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Semilattice Co-Congruence in Γ-Semigroups

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Abstract. As a generalization of semigroups, Sen, in 1981, introduced the concept of Γ-semigroups. In the author's paper (D A. Romano. Γ-semigroups with apartness. *Bull. Allahabad Math. Soc.*, 34(1)(2019), 71–83.) it is introduced and analyzed the concept of Γ-semigroups with apartness in Bishop's constructive framework. In this article, as a continuation of previous research, the concept of co-congruences in Γ-semigroups is introduced and analyzed. Additionally, it is investigated (co-ordered) semillatice co-congruence on (co-ordered) Γ-semigroup with apartness.

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1. Introduction

Let (S, w) be a grupoid, where the set S is the carrier of this structure and $w : S \times S \longrightarrow S$ is a total function. If w is associative, then the structure is (S, w) a semigroup. Let us suppose that the logical environment in which we analyze this algebraic structure is the Intuitionistic logic **IL** [20]. This assumption implies that the axiom 'Principe TND' (tertium non datur - the logical principle of 'the exclusion of the third') is not valid in this setting. In this logic, the diversity is a fundamental concept equal to the concept of equality in the Classical logic. Commitments under which we will obey in this text is the Bishop's principled-philosophical orientation **Bish** (see, for example: [1, 2, 9]). Now, we look at the carrier S as a relational system $(S, =, \neq)$, where $' = '$ is the standard equality, and $' \neq '$ is an apartness [10]:

$(\forall x, y \in S)(x \neq y \implies \neg(x = y))$ (consistency);
$(\forall x, y \in S)(x \neq y \implies y \neq x)$ (symmetry);
$(\forall x, y, z \in S)(x \neq z \implies (x \neq y \vee y \neq z))$ (co-transitivity);

This relation is extensive with respect to the equality in the standard way

$$= \circ \neq \subseteq \neq \text{ and } \neq \circ = \subseteq \neq$$

where $' \circ '$ is the standard composition of relations. In addition, any relation R on S, any functions f between such sets and any operation w in S appearing in this article are strongly extensional relative to the apartness (see, for example: [10, 13]). For a strongly extensional mapping we will hereafter briefly write 'se-mapping'. Because of the

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specificity of **IL**, for some subsets of the S, their strictly extensive doubles will appears. For example, a (strongly extensional) subset K of a semigroup S with apartness is a *co-ideal* of S if holds

$$(\forall x, y \in S)(xy \in K \implies x \in K \vee y \in K).$$

It is not difficult to show that a strong complement

$$K^{\triangleleft} = \{x \in S : (\forall s \in K)(x \neq s)\}$$

of co-ideal K is an ideal in S. Conversely, in the general case, it does not have to be valid. Thus, the observed structure $S = ((S, =, \neq), w)$ is a *semigroup with apartness*. In the last 40 years, this author is alone, or in collaboration with other authors, has investigated structures of various types of semigroups with apartnesses [4, 5, 12]. In the article [3], Cherubini and Frigeri introduce the concept of 'inverse semigroups with apartness'. One of the main problems in these researches was "How to find and describe the duals of classical algebraic concepts?" in well-known algebraic structures.

In this paper we interested Γ - semigroups with apartness as a continuation of our research [12,15]. Also, we will find and analyze some doubles of substructures of these semigroups. Our investigation the concept of Γ-semigroups with appartness consists of the observation of specificities that arise by placing the classically defined algebraic structure of Γ-semigroups ([6–8, 16–19]) into a different logical environment and using specific Bishop's constructive algebra tools.

The rest of the paper is organized as follows: In Section 2, the concept of Γ-semigroup with apartness (Definition 2.1) and the concept of Γ-cosubsemigroup with apartness (Definition 2.2) are given. In Section 3, the notion of ordered Γ-semigroup with apartess ordered under a co-order relation is given (Definition 3.1). Additionally, this section analyzes the concept of semilattice co-congruence on (co-ordered) Γ-semigroups with apartness.

2. Preliminaries

2.1. Γ-semigroup with apartness.

To explain the notions and notations used in this article, which but not previously described, we instruct the reader to look at the articles [7, 8, 16–19]. Here we will introduce some specific substructures of this semigroups that appear only in the **Bish** version.

Definition 2.1 ([12], Definition 2.1)**.** Let $(S, =, \neq)$ and $(\Gamma, =, \neq)$[1] be two non-empty sets with apartness. Then S is called a Γ-*semigroup with apartness* if there exist a strongly extensional mapping from $S \times \Gamma \times S \ni (x, a, y) \longmapsto xay \in S$ satisfying the condition

$$(\forall x, y, z \in S)(\forall a, b \in \Gamma)((xay)bz = xa(ybz)).$$

We recognize immediately that the following implications

$$(\forall x, y, u, v \in S)(\forall a, b \in \Gamma)(xay \neq ubv \implies (x \neq u \vee a \neq b \vee y \neq v)),$$

$$(\forall x, y \in S)(\forall a, b \in \Gamma)(xay \neq xby \implies a \neq b)$$

are valid, because f is a strongly extensional function.

Example 2.2. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N}$ be a set of natural numbers and $\Gamma = \mathbb{Q}$ be the set of rational numbers. It is well known (see, for example [1]) that in $\mathbb{R}$ there is an apartness relation '≠' defined by

$$(\forall a, b \in \mathbb{R})(a \neq b \iff (\exists k \in \mathbb{N})(|a - b| > \frac{1}{k})).$$

Define a mapping from $\mathbb{R} \times \Gamma \times \mathbb{R}$ to $\mathbb{R}$ by $a\alpha b$ = usual product of a, α, b for $a, b \in \mathbb{R}$ and $\alpha \in \mathbb{Q}$. Then $\mathbb{R}$ is a Γ-semigroup with apartness.

Example 2.3. Let S be the set $M_{2\times3}(\mathbb{R})$ of all 2×3 matrices over the set of real numbers $\mathbb{R}$ and Γ be the set $M_{3\times2}(\mathbb{R})$ of all 3×2 matrices over $\mathbb{R}$. Apartness '≠' is defined in $M_{2\times3}(\mathbb{R})$ in the standard way

$$(\forall A, B \in M_{2\times3}(\mathbb{R}))(A \neq B \iff (\exists (i, j) \in \{1, 2\} \times \{1, 2, 3\})(a_{ij} \neq b_{ij})).$$

The apartness relation in $M_{3\times2}(\mathbb{R})$ is defined analogously. Define $A\alpha B$ = usual matrix product of A, α, B for all $A, B \in S$ and for all $\alpha \in \Gamma$. Then S is a Γ-semigroup with apartness. Note that S is not a semigroup.

[1] We will not write different apartness relations in different sets on different way, unless it is needed. From the context and to the use of different variables of marking of elements of different sets, it will be clear what type of apartness relation is involved.

Definition 2.4 ([12], Definition 2.2)**.** Let S be a Γ-semigroup with apartness. A subset T of S is said to be a Γ-*cosubsemigroup* of S if the following holds

$$(\forall x, y \in S)(\forall a \in \Gamma)(xay \in T \implies (x \in T \lor y \in T)).$$

We will assume that the empty set $\emptyset$ is a Γ-cosubsemigroup of a Γ-semigroup S by definition.

Our first proposition in this section is the following:

Proposition 2.5 ([12], Proposition 2.1)**.** *If T is a Γ-cosubsemigroup of a Γ - semigroup with apartness S, then the set $T^{\lhd}$ is a Γ - subsemigroup of S.*

The following definition introduced the concept of co-ideal in these semigroups.

Definition 2.6 ([12], Definition 2.5)**.** A strongly extensional subset B of a Γ-semigroup with apartness S is said to be a Γ-*coideal* of S if the following implication holds

$$(\forall x, y \in S)(\forall a \in \Gamma)(xay \in B \implies x \in B \land y \in B).$$

Proposition 2.7 ([12], Proposition 2.6)**.** *If B is a Γ-coideals of a Γ-semigroup with apartness S, then the set $B^{\lhd}$ is a Γ-ideal of S.*

2.2. **Co-ordered Γ-semigroup with apartness.** The relation α is said to be a co-order on the set X with apartness if it is consistent $\alpha \subseteq \neq$, co-transitive $\alpha \subseteq \alpha * \alpha$, i.e.

$$(\forall c, y, z \in X)((x, z) \in \alpha \implies ((x, y) \in \alpha \lor (y, z) \in \alpha))$$

and linear in the following sense: $\neq \subseteq \alpha \cup \alpha^{-1}$. α is said to be a co-quasiorder on X if it is a consistent and co-transitive relation on X. An interested reader can find more information about this type of relations in sets (in algebraic structures) in one of our following papers [10,11,13]. A brief recapitulation of a number of algebraic structures ordered by co-quasiorder relation is presented in the article [13].

In the following definition we introduce the concept of co-order relations in Γ-semigroup with apartnesst.

Definition 2.8 ([15], Definition 3.1)**.** Let S be a Γ-semigroup with apartness. A co-order relation $\nleqslant$ on S is *compatible* with the semigroup operations in S if the following holds

$$(\forall x, y, z \in S)(\forall a \in \Gamma)((xaz \nleqslant yaz \lor zax \nleqslant zay) \implies x \nleqslant y).$$

In this case it is said that S is an *ordered* Γ-semigroup *under co-order* $\nleqslant$ or it is *co-ordered* Γ-semigroup.

Example 2.9. Let S and Γ be as in Example 2.3. Also, S and Γ are co-ordered sets with respect to $\nleqslant$, defined by

$$(\forall A, B \in M_{2\times 3}(\mathbb{R}))(A \nleqslant B \iff (\exists (i, j) \in \{1, 2\} \times \{1, 2, 3\}(a_{ij} \nleqslant b_{ij})).$$

The co-order relation in Γ can be determined in an analogous way. Then S is a co-ordered Γ-semigroup with apartness.

Example 2.10. Let S be a set of all reverse isotone se-mappings from co-ordered set $((X, =, \neq), \nleqslant)$ into another co-ordered set $((Y, =, \neq), \nleqslant)$ and Γ be the set of all reverse isotone se-mappings from Y into X. Let $f, g \in S$ and $\alpha \in \Gamma$. Denote by $f \cdot \alpha \cdot g$ the usual mapping composition $f \circ \alpha \circ g$. Then S is a Γ-semigroup with apartness. We can define a relation '$\nleqslant$' on S by $f \nleqslant g$ if and only if there exists an element $x \in S$ such that $f(x) \nleqslant g(x)$. This relation is a co-order on S. The co-order relation on the set Γ can be determined analogously. Hence, S is a co-ordered semigroup with apartness.

The concept of ordered co-ideals in such semigroups is introduced as follows.

Definition 2.11. For Γ-coideal B in a Γ-semigroups with apartness S ordered under co-order $\nleqslant$ it is said that an *ordered* Γ-*coideal* in S if

$$(\forall x, y \in S)(x \in B \implies (y \in B \lor x \nleqslant y)).$$

We take the notion of co-filters from the article [14].

Definition 2.12 ([14], Definition 3.2)**.** Let S be an ordered Γ-semigroup with apartness ordered under co-order $\nleqslant$. A Γ-cosubsemigroup G of S is called a *co-filter* of S if

(a) $(\forall x, y \in S)(\forall a \in \Gamma)(x \in G \implies (xay \in G \land yax \in G))$,
(b) $(\forall x, y \in S)(y \in G \implies (x \in G \lor x \nleqslant y))$.

Proposition 2.13 ([14], Proposition 3.4)**.** *Let G be a co-filter in an ordered Γ-semigroup S with apartness under co-order $\nleqslant$. Then $G^{\triangleleft}$ is a filter in S.*

3. The Concept of Semillatice Co-Congruences

The notion of the co-equality relation in sets with apartness introduced and analyzed by this author (see, for example, [10, 14]). The relation q is a co-equality on a set $(S, =, \neq)$ if it is a consistent, symmetric and co-transitive relation on S. A co-congruence on some algebraic structure $((S, =, \neq), \cdot)$ is a coequality relation on S which is compatible in one very specific sense with the internal operation in S. The reader can look at these specific features in our previously published texts [3, 10, 13, 14]. A co-equality relation q on a groupoid S with apartness is a co-congruence in S if the following holds

$$(\forall x, y, u, v \in S)((xu, yv) \in q \implies ((x, y) \in q \vee (u, v)).$$

The concept of Γ-cocongruence on Γ-semigroups with apartness was introduced in [12] by the following definition

Definition 3.1 ([12], Definition 2.6)**.** Let S be a Γ - semigroup with apartness. A coequality relation $q \subseteq S \times S$ is called a Γ - *cocongruence* on S if the following holds

(1) $((xau, ybv) \in q \implies ((x, y) \in q \vee a \neq b \vee (u, v) \in q))$

for any $x, y, u, v \in S$ and all $a, b \in \Gamma$.

Lemma 3.2. *The condition* (1) *in Definition 3.1 is equivalent to the following two conditions*

(2) $(\forall x, y, z \in S)(\forall a, b \in \Gamma)((zax, zby) \in q, \implies ((x, y) \in q \vee a \neq b))$ *and*

(3) $(\forall x, y, z \in S)(\forall a, b \in \Gamma)((xaz, ybz) \in q \implies ((x, y) \in q \vee a \neq b))$

Proof. (1) $\implies$ (2) $\wedge$ (3). Let q be a Γ-cocongruence on a Γ-semigroups S with apartness and let $x, y, z \in S$ and $a \in \Gamma$ be arbitrary elements such that $(zax, zay) \in q$. Then $(x, y) \in q$ according to previous definition. From $(xaz, yaz) \in q$ follows $(x, y) \in q$ by analogy, too.

(2) $\wedge$ (3) $\implies$ (1). Conversely, let us prove that from a given conditions (2) and (3) for co-equality q on S it follows that q is a Γ-cocongruence on Γ-semigroup S. Let $x, y, u, v \in S$ and $a, b \in \Gamma$ are arbitrary elements such that $(xau, ybv) \in q$. Then $(xau, xav) \in q$ or $(xav, ybv) \in q$ by co-transitivity of q. Thus $(u, v) \in q$ by (2) and $(x, y) \in q \vee a \neq b$ by (3). □

The concept of semilattice co-congruences on an ordered Γ-semigroup with apartness S under a co-order $'\nleqslant'$ is given in the following definition

Definition 3.3. A co-congruence q on a co-ordered Γ-semigroup with apartness S is a *semillattice co-congruence* on S if holds

(4) $(\forall x, y \in S)(\forall a \in \Gamma)((x, xay) \triangleleft q)$ and

(5) $(\forall x, y \in S)(\forall a \in \Gamma)((xay, yax) \triangleleft q)$.

A semilattice co-congruence q on S is called a *co-ordered semilattice co-congruence* if the following holds

(6) $(\forall x, y \in S)(\forall a \in \Gamma)((x, xay) \in q \implies x \nleqslant y)$.

Lemma 3.4. *The condition* (6) *implies condition*

(7) $(\forall x, y \in S)(\forall a \in \Gamma)(x \nleqslant^{\triangleleft} y \implies (x, xay) \triangleleft q)$.

Proof. Let $x, y, u, v \in S$ and $a \in \Gamma$ be arbitrary elements such that $x \nleqslant^{\triangleleft}$ and $(u, v) \in q$. Then $(u, x) \in q$ or $(x, xay) \in q$ or $(xay, v) \in q$ by co-transitivity of relation q. From the second option $(x, xay) \in q$ immediately follows $x \nleqslant y$ by (6). It is in contradiction with hypothesis. So, it have to be $x \neq x \vee xay \neq v$ by consistency of q. This means $(x, xay) \neq (u, v) \in q$. So, $(x, xay) \triangleleft q$. This proves conditions (7). □

In the following proposition we give a description of the q-class xq of the relation q for any $x \in S$.

Proposition 3.5. *If q is a semillatice co-congruence on a Γ-semigroups with apartness S, then for any $x \in S$, the q-class $xq = \{y \in S : (x, y) \in q\}$ is a strongly extensional subset in S and the following hold*

(8) $(\forall u, v \in S)(\forall a \in \Gamma)(uav \in xq \implies (u \in xq \wedge v \in xq))$ *and*

(9) $(\forall u, v \in S)(\forall a \in \Gamma)(u \in xq \implies (uav \in xq \wedge vau \in xq))$.

Proof. Let $u, v \in S$ be arbitrary elements such that $u \in xq$. Then, from $(x, u) \in q$ follows $(x, v) \in q \vee (v, u) \in q$ by co-transitivity of q. Thus $v \in xq \vee u \neq v$. So, the set xq is a strongly extensional subset in S.

Let $u, v \in S$ and $a \in \Gamma$ be arbitrary elements such that $uav \in xq$. Then $(x, uav) \in q$. Thus $(x, xav) \in q \vee (xav, uav) \in q$ by co-transitivity of q. Now, we have $(x, u) \in q$ by (4) and (3). So, $u \in xq$. Similarly, from $(x, uav) \in q$ follows $(x, xau) \in q \vee (xau, vau) \in q \vee (vau, uav) \in q$. Since the first and second options are impossible by (4) and (5), we have to have $(xau, vau) \in q$. From this follows $(x, v) \in q$ by (3). So, $v \in xq$. Finally, we have $u \in xq \wedge v \in xq$.

Let $u, v \in S$ and $a \in \Gamma$ be arbitrary elements such that $u \in xq$. Then $(x, u) \in q$. Thus $(x, uav) \in q \vee (uav, u) \in q$. Since the second option is impossible by (4), we have $(x, uav) \in q$. Similarly, $(x, u) \in q \implies (x, vau) \in q \vee (vau, u) \in q$. Thus $vau \in xq$ by (4) too. With this we have proved (9). □

Proposition 3.6. *If q is a co-ordered semillatice co-congruence on a Γ-semi- groups with apartness S, then for any $x \in S$, the q-class xq the following satisfies*

$$(\forall u, v \in S)(u \in xq \implies (v \in xq \vee v \nleqslant u \vee u \nleqslant v)).$$

Proof. If $u \in xq$, i.e if $(x, u) \in q$, then $(x, v) \in q \vee (v, vau) \in q \vee (vau, uav) \in q \vee (uav, u) \in q$. Thus $c \in xq$ or $v \nleqslant u$ or $u \nleqslant v$ by (6). □

Example 3.7. First, recall (see for example [1]) that a subset of M of S is said to be detachable in S if

$$(\forall x \in S)(x \in M \vee x \lhd M).$$

1. Let M be a detachable subset in a Γ-semigroup S. Then the relation q on S defined by

$$(\forall x, y \in S)((x, y) \in q \iff ((x \in M \wedge y \lhd M) \vee (x \lhd M \wedge y \in M)))$$

is a co-equality on S. Obviously, the relationship defined in this way is consistent and symmetric. Let us prove the co-transitivity of the relation q. Let $x, y, z \in S$ be arbitrary elements such that $(x, z) \in q$. Then $x \in M \wedge z \lhd M$ or $x \lhd M \wedge z \in M$. For the element y is true $y \in M$ or $y \lhd M$ since M is a detachable subset in S. Now

$x \in M \wedge z \lhd M \wedge y \in M \implies (y, z) \in q,$
$x \in M \wedge z \lhd M \wedge y \lhd M \implies (x, y) \in q,$

or

$x \lhd M \wedge z \in M \wedge y \in M \implies (x, y) \in q$ and
$x \lhd M \wedge z \in M \wedge y \lhd M \implies (y, z) \in q.$

2. Second, in addition to the previous hypothesis, suppose that M is a co-ideal and a co-filter in S synchronically. Let us prove that q is a co-congruence on S. Let $x, y, z \in S$ and $a, b \in \Gamma$ be arbitrary elements such that $(xaz, yaz) \in q$. Them

$$(xaz \in M \wedge yaz \lhd M) \vee (xaz \lhd M \wedge yaz \in M).$$

Since $M^{\lhd}$ is an ideal and a filter in S also, we have

$(xaz \in M \wedge yaz \lhd M) \implies (x \in M \wedge y \lhd N)$ and
$(xaz \lhd M \wedge yaz \in M) \implies (x \lhd M \wedge y \in M).$

So, the relation q is a co-congruence on S.

Let $x, y, u, v \in S$ and $a \in \Gamma$ be arbitrary elements such that $(u, v) \in q$. Then $u \in M \wedge v \lhd M$ or $u \lhd M \wedge v \in M$. Thus

$$((u \neq xay \vee xay \in M) \wedge v \lhd M) \vee (u \lhd M \wedge (v \neq yax \vee yax \in M)).$$

Further on, we have

$$(u \neq xay \vee (xay \in M \wedge v \lhd M)) \vee ((u \lhd M \wedge yax \in M) \vee v \neq yax)$$

and

$$(xay, yax) \neq (u, v) \in q \vee (xay, v) \in q \vee (y, yax) \in q.$$

Finally, we have

$$(xay, yax) \neq (u, v) \in q \vee (xay, yax) \neq (v, u) \in q.$$

So, for the co-congruence q on S holds (5).

Using the technique analogous to the technique used in the previous verification, it can be proven (4). Therefore, the relation q constructed in this way is a semilattice co-congruence on S.

Let S be a Γ-semigroup and $a \in \Gamma$. Then $x \in S$ is said to be an a-idempotent if $xax = x$. The set of all a-idempotents is denoted by E_a and we denote $\bigcup_{a\in\Gamma} E_a$ by $E(S)$. The elements of $E(S)$ are called idempotent element of S.

Theorem 3.8. *If q is a semillatice co-congruence on a Γ-semigroups S with apartness, then the set $S/(q^{\triangleleft}, q) = \{xq^{\triangleleft} : x \in S\}$ is an idempotent commutative Γ-semigroup with*

$$xq^{\triangleleft} = yq^{\triangleleft} \iff (x, y) \triangleleft q, \quad xq^{\triangleleft} \neq yq^{\triangleleft} \iff (x, y) \in q,$$

and multiplication $' \cdot '$ defined by

$$(\forall x, y \in S)(\forall a \in \Gamma)(xq^{\triangleleft} \cdot a \cdot yq^{\triangleleft} = (xay)q^{\triangleleft}).$$

Proof. Let $x, y, z \in S$ and $a, b \in \Gamma$ be arbitrary elements.

(i) Then

$$\begin{aligned}(xq^{\triangleleft} \cdot a \cdot yq^{\triangleleft}) \cdot b \cdot zq^{\triangleleft} &= (xay)q^{\triangleleft} \cdot b \cdot zq^{\triangleleft} = ((xay)bz)q^{\triangleleft} \\ &= (xa(ybz))q^{\triangleleft} = xq^{\triangleleft} \cdot a \cdot (ybz)q^{\triangleleft} \\ &= xq^{\triangleleft} \cdot a \cdot (yq^{\triangleleft} \cdot b \cdot zq^{\triangleleft}).\end{aligned}$$

So, the set $S/(q^{\triangleleft}, q)$ is a Γ-semigroup.

(ii) By (5), we have

$$xq^{\triangleleft} \cdot a \cdot yq^{\triangleleft} = (xay)q^{\triangleleft} = (yax)q^{\triangleleft} = yq^{\triangleleft} \cdot a \cdot xq^{\triangleleft}.$$

So, the Γ-semigroup with apartness $S/(q^{\triangleleft}, q)$ is commutative.

(iii) According to (4), we have

$$xq^{\triangleleft} \cdot a \cdot xq^{\triangleleft} = (xax)q^{\triangleleft} = xq^{\triangleleft}.$$

So every element in $S/(q^{\triangleleft}, q)$ is an idempotent. □

The following theorem is one of the specificities of the Bishop's orientation and has no counterpart in classical algebra. Since its proof can be demonstrated by analogy with the proof in the previous theorem, we will omit it.

Theorem 3.9. *If q is a semillatice co-congruence on a Γ-semigroups with apartness S, then the set $[S : q] = \{xq : x \in S\}$ is an idempotent commutative Γ-semigroup with*

$$xq = yq \iff (x, y) \triangleleft q, \quad xq \neq yq \iff (x, y) \in q,$$

and multiplication $'' \star ''$ defined by

$$(\forall x, y \in S)(\forall a \in \Gamma)(xq \star a \star yq = (xay)q).$$

4. Final Comment

Although co-ordered Γ-semigroup with apartness is the focus of this report, our attention is more focused on the concept of semilattice co-congruences on these semigroups. As an extension of this research, we could look for answers to the requirement that co-ordered semilattice co-congruence on co-ordered Γ-semigroup with apartness be some of the recognizable type of co-congruences.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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