds), v = sinh(-\int \kappa(s)ds),$

if both α and γ are timelike curves or α is a spacelike curve of type 1 and γ is a spacelike curve type 2,

iii) $u = cos(-\int \kappa(s)ds), v = sin(-\int \kappa(s)ds),$

if α is a spacelike curve of type 2 and γ is a spacelike curve type 1.

Proof: Since γ is the osculating-direction curve of α, from Definition (3.1) we have

$$X(s) = u(s)T(s) + v(s)N(s), \quad (4)$$

and

$$\varepsilon_T u^2(s) + \varepsilon_N v^2(s) = \pm 1, \quad (5)$$

Differentiating (4) with respect to s and by using the Frenet formulas, it follows

$$X'(s) = (u' + \varepsilon_B \kappa v)T + (v' + \kappa u)N + v\tau B \tag{6}$$

Since we have that X' and B are linearly dependent, $X'(s) \notin Sp\{T, N\}$. Then from (6) we can write

$$\begin{cases} u' + \varepsilon_B v\kappa = 0, \\ v' + u\kappa = 0, \\ v\tau \neq 0. \end{cases} \tag{7}$$

Let now consider the Lorentzian casual characters of the curves to solve the system (7). Since $\gamma: I \rightarrow E_1^3$ is an integral curve of α we have that the character of γ is same with X. Here, it's not considered some cases related to casual characters of the curves α and γ which contrary to rule that two timelike vectors cannot be perpendicular to each other.

i) If α is a timelike curve and γ is a spacelike curve type 2, then $-u^2(s) + v^2(s) = 1$. So, $-uu' + vv' = 0$ and from (7) we have

$$\begin{cases} u' + v\kappa = 0, \\ v' + u\kappa = 0, \\ v\tau \neq 0. \end{cases} \tag{8}$$

From the first and second equations of system (8) we have,

$$u = sinh\left(-\int \kappa(s)ds\right), \quad v = cosh\left(-\int \kappa(s)ds\right) \tag{9}$$

Besides, if α is a spacelike curve of type 1 and γ is a timelike curve, then system (7) becomes similarly the system (10) and $u^2(s) - v^2(s) = -1$. So, we get same results with (9).

ii) If both α and γ are timelike curves, then we have the same system (8) with $-u^2(s) + v^2(s) = -1$. It has the solution

$$u = cosh\left(-\int \kappa(s)ds\right), \quad v = sinh\left(-\int \kappa(s)ds\right). \tag{10}$$

Also, if α is a spacelike curve of type 1 and γ is a spacelike curve type 2, then system (7) becomes the system (8) and $u^2(s) - v^2(s) = 1$. Then we have again equations in (10).

iii) If α is a spacelike curve of type 2 and γ is a spacelike curve type 1, then system (7) becomes

$$\begin{cases} u' - v\kappa = 0, \\ v' + u\kappa = 0, \\ v\tau \neq 0. \end{cases}$$

and $u^2(s) + v^2(s) = 1$. Therefore, we have following solutions

$$u = cos\left(-\int \kappa(s)ds\right), \quad v = sin\left(-\int \kappa(s)ds\right). \tag{11}$$

Theorem 3.2. Let α be a non-lightlike Frenet curve in E_1^3. If γ is the osculating-direction curve of α, then γ is a Mannheim curve of α.

Proof: Since γ is an integral curve of X, we have $\gamma^{'} = X$. Denote the Frenet frame of γ by $\{\bar{T}, \bar{N}, \bar{B}\}$. Differentiating $\gamma' = \overline{T} = X$ with respect to s and by using Frenet formulas we get

$$X' = \bar{T}' = \bar{\kappa}\bar{N} \tag{12}$$

Moreover, we know that X' and B are linearly dependent. Then from (12) we get $\bar{N}$ and B are linearly dependent, i.e, γ is a Mannheim curve of α.

Theorem 3.3. Let α be a non-lightlike Frenet curve in E_1^3. If γ is the osculating-direction curve of α, then the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ of the curve γ are given as follows,

i) $\overline{\kappa} = \tau(s)cosh(-\int \kappa(s)ds), \overline{\tau} = -\tau(s)sinh(-\int \kappa(s)ds),$ (13)

where α is a timelike curve and γ is a spacelike curve type 2 or α is a spacelike curve of type 1 and γ is a timelike curve,

ii) $\overline{\kappa} = \tau(s)sinh(-\int \kappa(s)ds), \overline{\tau} = -\tau(s)cosh(-\int \kappa(s)ds),$ (14)

where both α and γ are timelike curves or α is a spacelike curve of type 1 and γ is a spacelike curve type 2,

iii) $\overline{\kappa} = \tau(s)sin(-\int \kappa(s)ds),\ \overline{\tau} = \tau(s)cos(-\int \kappa(s)ds),$ (15)

where α is a spacelike curve of type 2 and γ is a spacelike curve type 1.

Proof: i) From (6) and (12), we have

$$\bar{\kappa}\bar{N} = v\tau B. \tag{16}$$

By considering (9) and (16), we obtain

$$\overline{\kappa}\overline{N} = \tau(s)cosh\left(-\int \kappa(s)ds\right)B \tag{17}$$

which gives us

$$\overline{\kappa} = \tau(s)cosh\left(-\int \kappa(s)ds\right). \tag{18}$$

Besides, from (17) and (18) we can write

$$\overline{N} = B. \tag{19}$$

Then, we have

$$\overline{B} = \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}\overline{T} \times \overline{N} = \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}\left(sinh\left(-\int \kappa(s)ds\right)T + cosh\left(-\int \kappa(s)ds\right)N\right) \times B$$

$$\overline{B} = \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}cosh\left(-\int \kappa(s)ds\right)T - \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}sinh\left(-\int \kappa(s)ds\right)N. \tag{20}$$

Differentiating (20) with respect to s, we have

$$\overline{B}' = \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}\left(-2\kappa(s)sinh\left(-\int \kappa(s)ds\right)T + 2\kappa(s)cosh\left(-\int \kappa(s)ds\right)N - \tau(s)sinh\left(-\int \kappa(s)ds\right)B\right). \tag{21}$$

Since $\overline{\tau} = \varepsilon_{\overline{T}}\langle\overline{B}', \overline{N}\rangle = \varepsilon_{\overline{T}}\langle\overline{B}', B\rangle$ and $\varepsilon_{\overline{N}} = \langle\overline{N}, \overline{N}\rangle = 1$, (21) gives us

$$\overline{\tau} = -\tau(s)sinh\left(-\int \kappa(s)ds\right). \tag{22}$$

ii) By considering (10) and (16), we obtain

$$\overline{\kappa}\overline{N} = \tau(s)sinh\left(-\int \kappa(s)ds\right)B \tag{23}$$

which gives us

$$\overline{\kappa} = \tau(s)sinh\left(-\int \kappa(s)ds\right). \tag{24}$$

Besides, from (23) and (24) we can write

$$\overline{N} = B. \tag{25}$$

Then, we have

$$\overline{B} = \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}\overline{T} \times \overline{N} = \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}\left(cosh\left(-\int \kappa(s)ds\right)T + sinh\left(-\int \kappa(s)ds\right)N\right) \times B$$

$$\overline{B} = \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}sinh\left(-\int \kappa(s)ds\right)T - \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}cosh\left(-\int \kappa(s)ds\right)N. \tag{26}$$

Differentiating (26) with respect to s, we have

$$\overline{B}' = \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}\left(-2\kappa(s)cosh\left(-\int \kappa(s)ds\right)T + 2\kappa(s)sinh\left(-\int \kappa(s)ds\right)N - \tau(s)cosh\left(-\int \kappa(s)ds\right)B\right). \tag{27}$$

Since $\overline{\tau} = \varepsilon_{\overline{T}}\langle\overline{B}', \overline{N}\rangle = \varepsilon_{\overline{T}}\langle\overline{B}', B\rangle$ and $\varepsilon_{\overline{N}} = \langle\overline{N}, \overline{N}\rangle = 1$, (27) gives us

$$\overline{\tau} = -\tau(s)cosh\left(-\int \kappa(s)ds\right). \tag{28}$$

iii) By considering (11) and (16), we obtain

$$\overline{\kappa}\overline{N} = \tau(s)sin\left(-\int \kappa(s)ds\right)B \tag{29}$$

which gives us

$$\overline{\kappa} = \tau(s)sin\left(-\int \kappa(s)ds\right). \tag{30}$$

Besides, from (29) and (30) we can write

$$\overline{N} = B. \tag{31}$$

Then, we have

$$\overline{B} = \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}\overline{T} \times \overline{N} = \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}\left(cos\left(-\int \kappa(s)ds\right)T + sin\left(-\int \kappa(s)ds\right)N\right) \times B$$

$$\overline{B} = \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}sin\left(-\int \kappa(s)ds\right)T - \varepsilon_{\overline{T}}\varepsilon_{\overline{N}}cos\left(-\int \kappa(s)ds\right)N. \tag{32}$$

Differentiating (32) with respect to s, we have

$$\overline{B}' = -\varepsilon_{\overline{T}}\varepsilon_{\overline{N}}\tau(s)cos\left(-\int \kappa(s)ds\right)B. \tag{33}$$

Since $\overline{\tau} = \varepsilon_{\overline{T}}\langle \overline{B}', \overline{N}\rangle = \varepsilon_{\overline{T}}\langle \overline{B}', B\rangle$ and $\varepsilon_{\overline{N}} = \langle \overline{N}, \overline{N}\rangle = -1$, (33) gives us

$$\overline{\tau} = \tau(s)cos\left(-\int \kappa(s)ds\right). \tag{34}$$

Hence, the proof is completed.

Corollary 3.1. Let α be a non-lightlike Frenet curve in E_1^3. If γ is the osculating-direction curve of α, then

i) $\overline{T} = -\frac{\overline{\tau}}{\tau}T + \frac{\overline{\kappa}}{\tau}N, \quad \overline{N} = B, \quad \overline{B} = \frac{\overline{\kappa}}{\tau}T + \frac{\overline{\tau}}{\tau}N$

where α is a timelike curve and γ is a spacelike curve type 2 or α is a spacelike curve of type 1 and γ is a timelike curve,

ii) $\overline{T} = -\frac{\overline{\tau}}{\tau}T + \frac{\overline{\kappa}}{\tau}N, \quad \overline{N} = B, \quad \overline{B} = \frac{\overline{\kappa}}{\tau}T + \frac{\overline{\tau}}{\tau}N$

where both α and γ are timelike curves or α is a spacelike curve of type 1 and γ is a spacelike curve type 2,

iii) $\overline{T} = \frac{\overline{\tau}}{\tau}T + \frac{\overline{\kappa}}{\tau}N, \quad \overline{N} = B, \quad \overline{B} = \frac{\overline{\kappa}}{\tau}T - \frac{\overline{\tau}}{\tau}N,$

where α is a spacelike curve of type 2 and γ is a spacelike curve type 1.

Proof: The proof is clear from Theorem 3.3.

Theorem 3.4. Let γ be a non-lightlike osculating-direction curve of α with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$. Then the curvature κ and the torsion τ of α are given by

i) $\kappa = \frac{\overline{\kappa}^2}{\overline{\kappa}^2 - \overline{\tau}^2}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'$, $\quad \tau = \sqrt{\overline{\kappa}^2 - \overline{\tau}^2}$

where α is a timelike curve and γ is a spacelike curve type 2 or α is a spacelike curve of type 1 and γ is a timelike curve,

ii) $\kappa = \frac{\overline{\tau}^2}{\overline{\tau}^2 - \overline{\kappa}^2}\left(\frac{\overline{\kappa}}{\overline{\tau}}\right)'$, $\quad \tau = \sqrt{\overline{\tau}^2 - \overline{\kappa}^2}$

where both α and γ are timelike curves or α is a spacelike curve of type 1 and γ is a spacelike curve type 2,

iii) $\kappa = \frac{\overline{\kappa}^2}{\overline{\kappa}^2 + \overline{\tau}^2}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'$, $\quad \tau = \sqrt{\overline{\kappa}^2 + \overline{\tau}^2}$

where α is a spacelike curve of type 2 and γ is a spacelike curve type 1.

Proof: i) From (18) and (22), we easily get

$$\tau = \sqrt{\overline{\kappa}^2 - \overline{\tau}^2}. \tag{35}$$

Substituting (35) into (18) and (22), we obtain respectively

$$cosh\left(-\int \kappa(s)ds\right) = \frac{\overline{\kappa}}{\sqrt{\overline{\kappa}^2 - \overline{\tau}^2}} \tag{36}$$

$$sinh\left(-\int \kappa(s)ds\right) = -\frac{\overline{\tau}}{\sqrt{\overline{\kappa}^2 - \overline{\tau}^2}} \tag{37}$$

Differentiating (37) with respect to s, we have

$$-\kappa cosh\left(-\int \kappa(s)ds\right) = -\frac{\overline{\tau}'}{\sqrt{\overline{\kappa}^2 - \overline{\tau}^2}} + \frac{\overline{\tau}\left(\overline{\kappa}\,\overline{\kappa}' - \overline{\tau}\,\overline{\tau}'\right)}{\left(\overline{\kappa}^2 - \overline{\tau}^2\right)^{3/2}}. \tag{38}$$

From (18), (35) and (38), it follows

$$\kappa = \frac{\overline{\kappa}\,\overline{\tau}' - \overline{\tau}\,\overline{\kappa}'}{\overline{\kappa}^2 - \overline{\tau}^2}$$

or equivalently, we get

$$\kappa = \frac{\overline{\kappa}^2}{\overline{\kappa}^2 - \overline{\tau}^2}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'. \tag{39}$$

ii) From (24) and (28), we easily get

$$\tau = \sqrt{\overline{\tau}^2 - \overline{\kappa}^2}. \tag{40}$$

Substituting (40) into (24) and (28), we obtain respectively

$$sinh\left(-\int \kappa(s)ds\right) = \frac{\overline{\kappa}}{\sqrt{\overline{\tau}^2 - \overline{\kappa}^2}} \tag{41}$$

$$cosh\left(-\int \kappa(s)ds\right) = -\frac{\overline{\tau}}{\sqrt{\overline{\tau}^2 - \overline{\kappa}^2}} \tag{42}$$

Differentiating (42) with respect to s, we have

$$-\kappa sinh\left(-\int \kappa(s)ds\right) = -\frac{\overline{\tau}'}{\sqrt{\overline{\tau}^2 - \overline{\kappa}^2}} + \frac{\overline{\tau}\left(\overline{\tau}\,\overline{\tau}' - \overline{\kappa}\,\overline{\kappa}'\right)}{\left(\overline{\tau}^2 - \overline{\kappa}^2\right)^{3/2}}. \tag{43}$$

From (24), (40) and (43), it follows

$$\kappa = \frac{\overline{\tau}\,\overline{\kappa}' - \overline{\kappa}\,\overline{\tau}'}{\overline{\tau}^2 - \overline{\kappa}^2}$$

or equivalently, we get

$$\kappa = \frac{\overline{\tau}^2}{\overline{\tau}^2 - \overline{\kappa}^2}\left(\frac{\overline{\kappa}}{\overline{\tau}}\right)'. \tag{44}$$

iii) From (32) and (36), we easily get

$$\tau = \sqrt{\overline{\kappa}^2 + \overline{\tau}^2}. \tag{45}$$

Substituting (45) into (30) and (34), we obtain respectively

$$sin\left(-\int \kappa(s)ds\right) = \frac{\overline{\kappa}}{\sqrt{\overline{\kappa}^2 + \overline{\tau}^2}} \tag{46}$$

$$cos\left(-\int \kappa(s)ds\right) = \frac{\overline{\tau}}{\sqrt{\overline{\kappa}^2 + \overline{\tau}^2}} \tag{47}$$

Differentiating (47) with respect to s, we have

$$\kappa sin\left(-\int \kappa(s)ds\right) = \frac{\overline{\tau}'}{\sqrt{\overline{\kappa}^2 + \overline{\tau}^2}} - \frac{\overline{\tau}\left(\overline{\kappa}\,\overline{\kappa}' + \overline{\tau}\,\overline{\tau}'\right)}{\left(\overline{\kappa}^2 + \overline{\tau}^2\right)^{3/2}}. \tag{48}$$

From (30), (45) and (48), it follows

$$\kappa = \frac{\overline{\kappa}\,\overline{\tau}' - \overline{\tau}\,\overline{\kappa}'}{\overline{\kappa}^2 + \overline{\tau}^2}$$

or equivalently, we get

$$\kappa = \frac{\overline{\kappa}^2}{\overline{\kappa}^2 + \overline{\tau}^2}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'. \tag{49}$$

Hence, the proof is completed.

Corollary 3.2. Let γ be a non-lightlike osculating-direction curve of α with the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$. Then the curvature κ and the torsion τ of α satisfy following equations

i) $$\frac{\kappa}{\tau} = \frac{\overline{\kappa}^2}{\left(\overline{\kappa}^2 - \overline{\tau}^2\right)^{3/2}} \left(\frac{\overline{\tau}}{\overline{\kappa}}\right)',$$

where α is a timelike curve and γ is a spacelike curve type 2 or α is a spacelike curve of type 1 and γ is a timelike curve,

ii) $$\frac{\kappa}{\tau} = \frac{\overline{\tau}^2}{\left(\overline{\tau}^2 - \overline{\kappa}^2\right)^{3/2}} \left(\frac{\overline{\kappa}}{\overline{\tau}}\right)',$$

where both α and γ are timelike curves or α is a spacelike curve of type 1 and γ is a spacelike curve type 2,

iii) $$\frac{\kappa}{\tau} = \frac{\overline{\kappa}^2}{\left(\overline{\kappa}^2 + \overline{\tau}^2\right)^{3/2}} \left(\frac{\overline{\tau}}{\overline{\kappa}}\right)',$$

where α is a spacelike curve of type 2 and γ is a spacelike curve type 1.

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